

PERTURBATION OF PURE-CUBIC OPTICAL SOLITONS WITH KERR AND POWER-LAWS OF SELF-PHASE MODULATION

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Abstract. This study investigates optical soliton dynamics governed by a third-order nonlinear Schrödinger equation (NLSE) in which the chromatic dispersion term is neglected, a regime that has recently attracted significant attention in nonlinear optics. Both the Kerr and power-law nonlinear forms of the NLSE are considered to model pure-cubic optical nonlinearity under the influence of perturbative effects. Analytical solutions are constructed using the new Kudryashov method, the unified Riccati equation expansion method, and the sub-version of the auxiliary equation expansion method. The applied methods enable the efficient derivation of fundamental localized wave structures, namely bright and dark optical solitons, which play a central role in nonlinear wave propagation in optical media. In addition, a detailed parametric analysis examines how key physical parameters affect the amplitude, shape, and propagation characteristics of the obtained soliton solutions. The results highlight the model's ability to support diverse nonlinear optical regimes and provide further insight into soliton behavior in dispersionless or weakly dispersive media.

Keywords: third-order NLSE, Hamiltonian perturbation, dark and bright soliton

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1. Introduction

The evolution of soliton theory is generally linked to three major milestones: 1834, 1895, and 1965. In 1834, John Scott Russell reported the observation of the “wave of translation,” revealing that a solitary wave can propagate over long distances without altering its shape [1]. Later, in 1895, Diederik Korteweg and Gustav de Vries formulated the Korteweg–de Vries equation, which established the theoretical foundation of solitons through the balance between nonlinearity and dispersion [2]. In 1965, Norman Zabusky and Martin Kruskal numerically demonstrated soliton interactions and introduced the term “soliton” [3]. Subsequent progress led to the development of powerful analytical techniques, most notably the inverse scattering transform and related approaches for solving nonlinear evolution equations [4–7]. With the rapid advancement of nonlinear optics, solitons have become essential in optical fiber communications and a wide range of physical applications [8–15]. In this context, the Schrödinger equation, originally introduced by Erwin Schrödinger in 1926, provides a fundamental framework for modeling wave propagation and plays a central role in optical soliton theory [16].

More recent studies extend these developments by addressing both deterministic and stochastic models of soliton dynamics. In [17], dual-power-law nonlinear structures and their associated conservation laws are analyzed, confirming the stability and physical relevance of the resulting solitons. Paper [18] examines the role of stochastic perturbations, showing that multiplicative noise significantly influences soliton propagation characteristics. In addition, [19] employs advanced analytical techniques to construct dark and singular soliton solutions in higher-order nonlinear systems. Meanwhile, [20] focuses on controlling soliton interactions through the interplay between dispersion and nonlinear effects. Collectively, these contributions highlight the importance of integrating analytical methods with physical insights to better understand and control soliton behavior in complex optical systems.

In nonlinear optics, its modified form, the nonlinear Schrödinger equation (NLSE), is widely used to describe the propagation of optical pulses in nonlinear media [21–23]:

$$i \frac{\partial u}{\partial t} + \frac{\partial^2 u}{\partial x^2} + |u|^2 u = 0, \quad (1)$$

where i denotes the imaginary unit and $u(x, t)$ represents the complex envelope of the optical field. The term iu_t describes temporal evolution, u_{xx} accounts for group velocity dispersion (GVD), and $|u|^2 u$ represents the Kerr nonlinearity, corresponding to intensity-dependent phase modulation. The variables x and t denote normalized propagation distance and retarded time, respectively. Although the equation is generally non-integrable, it admits exact solutions in special cases and possesses a Hamiltonian structure [24].

Following the rapid development of optical communication technologies, several extensions of the NLSE have been proposed to model pulse propagation in realistic fiber systems, governed by the balance between chromatic dispersion (CD) and self-phase modulation (SPM). In 2016, Alberto Blanco-Redondo et al. introduced pure-quartic solitons in the absence of CD [25], stimulating further studies of alternative dispersion regimes. Subsequently, researchers widely investigated pure-cubic NLSE models involving third-order dispersion (TOD) without CD [26–30].

Motivated by these developments, the present work considers the following generalized models:

$$i \left(\frac{\partial u}{\partial t} + \alpha \frac{\partial^3 u}{\partial x^3} \right) + \beta |u|^2 u = i \left(\mu |u|^2 \frac{\partial u}{\partial x} + \sigma \frac{\partial |u|^2}{\partial x} u \right), \quad (2)$$

$$i \left(\frac{\partial u}{\partial t} + \gamma \frac{\partial^3 u}{\partial x^3} \right) + \rho |u|^{2n} u + i \left(c_1 |u|^{2n} \frac{\partial u}{\partial x} + c_2 \frac{\partial |u|^{2n}}{\partial x} u \right) = 0. \quad (3)$$

The terms iu_t , $i\alpha u_{xxx}$, and $\beta |u|^2 u$ correspond to linear temporal evolution, TOD, and Kerr-type SPM, respectively. The term $i\mu |u|^2 u_x$ corresponds to the self-steepening effect, which arises from the intensity dependence of the group velocity. This effect leads to an asymmetric distortion of the pulse profile and the formation of a steeper trailing edge during propagation. The term $i\sigma(|u|^2)_x u$ describes the nonlinear dispersion contribution associated with spatial variations of the pulse intensity. It accounts for the coupling between the intensity gradient and the complex

field envelope, influencing pulse shape evolution and spectral characteristics. Additionally, α, β, μ , and σ are real parameters governing dispersion and nonlinear response. In the absence of perturbations ($\mu = \sigma = 0$), the model reduces to the third-order NLSE with Kerr nonlinearity, which is a fundamental model in ultrafast nonlinear optics.

Eq. (3) extends this framework by introducing power-law nonlinearity, allowing the description of non-Kerr optical media where the refractive index depends on higher-order intensity effects. The governing terms represent linear evolution, TOD, nonlinear interaction, and perturbation-induced nonlinear dispersion, where γ, ρ, c_1, c_2 are real constants and $n \geq 1$ characterizes the order of nonlinearity. Eq. (2) has been widely studied, yielding bright, dark, and singular soliton solutions via various analytical methods [26–34], including recent extensions involving dual-power nonlinearity and stochastic effects [17,18]. Similarly, Eq. (3) has gained attention for modeling non-Kerr dynamics under different perturbative conditions [27,28]. Although Eqs. (2) and (3) have been investigated in several previous studies and various classes of optical solitons have been reported, most existing works focus primarily on obtaining particular exact solutions through specific analytical techniques. A comprehensive comparative analysis of these two models under a unified mathematical framework remains limited. In particular, the influence of third-order dispersion, self-steepening effects, nonlinear dispersion perturbations, and power-law nonlinearities on the existence conditions and parametric structures of different soliton families have not been systematically examined. The present work addresses this gap by employing the proposed analytical approach to derive new closed-form solutions and their corresponding existence constraints, enabling a unified investigation of both Kerr and non-Kerr nonlinear regimes. Furthermore, the results provide a broader classification of optical wave structures and reveal parameter conditions not explicitly reported in earlier studies. Albayrak et al. [35] reported recent progress on pure-cubic optical solitons, studying a Kerr-law nonlinear Schrödinger model incorporating third-order dispersion and nonlinear dispersion effects in the absence of chromatic dispersion. Using a simplified extended auxiliary equation method, they derived bright and singular soliton solutions and analyzed their dynamical behavior through graphical simulations. They also verified the stability of the obtained bright solitons using the Vakhitov–Kolokolov stability criterion. Their results demonstrated the significant influence of higher-order dispersive and nonlinear effects on pure-cubic pulse propagation in nonlinear optical media.

The paper is organized as follows. Section 2 presents the transformation and analysis; Section 3 describes the methods; Section 4 provides the solutions; Section 5 discusses the results; and Section 6 concludes the study.

2. Wave transformation and mathematical investigation

2.1. Mathematical investigation of Eq. (2)

Consider the following wave transformation:

$$\begin{aligned} u(x, t) &= U(\xi) e^{i\theta(x, t)}, \\ \theta(x, t) &= ax + bt + \phi_0, \quad \xi = kx + \omega t, \end{aligned} \quad (4)$$

where $U(\xi)$ is the real amplitude and $\theta(x, t)$ is the phase function. The parameters a , b , k , and ω denote the wave number, frequency, inverse width, and velocity, respectively, and ϕ_0 is the initial phase. All parameters are nonzero real constants. Substituting Eq. (4) into Eq. (2) yields:

$$\left[-k(\mu + 2\sigma)U(\xi)^2 - 3a^2\alpha k + \omega \right] U'(\xi) + \alpha k^3 U'''(\xi) = 0, \quad (5)$$

$$(a\mu + \beta)U(\xi)^3 + (a^3\alpha - b)U(\xi) - 3a\alpha k^2 U''(\xi) = 0. \quad (6)$$

Eq. (5) arises from the imaginary part, whereas Eq. (6) is obtained from the real part of the resulting equation. Integrating Eq. (5) with respect to ξ , while ignoring the integration constant, leads to the following nonlinear ordinary differential equation (NODE):

$$-k(\mu + 2\sigma)U(\xi)^3 + 3(-3a^2\alpha k + \omega)U(\xi) + 3\alpha k^3 U''(\xi) = 0. \quad (7)$$

Thus, applying the homogeneous balance rule (HBR) between Eqs. (6) and (7) yields:

$$\frac{k(\mu + 2\sigma)}{a\mu + \beta} = \frac{3(-3a^2\alpha k + \omega)}{a^3\alpha - b} = -\frac{k}{a}. \quad (8)$$

From Eq. (8), the following constraint relations are obtained:

$$a = \frac{\beta}{2\sigma}, \quad (9)$$

and

$$b = \frac{(3\omega\sigma^2 - 2\beta^2\alpha k)\beta}{2\sigma^3 k}. \quad (10)$$

Under these constraints, either Eq. (6) or Eq. (7) may serve as the governing NODE. In the present work, we adopt the NODE derived from the real part, Eq. (6).

2.2. Mathematical investigation of Eq. (3)

We begin by introducing the following wave transformation:

$$u(x, t) = U(\xi) e^{i(ax + bt + \phi_0)}, \quad \xi = k(x + \omega t). \quad (11)$$

where $U(\xi)$ denotes the real-valued amplitude function, while a , b , k , ω , and ϕ_0 represent the wave number, velocity, frequency, inverse width, and phase constant, respectively.

Substituting Eq. (11) into Eq. (3) yields:

$$\left[(c_1 + 2nc_2)U(\xi)^{2n} - 3a^2\gamma + \omega \right] U'(\xi) + \gamma k^2 U'''(\xi) = 0, \quad (12)$$

$$(a^3\gamma - b)U(\xi) + (\rho - ac_1)U(\xi)^{2n+1} - 3ak^2\gamma U''(\xi) = 0, \quad (13)$$

Integrating Eq. (12) with respect to ξ , and neglecting the integration constant, leads to:

$$\left(\frac{\omega - 3\gamma a^2}{k^2\gamma} \right) U(\xi) + \left(\frac{c_1 + 2nc_2}{k^2\gamma(2n+1)} \right) (U(\xi))^{2n+1} + U''(\xi) = 0, \quad (14)$$

To normalize the coefficient of the second-order derivative term in Eq. (13), in a manner consistent with Eq. (14), we rewrite Eq. (13) as:

$$\left(\frac{b - a^3\gamma}{3ak^2\gamma} \right) U(\xi) + \left(\frac{ac_1 - \rho}{3ak^2\gamma} \right) (U(\xi))^{2n+1} + U''(\xi) = 0. \quad (15)$$

This leads to:

$$\frac{3a(\omega - 3a^2\gamma)}{(b - a^3\gamma)} = \frac{3a(c_1 + 2nc_2)}{(2n+1)(ac_1 - \rho)} = 1. \quad (16)$$

From Eq. (16), the following constraint relations are obtained:

$$\begin{aligned}\omega &= \frac{8a^3\gamma + b}{3a}, \\ c_2 &= \frac{2ac_1(n-1) - \rho(2n+1)}{6an}.\end{aligned}\quad (17)$$

Applying the HBR in Eq. (13) between $U(\xi)^{2n+1}$ and $U''(\xi)$ suggests the transformation:

$$U(\xi) = [V(\xi)]^{\frac{1}{n}}. \quad (18)$$

Substituting Eq. (18) into Eq. (13) yields the following transformed NODE:

$$(n-1)V'^2(\xi) + \frac{n^2}{3k^2a\gamma}[(\rho - ac_1)V^2(\xi) + a^3\gamma - b]V^2(\xi) - nV(\xi)V''(\xi) = 0. \quad (19)$$

3. Presentation of the utilized methods

3.1. The new Kudryashov method

The new Kudryashov method (NKM) [36,37] constructs a series solution based on a balancing principle. Balancing $U''(\xi)$ with $U^3(\xi)$ in Eq. (6) yields:

$$N + 2 = 3N \Rightarrow N = 1.$$

Therefore, the solution of Eq. (6) is proposed in the form:

$$U(\xi) = A_0 + A_1 R(\xi), A_1 \neq 0. \quad (20)$$

where A_0 and A_1 are real constants to be determined, and $R(\xi)$ satisfies:

$$R'(\xi) = [\delta^2(R^2(\xi) - \chi R^4(\xi))]^{\frac{1}{2}}, \quad (21)$$

in which δ and χ are nonzero free parameters. Solving Eq. (21) yields:

$$R(\xi) = \frac{4L}{4L^2 e^{\delta\xi} + \chi e^{-\delta\xi}}, \quad (22)$$

where L is an arbitrary nonzero real constant. For analytical convenience, Eq. (22) can be expressed in a combined bright-singular soliton form as:

$$R(\xi) = \frac{4L}{(4L^2 + \chi)\cosh(\delta\xi) + (4L^2 - \chi)\sinh(\delta\xi)}. \quad (23)$$

Considering $\chi = 4L^2$, Eq. (23) reduces to the bright soliton solution:

$$R(\xi) = \frac{1}{2L} \operatorname{sech}(\delta\xi). \quad (24)$$

And taking $\chi = -4L^2$, Eq. (23) yields the singular soliton solution:

$$R(\xi) = \frac{1}{2L} \operatorname{csch}(\delta\xi). \quad (25)$$

3.2. The unified Riccati equation expansion method

The unified Riccati equation expansion method (UREEM) [38] applies the HBR to Eq. (6), balancing $U''(\xi)$ and $U^3(\xi)$, which yields:

$$N + 2 = 3N \Rightarrow N = 1.$$

Accordingly, the truncated series solution of Eq. (10) is proposed as:

$$U(\xi) = A_0 + A_1 R(\xi), A_1 \neq 0. \quad (26)$$

Here A_0 and A_1 are real coefficients and $R(\xi)$ satisfies:

$$R'(\xi) = r + qR(\xi) + pR^2(\xi), p \neq 0, \quad (27)$$

where r , q , and p are arbitrary real parameters. The dark and singular soliton solutions of Eq. (27) are given by:

$$R(\xi) = \begin{cases} -\frac{1}{2p} \left[q + \sqrt{q^2 - 4pr} \tanh\left(\frac{\sqrt{q^2 - 4pr}}{2} \xi\right) \right], & q^2 - 4pr > 0, \\ -\frac{1}{2p} \left[q + \sqrt{q^2 - 4pr} \coth\left(\frac{\sqrt{q^2 - 4pr}}{2} \xi\right) \right], & q^2 - 4pr > 0. \end{cases} \quad (28)$$

3.3. The sub-version of the auxiliary equation expansion method (SAEM26)

Applying the homogeneous balance between $V(\xi)^2$ and $V(\xi)V''(\xi)$ in Eq. (19) gives $N = 2$.

Hence, the solution is expressed as:

$$V(\xi) = A_0 + A_1 R(\xi) + A_2 R^2(\xi), A_2 \neq 0, \quad (29)$$

The coefficients A_0 , A_1 , and A_2 are real constants to be determined, and $R(\xi)$ satisfy [34]:

$$(R'(\xi))^2 = \delta^2 (R^2(\xi) - \chi R^6(\xi)). \quad (30)$$

where δ and χ are nonzero real parameters. Solving Eq. (30) yields the bright soliton solution:

$$R(\xi) = \sqrt{\operatorname{sech}(2\delta\xi)}, \quad (31)$$

and the singular soliton solution:

$$R(\xi) = \sqrt{\operatorname{csch}(2\delta\xi)}. \quad (32)$$

4. Application of the methods and deriving the solutions

This section applies the proposed methods to Eqs. (2) and (3). The NKM and UREEM methods are used for Eq. (2), while the SAEM26 method is applied to Eq. (3).

4.1. Application of the NKM for Eq. (2)

Substituting Eq. (20) and its derivatives into Eq. (6), together with Eq. (21), yields a polynomial in $R(\xi)$. Setting the coefficients of its powers to zero gives:

$$\begin{aligned} R^0: & A_0(4k\sigma^2(\mu + 2\sigma)A_0^2 + 9\beta^2\alpha k - 12\omega\sigma^2) = 0, \\ R^1: & -8A_0^2\sigma^3k + 4(\alpha\delta^2k^3 - k\mu A_0^2 + \omega)\sigma^2 - 3\beta^2\alpha k = 0, \\ R^2: & A_0(\mu + 2\sigma) = 0, \\ R^3: & (\mu + 2\sigma)A_1^2 + 6\alpha k^2\delta^2\chi = 0. \end{aligned} \quad (33)$$

Solving Eqs. (33) using algebraic or symbolic methods yields the following solution sets:

$$\text{Set}_{1,2} = \left\{ \begin{aligned} & \omega = -\frac{\alpha k}{\sigma^2} \left(\delta^2 k^2 \sigma^2 - \frac{3\beta^2}{4} \right), \\ & A_0 = 0, \\ & A_1 = \mp \frac{\sqrt{-6(\mu + 2\sigma)\alpha\chi\delta k}}{\mu + 2\sigma} \end{aligned} \right\}, \quad (34)$$

where $(\mu + 2\sigma)\alpha\chi < 0$. Combining Set 1 with Eqs. (4), (20), and (23) yields the combined bright-singular soliton solution of Eq. (2) as:

$$u(x,t) = \left[\frac{4LA_1}{(4L^2 + \chi) \cosh(\delta(kx + \omega t)) + (4L^2 - \chi) \sinh(\delta(kx + \omega t))} \right] \times \exp\left(i \left[\frac{\beta x}{2\sigma} - \frac{\beta}{2\sigma^3 k} (2\beta^2 \alpha k - 3\omega \sigma^2) t + \phi_0 \right]\right), \quad (35)$$

where ω, A_1 are defined as in Eq. (34). Eq. (35) reduces to the following bright soliton solution:

$$u(x,t) = \left[\frac{A_1}{2L} \operatorname{sech}(\delta(kx + \omega t)) \right] \exp\left(i \left[\frac{\beta x}{2\sigma} - \frac{\beta}{2\sigma^3 k} (2\beta^2 \alpha k - 3\omega \sigma^2) t + \phi_0 \right]\right), \quad (36)$$

and the following singular solution:

$$u(x,t) = \left[\frac{A_1}{2L} \operatorname{csch}(\delta(kx + \omega t)) \right] \exp\left(i \left[\frac{\beta x}{2\sigma} - \frac{\beta}{2\sigma^3 k} (2\beta^2 \alpha k - 3\omega \sigma^2) t + \phi_0 \right]\right). \quad (37)$$

4.2. Application of the UREEM for Eq. (2)

Substituting Eq. (26) and its derivatives into Eq. (6), together with Eq. (27), yields a polynomial in $R(\xi)$. Setting the coefficients of its powers to zero gives:

$$\begin{aligned} R^0: & -12\sigma^2(\alpha k^3 q r A_1 + \omega A_0) + 4A_0^3 \sigma^2 k(\mu + 2\sigma) + 9\beta^2 \alpha A_0 k = 0, \\ R^1: & (-8A_0^2 \sigma^3 k + 4(\alpha(pr + q^2)k^3 - kA_0^2 \mu + \omega)\sigma^2 - 3\beta^2 \alpha k) = 0, \\ R^2: & 3\alpha k^2 p q - (\mu + 2\sigma)A_0 A_1 = 0, \\ R^3: & +6k^2 p^2 \alpha - (\mu + 2\sigma)A_1^2 = 0. \end{aligned} \quad (38)$$

Solving Eq. (38) yields:

$$\text{Set}_{3,4} = \left\{ \begin{aligned} \omega &= \frac{\alpha k(2k^2 \sigma^2(q^2 - 4pr) + 3\beta^2)}{4\sigma^2}, \\ A_0 &= \mp \frac{qk\sqrt{6(\mu + 2\sigma)\alpha}}{2(\mu + 2\sigma)}, \\ A_1 &= \mp \frac{pk\sqrt{6(\mu + 2\sigma)\alpha}}{\mu + 2\sigma} \end{aligned} \right\}, \quad (39)$$

where $(\mu + 2\sigma)\alpha > 0$. Combining Set 3 with Eqs. (4), (26), and (28) gives the dark and singular soliton solutions, respectively:

$$u(x,t) = \left\{ A_0 - \frac{A_1}{2p} \left[q + \sqrt{\Delta} \tanh\left(\frac{\sqrt{\Delta}}{2}(kx + \omega t)\right) \right] \right\} \times \exp\left(i \left[\frac{\beta x}{2\sigma} - \frac{\beta}{2\sigma^3 k} (2\beta^2 \alpha k - 3\omega \sigma^2) t + \phi_0 \right]\right), \quad (40)$$

$$u(x,t) = \left\{ A_0 - \frac{A_1}{2p} \left[q + \sqrt{\Delta} \coth\left(\frac{\sqrt{\Delta}}{2}(kx + \omega t)\right) \right] \right\} \times \exp\left(i \left[\frac{\beta x}{2\sigma} - \frac{\beta}{2\sigma^3 k} (2\beta^2 \alpha k - 3\omega \sigma^2) t + \phi_0 \right]\right), \quad (41)$$

provided $\Delta > 0$. In Eqs. (40) and (41) $\Delta = q^2 - 4pr$ and ω, A_0, A_1 are presented as in Eq. (39).

4.3. Utilization of the SAEM26 for Eq. (3)

To determine the unknown parameters, we obtain the following system of algebraic equations by substituting Eqs. (29) and (30) into Eq. (19):

$$\begin{aligned}
&(\Omega_1 A_0^2 + \Omega_2) A_0^2 = 0, \\
&(2n(2\Omega_1 A_0^2 + \Omega_2) - 3k^2 \gamma a \delta^2) A_0 A_1 = 0, \\
&3a\gamma \delta^2 k^2 (4nA_0 A_2 + A_1^2) - n^2 (2A_0 A_2 + A_1^2) \Omega_2 - 2n^2 A_0^2 (2A_0 A_2 + 3A_1^2) \Omega_1 = 0, \\
&3nA_1 A_2 k^2 \gamma a \delta^2 - 2n^2 A_1 A_2 \Omega_2 + 12A_1 A_2 (k^2 \gamma a \delta^2 - n^2 A_0^2 \Omega_1) - 4n^2 A_0 A_1^3 \Omega_1 = 0, \\
&12a\delta^2 k^2 \gamma A_2^2 - ((\Omega_2 + 6A_0^2 \Omega_1) A_2^2 + A_1^2 \Omega_1 (12A_0 A_2 + A_1^2)) n^2 = 0, \\
&9nk^2 \gamma \chi a \delta^2 A_0 A_1 + 4n^2 A_1 A_2 \Omega_1 (3A_0 A_2 + A_1^2) = 0, \\
&3a\gamma k^2 \delta^2 \chi [8nA_2 A_0 + A_1^2 (2n+1)] + 2n^2 \Omega_1 A_2^2 (2A_0 A_2 + 3A_1^2) = 0, \\
&3A_1 a\gamma \chi k^2 \delta^2 (7n+4) + 4n^2 A_1 A_2^2 \Omega_1 = 0, \\
&n^2 \Omega_1 A_2^2 + 12k^2 \gamma \chi a \delta^2 (n+1) = 0.
\end{aligned} \tag{42}$$

Where $\Omega_1 = (\rho - ac_1)$, $\Omega_2 = (a^3 \gamma - b)$. A representative solution set obtained via symbolic or algebraic methods is:

$$Set_{1,2} = \left\{ \omega = \frac{a\gamma(a^2 n^2 - 12\delta^2 k^2)}{n^2}, A_0 = A_1 = 0, A_2 = \pm \frac{\delta k \sqrt{-12\Omega_1 a \chi \gamma (n+1)}}{n\Omega_1} \right\}. \tag{43}$$

Combining Eqs. (11), (18), (29), and (31) yields the bright and singular soliton solutions of Eq. (3):

$$\begin{aligned}
u(x, t) = & \left[\frac{\delta k \sqrt{12(ac_1 - \rho)a\chi\gamma(n+1)}}{n(ac_1 - \rho)} \operatorname{sech}(2\delta k(x + \Theta t)) \right]^{\frac{1}{n}} \\
& \times \exp \left(i \left[ax + \frac{a\gamma(a^2 n^2 - 12\delta^2 k^2)t}{n^2} + \phi_0 \right] \right),
\end{aligned} \tag{44}$$

$$\begin{aligned}
u(x, t) = & \left[\frac{\delta k \sqrt{12(ac_1 - \rho)a\chi\gamma(n+1)}}{n(ac_1 - \rho)} \operatorname{csch}(2\delta k(x + \Theta t)) \right]^{\frac{1}{n}} \\
& \times \exp \left(i \left[ax + \frac{a\gamma(a^2 n^2 - 12\delta^2 k^2)t}{n^2} + \phi_0 \right] \right),
\end{aligned} \tag{45}$$

where $\Theta = \frac{8n^2 a^2 \gamma + \gamma(a^2 n^2 - 12\delta^2 k^2)}{3n^2}$.

5. Results and discussion

Fig. 1 illustrates the bright optical soliton solution obtained from Eq. (36) under the selected parameter set. The chosen parameters provide the required balance between dispersion and nonlinearity, ensuring the solution is physically admissible.

Specifically, Figs. 1(a-c) present the three-dimensional spatiotemporal evolution of the absolute, real, and imaginary components of the complex field $u(x, t)$, respectively. The localized peak structure clearly confirms the formation of a bright soliton propagating along the positive spatial direction. The contour plots in Fig. 1(d-f) further emphasize the strong localization and stable energy confinement in the (x, t) -plane. Meanwhile, the two-dimensional profiles in Fig. 1(g-i), plotted at $t = 0, 2, 4, 6, 8, 10$, demonstrate that the waveform preserves its shape while translating in space, which is a key signature of soliton dynamics. The oscillatory behavior observed in the real and imaginary components reflects the phase evolution of the complex optical field, whereas the absolute profile remains smooth and localized.

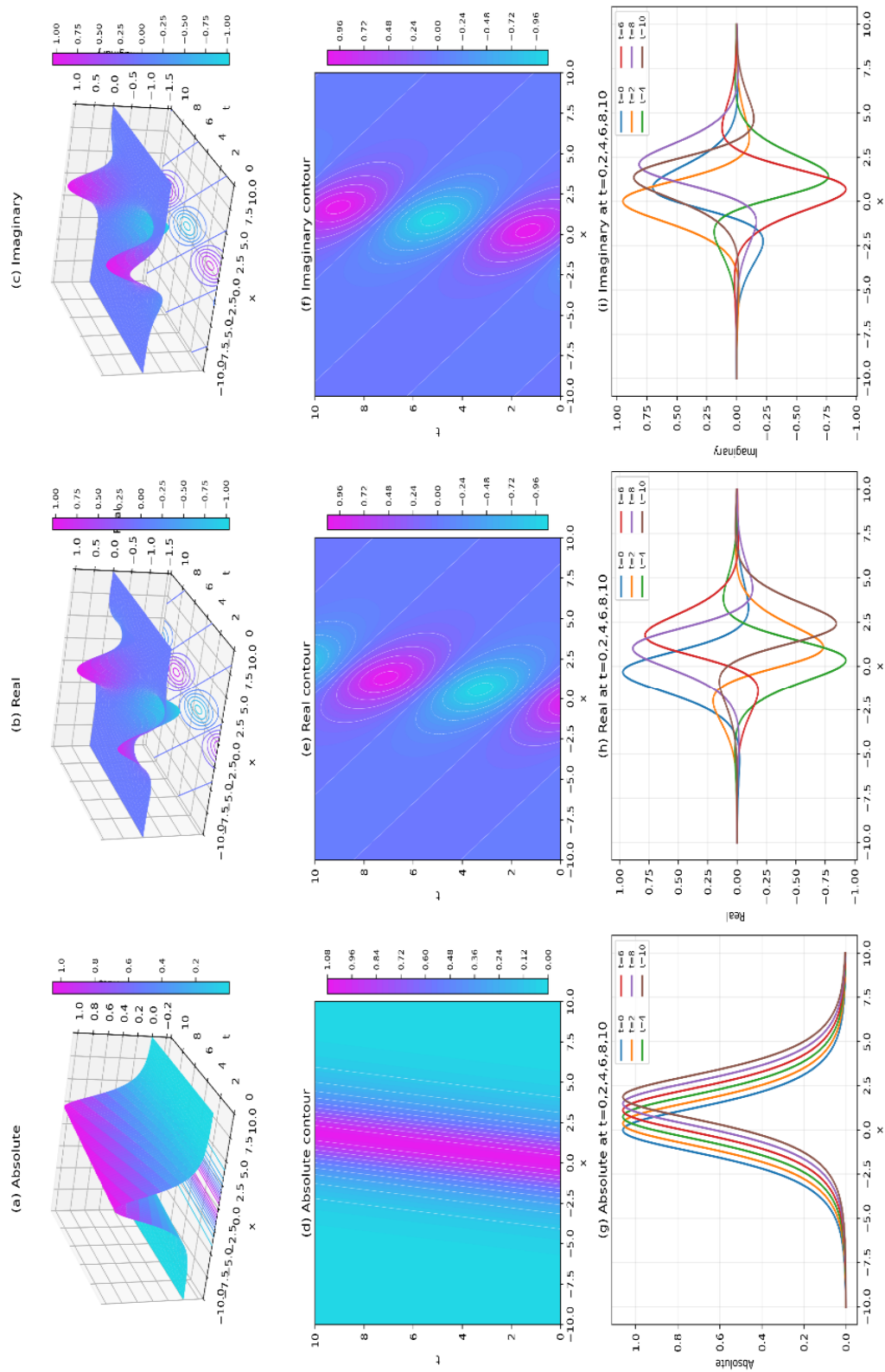


Fig. 1. Bright soliton profile of $u(x, t)$ governed by Eq. (36), for $L=1$, $\chi=4$, $k=0.75$, $\varphi_0=0.5$, $\alpha=-1$, $\beta=1$, $\mu=1$, and $\sigma=1$.

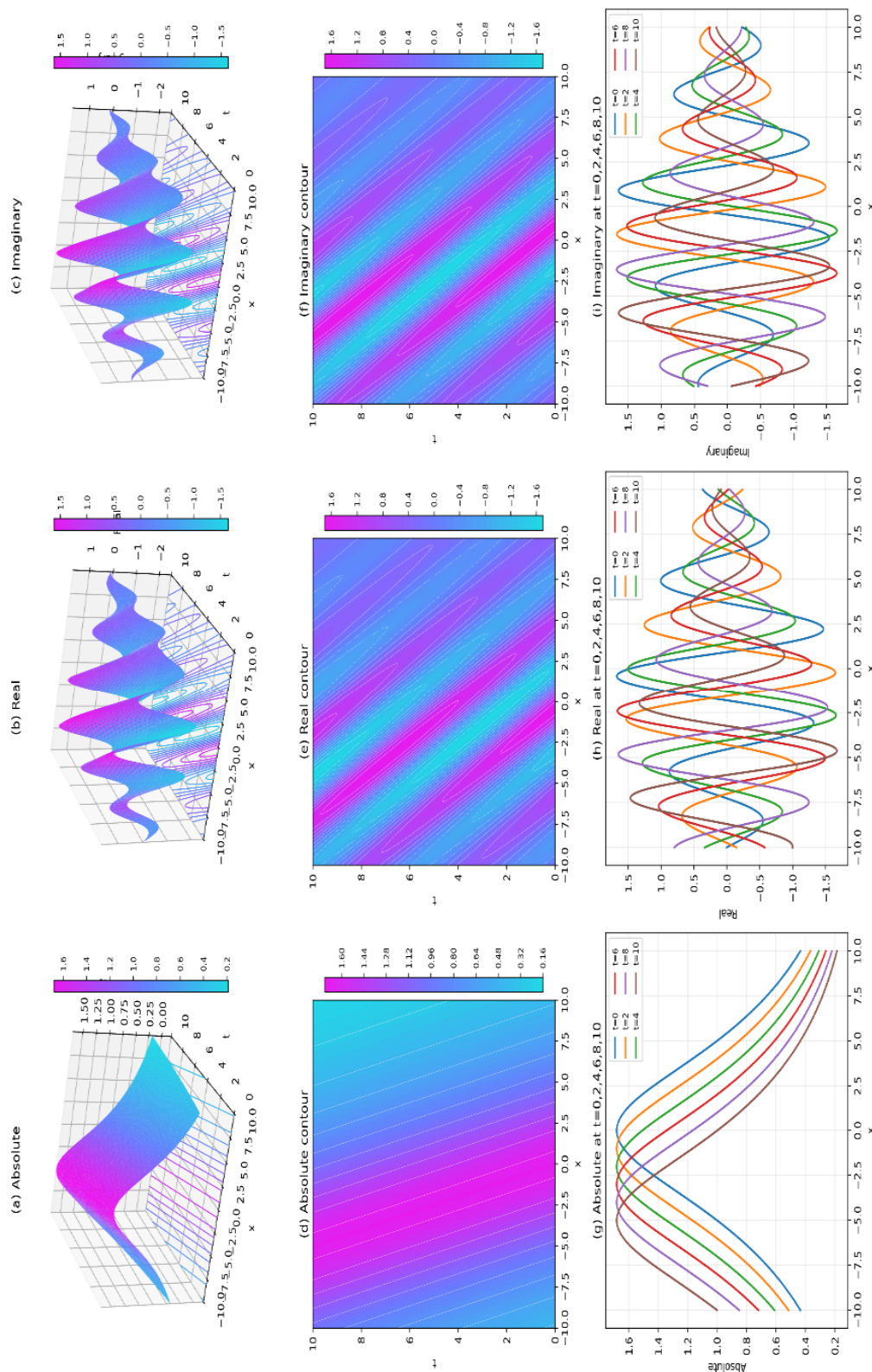


Fig. 2. Bright soliton solution of Eq. (44), obtained for $\delta=1$, $\chi=4$, $k=0.17$, $\varphi_0=0.5$, $\gamma=1$, $a=1.15$, $c_1=1$, $\rho=1$, and $n=2$.

Fig. 2 presents the bright soliton solution corresponding to the power-law nonlinear model described by Eq. (44). The parameter values are selected to ensure the existence of physically meaningful solutions in non-Kerr media. In Fig. 2(a–c), the three-dimensional plots of the absolute, real, and imaginary components reveal a localized structure similar to the standard Kerr-type soliton; however, noticeable differences in amplitude distribution and waveform sharpness can be observed due to the influence of the power-law nonlinearity. The contour representations in Fig. 2(d–f) highlight a more pronounced localization and slight modification in the propagation trajectory compared to the classical case. The two-dimensional slices in Fig. 2(g–i), evaluated at $t = 0, 2, 4, 6, 8, 10$, confirm that the solution maintains its coherent structure during propagation, although the peak amplitude and width are affected by the nonlinear exponent n .

Overall, the comparison between Figs. 1 and 2 demonstrates that while both models support stable bright soliton propagation, the inclusion of power-law nonlinearity introduces additional flexibility in controlling the soliton's amplitude, width, and propagation characteristics. This highlights the broader applicability of the generalized model in describing nonlinear optical wave dynamics beyond the standard Kerr regime.

It should be noted that the coefficients appearing in Eqs. (2) and (3) are normalized parameters representing the combined effects of dispersion and nonlinearity in optical waveguides. The parameters α and γ characterize third-order dispersion, β and ρ represent Kerr-type and generalized power-law nonlinear responses, while μ , σ , c_1 and c_2 describe higher-order nonlinear perturbations associated with self-steepening and nonlinear dispersion effects. The numerical values employed in the simulations were selected to satisfy the analytical existence conditions of the obtained solitons and to illustrate their propagation characteristics. Although these values do not correspond to a specific material system, similar normalized parameter regimes can be realized in highly nonlinear fibers, photonic crystal fibers, semiconductor waveguides, and chalcogenide-based optical structures where higher-order dispersive and nonlinear effects play an important role.

6. Conclusion

In this study, the Kerr and power-law forms of the nonlinear Schrödinger equation (NLSE)—often referred to as the third-order or pure-cubic Schrödinger model—are investigated in the presence of perturbation effects. Using the NKM, UREEM, and SAEM26 methods, we successfully derive the fundamental optical structures of nonlinear wave propagation: bright and dark solitons. We selected these analytical techniques for their computational efficiency, minimal redundancy, and ability to generate physically meaningful localized optical solutions.

The results confirm the existence of both bright and dark soliton regimes and demonstrate how key system parameters influence their amplitude and propagation characteristics. In particular, the investigation contributes to the ongoing study of soliton dynamics in regimes with negligible or absent chromatic dispersion, which remains a significant topic in modern nonlinear optics.

From the perspective of the governing models in Eqs. (2) and (3), several extensions can be envisioned, including fractional-order formulations, stochastic perturbations, multi-soliton and N -soliton interactions, as well as more complex structures such as breathers and lump-type solutions. These directions provide promising avenues for future research in advanced optical soliton theory.

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Анотація. У цьому дослідженні вивчається динаміка оптичних солітонів, що регулюється нелінійним рівнянням Шредінгера (НРШ) третього порядку, в якому нехтують членом хроматичної дисперсії, режим, який нещодавно привернув значну увагу в нелінійній оптиці. Як Керрівська, так і степенева нелінійні форми НРШ розглядаються для моделювання чисто кубічної оптичної нелінійності під впливом збурювальних ефектів. Аналітичні розв'язки побудовані з використанням нового методу Кудряшова (НКМ), уніфікованого методу розкладання рівнянь Ріккати (UREEM) та підверсії методу розкладання за допоміжним рівнянням (SAEM26). Застосовані методи дозволяють ефективно виводити фундаментальні локалізовані хвильові структури, а саме світлі та темні оптичні солітони, які відіграють центральну роль у поширенні оптичних хвиль в нелінійних середовищах. Крім того, проведено детальний параметричний аналіз для вивчення того, як ключові фізичні параметри впливають на амплітуду, форму та характеристики поширення отриманих солітонних розв'язків. Результати підкреслюють здатність розглянутої моделі підтримувати різноманітні нелінійні оптичні режими та надають подальше розуміння поведінки солітонів у середовищах без дисперсії або зі слабкою дисперсією.

Ключові слова: НРШ третього порядку, гамільтонове збурення, темний та світлий солітон