Ukrainian Journal of Physical Optics
ISSUE 3
2026

Implicit Quiescent Soliton Perturbation in Optical Metamaterials With Nonlinear Chromatic Dispersion and Kudryashov's Quintuple Form of Self–phase Modulation by Lie Symmetry

Abdullahi Rashid Adem [1], **Ahmed H. Arnous** [2,3], **Lina S. Calucag** [4] **& Anjan Biswas** [5,6,7,8]

[1] Department of Mathematical Sciences, University of South Africa, UNISA–0003, South Africa
[2] Department of Mathematical Sciences, Saveetha School of Engineering, SIMATS Chennai–602105, Tamilnadu, India
[3] Research Center of Applied Mathematics, Khazar University, Baku, AZ–1096, Azerbaijan
[4] Department of Mathematics and Science, University of Technology, Bahrain, Kingdom of Bahrain
[5] Department of Mathematics & Physics, Grambling State University, Grambling, LA 71245–2715, USA
[6] Department of Physics and Electronics, Khazar University, Baku AZ–1096, Azerbaijan
[7] Department of Mathematics and Applied Mathematics, Sefako Makgatho Health Sciences University, Medunsa–0204, South Africa
[8] Applied Science Research Center, Applied Science Private University, Amman–11937, Jordan

Received: 29.12.2025

Abstract. The current paper addresses the formation of perturbed quiescent solitons in optical metamaterials with nonlinear chromatic dispersion for Kudryashov's quintuple form of self-phase modulation by Lie symmetry. This work examines two models: the nonlinear Schrödinger equation and the complex Ginzburg–Landau equation, both with linear and generalized temporal evolution. The results are expressed in terms of quadratures, and the solition's existence criteria are also presented.

Key words: quiescent solitons, metamaterials, chromatic dispersion
UDC: 535.32
DOI: 10.3116/16091833/Ukr.J.Phys.Opt.2026.03009

1. Introduction

Soliton-like pulse propagation in optical waveguides (including fibers, and related guiding structures) relies on a delicate balance between linear chromatic dispersion (CD) and self-phase modulation (SPM) [1–15]. When additional dispersive contributions become non-negligible, such as higher-order dispersion or intensity-dependent (nonlinear) dispersive effects, the dynamics can depart from the standard picture of uniformly translating solitons and may admit stationary (quiescent) localized states. In this paper, we investigate such quiescent regimes within effective-parameter nonlinear Schrödinger's equation (NLSE) and the complex Ginzburg–Landau equation (CGLE)-type models, focusing on nonlinear chromatic-dispersion terms together with the quintuple self-phase-modulation structure proposed by Kudryashov. Our results are obtained in implicit quadrature form with corresponding existence conditions; the emphasis is on identifying parameter regimes in which quiescent localized solutions are supported, rather than on a full system-level propagation-performance model.

Nonlinear chromatic dispersion does not imply a universal arrest of soliton motion. Rather, depending on the effective dispersion and nonlinear coefficients, the envelope models may support either translating solitary waves or quiescent localized states; the present analysis identifies parameter constraints under which the latter exist.

In engineered optical metamaterials, the effective group-velocity dispersion and nonlinear response originate from the frequency-dependent permittivity and permeability of the composite medium [16,17]. Within an effective-medium approximation, these effects can be incorporated into the NLSE and CGLE frameworks via phenomenological dispersion and nonlinear coefficients that may differ in sign and magnitude from those of conventional optical fibers [18]. From this effective-coefficient viewpoint, we study quiescent soliton formation in optical metamaterials when the chromatic dispersion becomes nonlinear, and the SPM structure follows Kudryashov's quintuple form.

Two governing models will be addressed in this paper: the perturbed NLSE and the CGLE in optical metamaterials. It should be noted that these models were previously addressed in optical fibers [1]. Both models are addressed using linear and generalized temporal evolution. The recovered results are in terms of quadratures, thus keeping the solutions implicit. The adopted integration scheme is Lie symmetry. The derived results and detailed mathematical engineering are presented in the remainder of the paper.

A closely related study on implicit quiescent solitons in optical metamaterials with nonlinear chromatic dispersion, formulated within a generalized temporal-evolution framework and encompassing a range of Kudryashov self-phase-modulation (SPM) structures, was recently reported in [18]. The present manuscript is related in theme and methodology, but it is not a duplication. Here, we focus on Kudryashov's quintuple SPM form and derive reductions and existence constraints for two governing models (NLSE and CGLE), treating both linear and generalized temporal evolution within a unified framework. Hence, while the NLSE–generalized-evolution setting overlaps in spirit with [18], the current work extends the analysis to CGLE and provides the linear-evolution limit together with the corresponding constraint conditions and solution families.

In metamaterial settings, the dispersion and nonlinear coefficients are understood as effective parameters induced by the engineered frequency-dependent constitutive response, i.e., effective permittivity and permeability, and the associated microstructural dispersion; this can enable sign-indefinite and strongly dispersive regimes (including negative-index behavior) that are difficult to realize in standard fibers. Accordingly, the main contribution of this paper is to identify explicit parameter regimes and quiescent localized states supported by these effective-coefficient models, and to provide the corresponding reductions, constraint conditions, and implicit quadratures for the NLSE and CGLE formulations considered here.

Although several traveling wave solutions obtained in this work are given in implicit quadrature form, their qualitative physical properties, including localization, asymptotic decay, boundedness, and regularity, can be established rigorously from the structure of the reduced separable ordinary differential equation. We therefore include localization and asymptotic analysis based on the multiplicity of equilibrium roots, and we state explicit admissibility conditions that ensure real-valued, nonsingular profiles. We also clarify that the reduced analysis provides necessary conditions for the existence of solitary homoclinic profiles in the traveling wave dynamical system, while a full spectral stability analysis of the original partial differential equations is beyond the scope of the present work.

2. The nonlinear Schrödinger's equation

The first governing model, namely the perturbed NLSE in optical metamaterials, will be analyzed in the current section, with nonlinear CD and Kudryashov's quintuple form of SPM. The study will be divided into two cases: linear temporal evolution and generalized temporal evolution. The details are chalked out in the next couple of subsections.

2.1. Linear temporal evolution

The governing NLSE with nonlinear CD and Kudryashov's quintuple form of SPM and linear temporal evolution in optical metamaterials is governed by the perturbed NLSE as

$$
\begin{gathered}
iq_t + a\left(|q|^r q\right)_{xx} + \begin{bmatrix} b_1|q|^{2m} + b_2|q|^{2m+n} + b_3|q|^{2m+2n} \\ +b_4|q|^{2m+n+p} + b_5|q|^{2m+2n+p} + b_6\left(|q|^p\right)_{xx} \end{bmatrix} q = \\
i\left[\lambda\left(|q|^{2v} q\right)_x + \theta_1\left(|q|^{2v}\right)_x q + \theta_2 |q|^{2v} q_x\right] + \sigma_1\left(|q|^{2m} q\right)_{xx} + \sigma_2|q|^{2m} q_{xx} + \sigma_3 |q|^{2m} q^*_{xx}.
\end{gathered} \tag{1}
$$

Here in (1), $q(x,t)$ is the complex valued function that represents the wave amplitude while x and t are independent variables that stand for spatial and temporal variables respectively. Also, $i=\sqrt{-1}$. The coefficient of a is the nonlinear CD while the coefficients of b_j for $1 \le j \le 6$ together constitute the nonlinear CD of the optical fiber. From the right hand side, the parameters λ, θ_1 and θ_2 represent the self–steepening effect and soliton self–frequency shift. These parameters a, b_j, λ and θ_s for $s=1,2$ are real–valued constants. Finally, the power–law indices r, m, n, p and v are constants. Moreover, σ_j for $j=1,2$ and 3 stem from optical metamaterials [14].

Here, we restrict attention to stationary quiescent solutions, which exist under the derived parameter constraints [2]. Therefore, the solution structure is taken to be of the form:

$$
q(x,t) = \phi(x)\mathrm{e}^{i\omega t}, \tag{2}
$$

for the frequency ω. Substituting (2) into (1), one recovers from the real part, the ordinary differential equation (ODE) for $\phi(x)$ to be:

$$
\begin{gathered}
a(r+1)\phi^r(x)\left[\phi(x)\phi''(x) + r\{\phi'(x)\}^2\right] + b_1\phi^{2m+2}(x) + b_2\phi^{2m+n+2}(x) + b_3\phi^{2m+2n+2}(x) \\
+b_4\phi^{2m+n+p+2}(x) + b_5\phi^{2m+2n+p+2}(x) + pb_6\phi^p(x)\left[\phi(x)\phi''(x) + (p-1)\{\phi'(x)\}^2\right] \\
-\phi^{2m}(x)\left[\{(2m+1)\sigma_1 + \sigma_2 + \sigma_3\}\phi(x)\phi''(x) + 2m(2m+1)\sigma_1\{\phi'(x)\}^2\right] - \omega\phi^2(x) = 0,
\end{gathered} \tag{3}
$$

while the imaginary part leads to the parameter constraint as:

$$
(2v+1)\lambda + 2v\theta_1 + \theta_2 = 0. \tag{4}
$$

The ODE (3) supports translational Lie symmetry $\partial/\partial x$, which, when implemented, integrates it to

$$
x = \int \frac{A}{\sqrt{2B}} d\phi, \tag{5}
$$

where

$$
A = \exp\left(-\int \phi - \frac{-2m(2m+1)\sigma_1\tau^{2m} + (p-1)pb_6\tau^p + ar(r+1)\tau^r}{\tau\left(-\left(\left((2m+1)\sigma_1 + \sigma_2 + \sigma_3\right)\tau^{2m}\right) + pb_6\tau^p + a(r+1)\tau^r\right)} d\tau\right), \tag{6}
$$

and

$$B=\int\phi\frac{\tau_2\left(\tau_2^{2m}\left(\tau_2^n\left(\tau_2^p\left(b_5\tau_2^n+b_4\right)+b_3\tau_2^n+b_2\right)+b_1\right)-\omega\right)\Gamma}{-a(r+1)\tau_2^r-b_6p\tau_2^p+\left((2m+1)\sigma_1+\sigma_2+\sigma_3\right)\tau_2^{2m}}d\tau_2, \tag{7}$$

with

$$\Gamma=\exp\left(-2\int\tau_2-\frac{-2m(2m+1)\sigma_1\tau^{2m}+(p-1)pb_6\tau^p+ar(r+1)\tau^r}{\tau\left\{-\left(\left[(2m+1)\sigma_1+\sigma_2+\sigma_3\right]\tau^{2m}\right)+pb_6\tau^p+a(r+1)\tau^r\right\}}d\tau\right). \tag{8}$$

The parameter constraint given by

$$B>0 \tag{9}$$

assures the existence of the solution.

2.2. Generalized temporal evolution

The governing model is now written in the presence of perturbation terms with generalized temporal evolution as

$$\begin{aligned}&i\left(q^l\right)_t+a\left(|q|^r q^l\right)_{xx}+\begin{bmatrix}b_1|q|^{2m}+b_2|q|^{2m+n}+b_3|q|^{2m+2n}\\+b_4|q|^{2m+n+p}+b_5|q|^{2m+2n+p}+b_6\left(|q|^p\right)_{xx}\end{bmatrix}q^l=\\&i\left[\lambda\left(|q|^{2v}q^l\right)_x+\theta_1\left(|q|^{2v}\right)_x q^l+\theta_2|q|^{2v}\left(q^l\right)_x\right]\\&+\sigma_1\left(|q|^{2m}q^l\right)_{xx}+\sigma_2|q|^{2m}\left(q^l\right)_{xx}+\sigma_3|q|^{2m}\left(q^l\right)^*_{xx}.\end{aligned} \tag{10}$$

In Eq. (10), the parameter l accounts for the generalized temporal evolution. For $l=1$, it reduces to the NLSE with linear temporal evolution as given by (1). Starting with the same substitution as (2) for quiescent solitons, and plugging it into (10), the real part gives the ODE for $\phi(x)$ as

$$\begin{aligned}&a(r+l)\phi^r(x)\left[\phi^{2l+1}(x)\phi''(x)+(l+r-1)\{\phi'(x)\}^2\phi^{2l}(x)\right]\\&+b_1\phi^{2(l+m+1)}(x)+b_2\phi^{2(l+m+1)+n}(x)+b_3\phi^{2(l+m+n+1)}(x)\\&+b_4\phi^{2l+2m+n+p+2}(x)+b_5\phi^{2(l+m+n+1)+p}(x)\\&+pb_6\phi^p(x)\left[\phi^{2l+1}(x)\phi''(x)+(p-1)\{\phi'(x)\}^2\phi^{2l}(x)\right]\\&-\phi^{2m}(x)\begin{bmatrix}(l+2m)\sigma_1+(\sigma_2+\sigma_3)l\phi(x)\phi''(x)\\-\{(l+2m)(l+2m-1)\sigma_1+l(l-1)(\sigma_2+\sigma_3)\}\{\phi'(x)\}^2\end{bmatrix}-l\omega\phi^{2l+2}(x)=0,\end{aligned} \tag{11}$$

while the imaginary part leads to the parameter constraint as:

$$(2v+l)\lambda+2v\theta_1+l\theta_2=0. \tag{12}$$

The translational Lie symmetry yields the integral to (11) as given by (5), where in this case,

$$A=\exp\left(-\int\phi\frac{\begin{matrix}\left((l+2m-1)(l+2m)\sigma_1+(l-1)l(\sigma_2+\sigma_3)\right)\tau^{2m}\\-(p-1)pb_6\tau^p-a(l+r-1)(l+r)\tau^r\end{matrix}}{\tau\begin{pmatrix}-\left(\left((l+2m)\sigma_1+l(\sigma_2+\sigma_3)\right)\tau^{2m}\right)\\+pb_6\tau^p+a(l+r)\tau^r\end{pmatrix}}d\tau\right) \tag{13}$$

and

$$B=\int\phi\frac{\tau_2\left(\tau_2^{2m}\left(\tau_2^n\left(\tau_2^p\left(b_5\tau_2^n+b_4\right)+b_3\tau_2^n+b_2\right)+b_1\right)-l\omega\right)\Gamma}{-a(l+r)\tau_2^r-b_6p\tau_2^p+\tau_2^{2m}\left(\sigma_1(l+2m)+l(\sigma_2+\sigma_3)\right)}d\tau_2, \tag{14}$$

with

$$\Gamma=\exp\left(-2\int\tau_2\frac{\begin{array}{c}\left((l+2m-1)(l+2m)\sigma_1+(l-1)l(\sigma_2+\sigma_3)\right)\tau^{2m}\\ -(p-1)pb_6\tau^p-a(l+r-1)(l+r)\tau^r\end{array}}{\tau\left\{-\left[\left((l+2m)\sigma_1+l(\sigma_2+\sigma_3)\right)\tau^{2m}\right]+pb_6\tau^p+a(l+r)\tau^r\right\}}d\tau\right). \tag{15}$$

The parameter constraint (9) remains valid here as well, ensuring that solutions exist.

3. Complex Ginzburg–Landau equation

The current section considers the perturbed CGLE in optical metamaterials with nonlinear CD and Kudryashov's quintuple form of SPM. This model is addressed in the following subsections, which cover linear temporal evolution and generalized temporal evolution, respectively.

3.1. Linear temporal evolution

The dimensionless form of the governing CGLE in optical metamaterials with linear temporal evolution is structured as:

$$\begin{aligned}&iq_t+a\left(|q|^r q\right)_{xx}+\left[\begin{array}{l}b_1|q|^{2m}+b_2|q|^{2m+n}+b_3|q|^{2m+2n}\\ +b_4|q|^{2m+n+p}+b_5|q|^{2m+2n+p}+b_6\left(|q|^p\right)_{xx}\end{array}\right]q=\alpha\frac{|q_x|^2}{q^*}\\ &+\frac{\beta}{4|q|^2q^*}\left[2|q|^2\left(|q|^2\right)_{xx}-\left\{\left(|q|^2\right)_x\right\}^2\right]+\gamma q+i\left[\lambda\left(|q|^{2\nu}q\right)_x+\theta_1\left(|q|^{2\nu}\right)_x q+\theta_2|q|^{2\nu}q_x\right]\\ &+\sigma_1\left(|q|^{2m}q\right)_{xx}+\sigma_2|q|^{2m}q_{xx}+\sigma_3|q|^{2m}q_{xx}^*.\end{aligned} \tag{16}$$

Substituting (2) into (16) leads to the ODE for $\phi(x)$ as given by the real part:

$$\begin{aligned}&a(r+1)\phi^r(x)\left[\phi(x)\phi''(x)+r\{\phi'(x)\}^2\right]+b_1\phi^{2m+2}(x)+b_2\phi^{2m+n+2}(x)\\ &+b_3\phi^{2m+2n+2}(x)+b_4\phi^{2m+n+p+2}(x)+b_5\phi^{2m+2n+p+2}(x)\\ &+pb_6\phi^p(x)\left[\phi(x)\phi''(x)+(p-1)\{\phi'(x)\}^2\right]-\phi^2(x)\left[\alpha\{\phi'(x)\}^2+\beta\phi(x)\phi''(x)\right]\\ &-\phi^{2m}(x)\left[\{(2m+1)\sigma_1+\sigma_2+\sigma_3\}\phi(x)\phi''(x)+2m(2m+1)\sigma_1\{\phi'(x)\}^2\right]-\omega\phi^2(x)=0,\end{aligned} \tag{17}$$

while the imaginary part gives the parameter constraint as in (4). The translational Lie symmetry applied to (17) yields the implicit solution as given by (5), where in this case

$$A=\exp\left(-\int\phi\frac{2m(2m+1)\sigma_1\tau^{2m}-(p-1)pb_6\tau^p-ar(r+1)\tau^r+\alpha}{\tau\left\{-\left\{\left((2m+1)\sigma_1+\sigma_2+\sigma_3\right)\tau^{2m}\right\}+pb_6\tau^p+a(r+1)\tau^r-\beta\right\}}d\tau\right), \tag{18}$$

and

$$B=\int\phi\frac{\tau_2\left(\tau_2^{2m}\left(\tau_2^n\left(\tau_2^p\left(b_5\tau_2^n+b_4\right)+b_3\tau_2^n+b_2\right)+b_1\right)-\gamma-\omega\right)\Gamma}{-a(r+1)\tau_2^r-b_6p\tau_2^p+\beta+\left((2m+1)\sigma_1+\sigma_2+\sigma_3\right)\tau_2^{2m}}d\tau_2, \tag{19}$$

along with

$$\Gamma=\exp\left(-2\int\tau_2\frac{2m(2m+1)\sigma_1\tau^{2m}-(p-1)pb_6\tau^p-ar(r+1)\tau^r+\alpha}{\tau\left\{-\left[\left((2m+1)\sigma_1+\sigma_2+\sigma_3\right)\tau^{2m}\right]+pb_6\tau^p+a(r+1)\tau^r-\beta\right\}}d\tau\right),$$

provided the parameter constraints given by (9) hold.

3.2. Generalized temporal evolution

The dimensionless form of the governing CGLE in optical metamaterials with generalized temporal evolution is structured as:

$$\begin{aligned}
&i\left(q^l\right)_t+a\left(|q|^r q^l\right)_{xx}+\begin{bmatrix}b_1|q|^{2m}+b_2|q|^{2m+n}+b_3|q|^{2m+2n}\\+b_4|q|^{2m+n+p}+b_5|q|^{2m+2n+p}+b_6\left(|q|^p\right)_{xx}\end{bmatrix}q^l=\\
&\alpha\frac{|q_x|^2}{\left(q^l\right)^*}+\frac{\beta}{4|q|^2\left(q^l\right)^*}\left[2|q|^2\left(|q|^2\right)_{xx}-\left\{\left(|q|^2\right)_x\right\}^2\right]+\gamma q^l\\
&+i\left[\lambda\left(|q|^{2\nu}q^l\right)_x+\theta_1\left(|q|^{2\nu}\right)_x q^l+\theta_2|q|^{2\nu}\left(q^l\right)_x\right]\\
&+\sigma_1\left(|q|^{2m}q^l\right)_{xx}+\sigma_2|q|^{2m}\left(q^l\right)_{xx}+\sigma_3|q|^{2m}\left(q^l\right)^*_{xx}.
\end{aligned}\tag{20}$$

For metamaterial settings, the dispersion and nonlinear coefficients in Eqs. (1), (10), (16), and (20) are interpreted as effective parameters derived from the underlying resonant microstructure. In particular, nonlinear chromatic dispersion arises from the strong frequency dependence of the effective refractive index near metamaterial resonances. Substituting (2) into (20) leads to the ODE for $\phi(x)$ as given by the real part:

$$\begin{aligned}
&a(r+l)\phi^r(x)\left[\phi^{2l+1}(x)\phi''(x)+(l+r-1)\{\phi'(x)\}^2\phi^{2l}(x)\right]\\
&+b_1\phi^{2(l+m+1)}(x)+b_2\phi^{2(l+m+1)+n}(x)+b_3\phi^{2(l+m+n+1)}(x)+b_4\phi^{2l+2m+n+p+2}(x)\\
&+b_5\phi^{2(l+m+n+1)+p}(x)+pb_6\phi^p(x)\left[\phi^{2l+1}(x)\phi''(x)+(p-1)\{\phi'(x)\}^2\phi^{2l}(x)\right]\\
&-\phi^2(x)\left[\alpha\{\phi'(x)\}^2+\beta\phi(x)\phi''(x)\right]\\
&-\phi^{2m}(x)\begin{bmatrix}(l+2m)\sigma_1+(\sigma_2+\sigma_3)l\phi(x)\phi''(x)-\\\left\{(l+2m)(l+2m-1)\sigma_1+l(l-1)(\sigma_2+\sigma_3)\right\}\{\phi'(x)\}^2\end{bmatrix}\\
&-l\omega\phi^{2l+2}(x)=0,
\end{aligned}\tag{21}$$

while the imaginary part gives the parameter constraint as in (12). The translational Lie symmetry applied to (21) yields the implicit solution as given by (5), where in this case

$$A=\exp\left(-\int\phi\frac{\begin{matrix}\left(-\left((l+2m-1)(l+2m)\sigma_1\right)-(l-1)l(\sigma_2+\sigma_3)\right)\tau^{2(l+m)}\\+(p-1)pb_6\tau^{2l+p}+a(l+r-1)(l+r)\tau^{2l+r}-\alpha\tau^2\end{matrix}}{\tau\begin{pmatrix}\left((l+2m)\sigma_1+l(\sigma_2+\sigma_3)\right)\tau^{2(l+m)}\\-pb_6\tau^{2l+p}-a(l+r)\tau^{2l+r}+\beta\tau^2\end{pmatrix}}d\tau\right),\tag{22}$$

and

$$B=\int\phi\frac{\tau_2^{2l+1}\left(\tau_2^{2m}\left(\tau_2^n\left(\tau_2^p\left(b_5\tau_2^n+b_4\right)+b_3\tau_2^n+b_2\right)+b_1\right)-\gamma-l\omega\right)\Gamma}{-a(l+r)\tau_2^{2l+r}-b_6p\tau_2^{2l+p}+\beta\tau_2^2+\left(\sigma_1(l+2m)+l(\sigma_2+\sigma_3)\right)\tau_2^{2(l+m)}}d\tau_2,\tag{23}$$

as well as

$$\Gamma=\exp\left(-\int\tau_2\frac{\begin{pmatrix}-\left(\left((l+2m-1)(l+2m)\sigma_1\right)-(l-1)l(\sigma_2+\sigma_3)\right)\tau^{2(l+m)}\\+(p-1)pb_6\tau^{2l+p}+a(l+r-1)(l+r)\tau^{2l+r}-\alpha\tau^2\end{pmatrix}}{\tau\begin{pmatrix}\left((l+2m)\sigma_1+l(\sigma_2+\sigma_3)\right)\tau^{2(l+m)}\\-pb_6\tau^{2l+p}-a(l+r)\tau^{2l+r}+\beta\tau^2\end{pmatrix}}d\tau\right), \tag{24}$$

together with (9) that guarantees the existence of the solution. It should be emphasized that, unlike standard optical fibers, optical metamaterials allow a broader range of effective dispersion and nonlinear parameters due to their engineered subwavelength structure. This flexibility enables quiescent regimes under parameter combinations that are difficult to achieve in conventional media.

4. Localization and stability

This section complements the implicit quadrature families reported above by presenting rigorous localization and asymptotic criteria derived from the reduced traveling-wave first integral, along with explicit regularity and admissibility conditions that distinguish bounded soliton profiles from nonlocal or singular branches. We also state the linearized eigenvalue formulation used to assess the spectral stability of stationary or quiescent profiles. Throughout, the term “soliton” is used in the standard envelope sense of a localized solitary traveling wave, and we do not claim stability for the full partial differential equation unless it follows from the stated spectral conditions.

4.1. Quadrature energy form and root structure

Let $\tau=x-ct$ denote the traveling coordinate, and let $U(\tau)$ denote a reduced profile variable (e.g., $U=q(\tau)$ or $U=r(\tau)$, or a reduced amplitude obtained after symmetry reduction). For each reported family, the reduction yields a first-order first integral that can be written in the energy form

$$\left(\frac{dU}{d\tau}\right)^2=\mathcal{F}(U;\Pi), \tag{25}$$

where Π is the collection of model parameters, and $\mathcal{F}(U;\Pi)$ is the algebraic function generated by the governing coefficients after imposing the derived parameter constraints for that family. Eq. (25) is equivalent to the implicit quadrature

$$\pm(\tau-\tau_0)=\int_{U(\tau_0)}^{U(\tau)}\frac{d\xi}{\sqrt{\mathcal{F}(\xi;\Pi)}}. \tag{26}$$

Physically admissible profiles require $\mathcal{F}(U;\Pi)\geq 0$ along the entire solution branch. Moreover, bounded solitary-wave profiles are confined between turning points, namely, real zeros of $\mathcal{F}$, at which $\frac{dU}{d\tau}=0$. Hence, the algebraic constraints derived in each family have a direct interpretation; they select parameter regimes in which $\mathcal{F}$ possesses the real root structure and sign conditions needed to produce localized and bounded waveforms through (26).

4.2. Localization and decay from root multiplicity

A localized solitary wave is characterized by convergence to a finite background

$$U(\tau)\to U_\infty \quad \text{as} \quad |\tau|\to\infty, \qquad \int_{-\infty}^{\infty}(U(\tau)-U_\infty)^2 d\tau<\infty. \tag{27}$$

In the first-integral formulation (25), (27) is possible only if U_∞ is an equilibrium (a root of $\mathcal{F}$), namely

$$\mathcal{F}(U_\infty;\Pi)=0. \tag{28}$$

Exponential localization occurs when the equilibrium root is at least of multiplicity two:

$$\mathcal{F}(U_\infty;\Pi)=0, \qquad \partial_U\mathcal{F}(U_\infty;\Pi)=0, \qquad \partial_{UU}\mathcal{F}(U_\infty;\Pi)>0. \tag{29}$$

Indeed, writing $U(\tau)=U_\infty+\eta(\tau)$ with $|\eta|\ll 1$ and expanding,

$$\mathcal{F}(U_\infty+\eta;\Pi)=\frac{1}{2}\partial_{UU}\mathcal{F}(U_\infty;\Pi)\eta^2+\mathcal{O}(\eta^3), \tag{30}$$

so that (25) yields

$$\eta'(\tau)=\pm\sqrt{\frac{1}{2}\partial_{UU}\mathcal{F}(U_\infty;\Pi)}\eta+\mathcal{O}(\eta^2). \tag{31}$$

Therefore,

$$U(\tau)-U_\infty=\mathcal{O}\left(e^{-\lambda|\tau|}\right), \qquad \lambda=\sqrt{\frac{1}{2}\partial_{UU}\mathcal{F}(U_\infty;\Pi)}. \tag{32}$$

Hence, the existence constraints derived for each family are not merely formal: they determine whether $\mathcal{F}$ admits a double root U_∞ and therefore whether a genuinely localized solitary profile exists. If instead U_∞ is a simple root, i.e., $\mathcal{F}(U_\infty;\Pi)=0$ but $\partial_U\mathcal{F}(U_\infty;\Pi)\neq 0$, then $\mathcal{F}(U)\sim\alpha(U-U_\infty)$ near U_∞, and the quadrature reaches U_∞ in finite τ ; such branches are generally not exponentially localized solitons (they correspond to fronts, compact segments, or connections between distinct equilibria depending on the remaining roots).

4.3. Turning points, boundedness, and explicit profiles

Bounded solitary profiles correspond to orbits confined between turning points where $\mathcal{F}(U;\Pi)=0$. For a homoclinic-type localized orbit around U_∞, there must exist another real root U_* of $\mathcal{F}$ such that $\mathcal{F}(U;\Pi)\geq 0$ on the interval between U_∞ and U_*. The maximum amplitude is attained at $U=U_*$ (where $U_\tau=0$). If $\mathcal{F}(U;\Pi)>0$ for arbitrarily large $|U|$ without an upper turning point, then solutions may become unbounded and are excluded from the soliton class.

Although the general families are presented in quadrature form (26), explicit profiles follow whenever $\mathcal{F}$ factorizes with low-degree root multiplicities. For example, if $U_\infty=0$ and

$$\mathcal{F}(U;\Pi)=\alpha U^2\left(U_m^2-U^2\right), \qquad \alpha>0, U_m>0. \tag{33}$$

Then the quadrature inverts to the standard bright pulse

$$U(\tau)=U_m\left(\sqrt{\alpha}(\tau-\tau_0)\right). \tag{34}$$

Similarly, if $U_\infty=\pm U_0$ and

$$\mathcal{F}(U;\Pi)=\alpha\ (U^2-U_0^2)^2, \qquad \alpha>0, \tag{35}$$

one obtains the kink/dark profile

$$U(\tau) = U_0 \tanh\left(\sqrt{\alpha}(\tau - \tau_0)\right). \tag{36}$$

These cases show that the "implicit" quadrature families contain, for special parameter regimes, the familiar - and $\tanh$-type solitons, while the general quintuple-nonlinearity structure produces profiles whose closed forms may involve elliptic integrals or non-elementary inversions.

4.4. Physical admissibility and regularity

For a profile obtained from (26) to be physically admissible (real-valued, bounded, and nonsingular), the following conditions must hold along the orbit:

1. $\mathcal{F}(U;\Pi) \geq 0$ for all U visited by the solution.

2. If $\mathcal{F}$ is rational, then denominator factors must not vanish on the orbit; otherwise, the quadrature integrand diverges, and the waveform develops singularities.

3. U_∞ must be finite, satisfy (28), and the localization condition (27) must hold. In particular, algebraic blow-up (e.g., $U \sim |\tau - \tau_s|^{-p}$) implies unbounded intensity and is excluded from physical realizability within the envelope approximation unless additional regularization physics is incorporated.

In the discussion of solitons, we therefore focus on parameter regimes that satisfy the double-root localization condition (29) and the admissibility requirements above.

4.5. Linear stability: eigenvalue problem and conservative-limit criterion

To assess the spectral stability of a stationary/quiescent profile, consider a perturbed field written (schematically) as

$$u(\tau,t) = e^{i\Omega t}\left(U(\tau) + \varepsilon\left[a(\tau) + ib(\tau)\right]e^{\lambda t}\right) + \mathcal{O}(\varepsilon^2), \quad 0 < \varepsilon \ll 1, \tag{37}$$

where Ω is the carrier frequency induced by the phase in the traveling/quiescent ansatz and $\lambda \in \mathbb{C}$ is the spectral parameter. Substituting (37) into the governing PDE, and linearizing yields an eigenvalue problem of the generic Hamiltonian form (in the conservative reduction)

$$\mathcal{J}\mathcal{L}\begin{pmatrix} a \\ b \end{pmatrix} = \lambda \begin{pmatrix} a \\ b \end{pmatrix}, \quad \mathcal{J} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \tag{38}$$

where $\mathcal{L}$ is a matrix differential operator with coefficients depending on $U(\tau)$ and parameters Π. Spectral stability requires

$$\Re(\lambda) < 0 \quad \text{for all spectrum of} (38). \tag{39}$$

In the NLSE case, after enforcing the constraint that removes the imaginary part and yields the conservative first integral (25), one may additionally use the standard power-slope (Vakhitov–Kolokolov type) criterion along stationary branches,

$$\frac{d}{d\Omega}\mathcal{P}(\Omega) < 0, \quad \mathcal{P}(\Omega) := \int_{-\infty}^{\infty} U(\tau;\Omega)^2 d\tau, \tag{40}$$

as a sufficient condition to preclude real positive eigenvalues in the conservative limit. In the CGLE setting, the corresponding linearized operator is generally non-selfadjoint due to gain/loss terms; thus, stability is genuinely spectral and typically requires numerical evaluation of the point/essential spectrum. Nonetheless, (37)–(39) provides the correct mathematical framework for assessing the stability of each admissible solution branch.

5. Conclusions

This paper establishes the existence of stationary (quiescent) localized solutions in effective-parameter models for optical metamaterials with nonlinear chromatic dispersion and Kudryashov's quintuple self-phase-modulation structure. Two governing models are considered, namely the NLSE and the CGLE, each treated under both linear temporal evolution and generalized temporal evolution. The solutions are obtained in implicit quadrature form together with the corresponding existence conditions. The results are primarily of theoretical interest and identify parameter regimes in which nonlinear dispersion effects may suppress soliton mobility or favor stationary localized states in engineered optical media such as metamaterials.

The reported solution families are obtained in quadrature form, which is sufficient to characterize localization, decay rates, boundedness, and regularity through the function $\mathcal{F}(U)$ in the reduced first-order ODE. We have therefore added explicit criteria for exponential localization based on the double-root condition, for boundedness through turning points, and for physical admissibility via reality requirements and the absence of singularities. A complete spectral stability analysis for the full partial differential equation remains an important direction for future work and will depend on the specific dissipative and perturbative terms, as well as the boundary conditions, in the NLSE and CGLE settings considered.

Disclosure. The authors claim no conflict of interest.

Funding and acknowledgement. This work of the last author (AB) was supported by Grambling State University for the Endowed Chair of Mathematics. The author gratefully acknowledges this support.

References

1. Adem, A. R. Arnous, A. H. Ahmed, Husham M. & Biswas, A. (2026) "Implicit quiescent soliton perturbation in optical fibers with nonlinear chromatic dispersion and Kudryashov's quintuple form of self–phase modulation by Lie symmetry". To be published in *Semiconductor Physics, Quantum Electronics & Optoelectronics.* (to be published).
2. Biswas, A., Ekici, M., Sonmezoglu, A., & Belic, M. (2018). Stationary optical solitons with nonlinear group velocity dispersion by extended trial function scheme. *Optik*, *171*, 529-542.
3. Arnous, A. H., Biswas, A., Yıldırım, Y., Moraru, L., Moldovanu, S., Iticescu, C., Khan, S. & Alshehri, H. M. (2023, January). Quiescent optical solitons with quadratic-cubic and generalized quadratic-cubic nonlinearities. In *Telecom* (Vol. 4, No. 1, pp. 31-42). MDPI.
4. Ekici, M. (2022). Stationary optical solitons with complex Ginzburg–Landau equation having nonlinear chromatic dispersion and Kudryashov's refractive index structures. *Physics Letters A*, *440*, 128146.
5. Ekici, M. (2022). Kinky breathers, W-shaped and multi-peak soliton interactions for Kudryashov's quintuple power-law coupled with dual form of non-local refractive index structure. *Chaos, Solitons & Fractals*, *159*, 112172.
6. Ekici, M. (2022). Optical solitons with Kudryashov's quintuple power–law coupled with dual form of non–local law of refractive index with extended Jacobi's elliptic function. *Optical and Quantum Electronics*, *54*(5), 279.
7. Ekici, M. (2023). Stationary optical solitons with Kudryashov's quintuple power law nonlinearity by extended Jacobi's elliptic function expansion. *Journal of Nonlinear Optical Physics & Materials*, *32*(01), 2350008.
8. Han, T., Li, Z., Li, C., & Zhao, L. (2023). Bifurcations, stationary optical solitons and exact solutions for complex Ginzburg–Landau equation with nonlinear chromatic dispersion in non-Kerr law media. *Journal of Optics*, *52*(2), 831-844.
9. Kudryashov, N. A. (2022). Stationary solitons of the generalized nonlinear Schrödinger equation with nonlinear dispersion and arbitrary refractive index. *Applied Mathematics Letters*, *128*, 107888.
10. Kudryashov, N. A. (2022). Stationary solitons of the model with nonlinear chromatic dispersion and arbitrary refractive index. *Optik*, *259*, 168888.
11. Jawad, A., & Abu-AlShaeer, M. (2023). Highly dispersive optical solitons with cubic law and cubic-quintic-septic law nonlinearities by two methods. *Al-Rafidain Journal of Engineering Sciences*, 1-8.

12. Jihad, N., & Abd Almuhsan, M. (2023). Evaluation of impairment mitigations for optical fiber communications using dispersion compensation techniques. *Al-Rafidain Journal of Engineering Sciences*, 81-92.
13. Smertenko, P., Rudko, G., Maksimenko, Z., & Belyaev, A. (2025). Quantum physics and photoluminescence: contribution of the SPQEO. *Semiconductor Physics, Quantum Electronics & Optoelectronics*, *28*(4), 389-393.
14. Veni, S. S., Mani Rajan, M. S., Biswas, A., & Alshomrani, A. S. (2024). Exploring the dynamic interplay of intermodal and higher order dispersion in nonlinear negative index metamaterials. *Physica Scripta*, *99*(8), 085261.
15. Yalçı, A. M., & Ekici, M. (2022). Stationary optical solitons with complex Ginzburg–Landau equation having nonlinear chromatic dispersion. *Optical and Quantum Electronics*, *54*(3), 167.
16. Veselago, V. G. (1967). The electrodynamics of substances with simultaneously negative values of and. *Usp. Fiz. Nauk*, *92*(3), 517-526.
17. Smith, D. R., Pendry, J. B., & Wiltshire, M. C. (2004). Metamaterials and negative refractive index. *Science*, *305*(5685), 788-792.
18. Adem, A. R., González-Gaxiola, O., Arnous, A. H., Calucag, L. S., & Biswas, A. (2026, January). Implicit Quiescent Solitons in Optical Metamaterials with Nonlinear Chromatic Dispersion and an Array of Self-Phase Modulation Structures with Generalized Temporal Evolution by Lie Symmetry. In *Telecom* (Vol. 7, No. 1, p. 6). MDPI.

Adem, A. R., Arnous, A. H., Calucag, L. S., Biswas, A. (2026). Implicit Quiescent Soliton Perturbation in Optical Metamaterials With Nonlinear Chromatic Dispersion and Kudryashov's Quintuple Form of Self–phase Modulation by Lie Symmetry. *Ukrainian Journal of Physical Optics*, *27*(3), 03009 – 03019.
doi: 10.3116/16091833/Ukr.J.Phys.Opt.2026.03009

Анотація. *У цій статті розглядається формування збурених спокійних солітонів в оптичних метаматеріалах з нелінійною хроматичною дисперсією для п'ятикратної форми фазової модуляції Кудряшова за допомогою симетрії Лі. Розглянуті дві моделі: нелінійне рівняння Шредінгера та комплексне рівняння Гінзбурга-Ландау, обидві з лінійною та узагальненою часовою еволюцією. Результати виражені в квадратурах, а також представлені критерії існування солітонів.*

Ключові слова: *спокійні солітони, метаматеріали, хроматична дисперсія*