Ukrainian Journal of Physical Optics
ISSUE 3
2026

EXACT SOLUTIONS AND DYNAMICAL BEHAVIOR OF THE FRACTIONAL PURE-QUARTIC NONLINEAR SCHRÖDINGER EQUATION

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Received: 30.05.2026

Abstract. The fractional pure-quartic nonlinear Schrödinger equation (FPQNLSE) plays an important role in modeling nonlinear wave propagation in complex dispersive media where quartic dispersion dominates the conventional second-order effects. This model has significant applications in nonlinear optics, optical fibers, plasma physics, and other systems with memory and nonlocal properties. In this work, exact analytical solutions of the FPQNLSE with the M-truncated fractional derivative are obtained using the mapping method. Different types of solutions, including elliptic, rational, trigonometric, and hyperbolic forms, are derived. These solutions provide important insights into nonlinear phenomena such as optical pulse propagation, signal transmission, and solitary-wave dynamics. Moreover, MATLAB simulations are presented to illustrate the physical behavior of the obtained solutions and to investigate the influence of the fractional-order parameter on wave propagation.

Keywords: Schrödinger equation, optical solitons, M-truncated derivative, exact solutions, simulation, mapping method
UDC: 535.3
DOI: 10.3116/16091833/Ukr.J.Phys.Opt.2026.03139
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1. Introduction

The investigation of fractional nonlinear evolution equations (FNLEEs) has attracted growing interest due to their enhanced ability to capture intricate dynamical behaviors beyond the scope of classical integer-order models. By incorporating fractional-order derivatives, these equations inherently account for memory and hereditary characteristics, which are crucial for accurately describing many natural and engineered systems. Consequently, FNLEEs offer a more realistic and flexible framework for modeling processes that exhibit nonlocal interactions, long-range dependence, and anomalous diffusion. In recent years, significant progress has been made in both the analytical and numerical treatment of such equations, leading to the development of various exact solution techniques and efficient computational methods. Furthermore, FNLEEs have demonstrated their effectiveness across a wide range of applications, including nonlinear optics, quantum mechanics, biological systems, chemical kinetics, financial modeling, and economic dynamics. This broad applicability underscores their importance as powerful tools for understanding complex phenomena and advancing research across interdisciplinary scientific fields [1-5].

In physics, these equations have been widely employed to characterize diffusion in heterogeneous or fractal media, wave propagation in viscoelastic environments, and heat transfer in materials exhibiting anomalous thermal responses. In biological applications, fractional nonlinear evolution equations are effective tools for modeling complex processes such as population dynamics, tumor development, and the spread of infectious diseases, in which memory and nonlocal interactions play significant roles. Their utility also extends to chemical systems, particularly for analyzing reaction–diffusion mechanisms and kinetic processes influenced by history-dependent effects.

In light of the increasing importance of FNLEEs in modeling nonlinear and memory-dependent processes, a wide range of analytical methods has been used to derive their exact solutions. Notable examples include the Riccati expansion method [6], sine-cosine method [7], (G'/G)-expansion [8, 9], extended tanh–coth method [10], first-integral method [11], exp-function method [12], generalized Kudryashov method [13], Jacobi elliptic function expansion [14], sine-Gordon expansion technique [15], etc.

In this work, we consider the fractional pure-quartic nonlinear Schrödinger equation (FPQNLSE) formulated with the M-truncated derivative, given by [16]:

$$i\mathcal{M}_{k,t}^{\alpha,\beta}\mathcal{Z}+\frac{1}{24}\gamma_1\mathcal{M}_{k,xxxx}^{4\alpha,\beta}\mathcal{Z}+\gamma_2|\mathcal{Z}|^2\mathcal{Z}+\gamma_3\mathcal{Z}\mathcal{M}_{k,xx}^{2\alpha,\beta}\left(|\mathcal{Z}|^2\right)=0, \tag{1}$$

where $\mathcal{Z}=\mathcal{Z}(x,t)$ represents the complex wave function (or envelope), describing both its amplitude and phase. The term $\mathcal{M}_{k,t}^{\alpha,\beta}\mathcal{Z}$ denotes the M-truncated fractional derivative of order α with respect to time; $i=\sqrt{-1}$. The term $\frac{1}{24}\gamma_1\mathcal{M}_{k,xxxx}^{4\alpha,\beta}\mathcal{Z}$ represents the fractional fourth-order spatial dispersion; γ_1 is a real coefficient controlling the strength of dispersion. The term $\gamma_2|\mathcal{Z}|^2\mathcal{Z}$ represents to the cubic Kerr nonlinearity, accounting for self-phase modulation and interactions among waves in a nonlinear medium; γ_2 determines the strength and type of the nonlinearity. The term $\gamma_3\mathcal{Z}\mathcal{M}_{k,xx}^{2\alpha,\beta}\left(|\mathcal{Z}|^2\right)$ represents a nonlinear dispersive effect, describing how the dispersion depends on the wave intensity and capturing nonlocal nonlinear behavior in complex media; γ_3 controls its contribution.

The pure-quartic nonlinear Schrödinger equation (PQNLSE) plays an important role in modeling wave propagation in media where higher-order dispersion dominates over conventional second-order effects [17, 18]. In particular, it is highly relevant in nonlinear optics, especially in engineered optical fibers and waveguides designed to suppress quadratic dispersion, allowing quartic dispersion to govern pulse dynamics. This equation enables the study of novel wave structures, including ultra-broadband pulses and stable solitary waves, which are not captured by the standard nonlinear Schrödinger equation. Beyond optics, the PQNLSE also finds applications in plasma physics and other complex dispersive systems, providing a more accurate framework for understanding nonlinear wave interactions and advanced signal transmission phenomena.

Due to the significance of the PQNLSE, it has attracted considerable attention in the literature. For example, Dai et al. [19] performed a quantitative investigation of equilibrium and oscillatory bound states of pure-quartic solitons by deriving an effective interaction potential and formulating a particle-like model to describe their dynamics. Kevian et al. [20] performed a numerical study of a generalized nonlinear Schrödinger equation and reported a family of pure-quartic solitons sustained by a balance between positive Kerr nonlinearity and negative quartic dispersion. Deng et al. [21] studied weak interactions of well-separated pure-quartic solitons using an asymptotic approach to track their separation and phase evolution, identify bound states, and evaluate stability through spectral methods. Soltani et al. [22-23] analyzed the dynamics and modulational instability of the PQNLSE with quartic dispersion and cubic–quintic nonlinear effects, while Irshad et al. [24] investigated the stability properties of the solutions. Wazwaz [25] obtained exact solutions to the PQNLSE using two methods: the tanh method and the sine–cosine method. Recently, Soliman et al. [16] investigated exact solutions to the PQNLSE with a β-fractional derivative using the improved modified extended tanh-function method.

In the present study, the M-truncated fractional derivative is adopted as an alternative framework to describe fractional-order dynamics. Unlike classical operators such as the Caputo or Atangana–Baleanu derivatives, which are defined through singular or non-singular memory kernels, the M-truncated derivative provides a modified local formulation that retains several key properties of standard differentiation while incorporating fractional effects. This feature makes it particularly suitable for analytical investigations and for constructing exact solutions. Although its representation of memory differs from that of kernel-based approaches, it offers an effective way to capture nonlocal behavior via the fractional-order parameter. Consequently, the use of the M-truncated derivative in this work provides a complementary perspective on fractional nonlinear wave propagation and facilitates comparison with results obtained using other fractional formulations.

The primary objective of this work is to derive exact analytical solutions of the FPQNLSE (1) by employing the mapping method within the framework of the M-truncated fractional derivative. Although the solution technique is well established, the present study focuses on obtaining new solution structures arising from the use of this alternative fractional operator, in contrast to earlier works that adopt the β-fractional derivative.

In this setting, we construct several classes of exact solutions, including elliptic, rational, trigonometric, and hyperbolic forms. These solutions extend the existing solution space reported for fractional nonlinear Schrödinger equations and are not, in general, reducible to previously published results via simple parameter transformations. The present work also provides a comparative perspective on how different fractional derivatives influence the analytical structure of the solutions.

Moreover, this study extends the results stated in [16] by introducing the M-truncated fractional derivative as an alternative modeling framework and by deriving a broader range of analytical solutions for the FPQNLSE. The physical relevance of the obtained solutions is illustrated through applications in heat pulse propagation, signal transmission, and nonlinear optical phenomena. In addition, graphical simulations using MATLAB are presented to visualize the solutions and to examine the influence of the fractional operator on their qualitative behavior.

The remainder of the paper is organized as follows. Section 2 introduces the M-truncated derivative (MTD) along with its main properties. In Section 3, the corresponding wave equation for the FPQNLSE (1) is derived. Section 4 is devoted to constructing the exact solutions of the FPQNLSE (1). In Section 5, we discuss the physical meaning of the solutions. Section 6 is devoted to examining the influence of the M-truncated derivative (MTD) on the obtained solutions. Finally, the paper concludes with a summary of the main findings.

2. The M-truncated derivative

Fractional calculus has emerged as an effective and rigorous mathematical framework for describing complex dynamical phenomena that are not adequately captured by classical integer-order models. In contrast to standard derivatives, fractional operators possess an intrinsic nonlocal character, enabling them to incorporate memory and hereditary properties into the formulation of physical and engineering processes. This capability has led to their successful application in diverse areas such as anomalous diffusion, viscoelasticity, fluid dynamics, and nonlinear wave propagation. Over the years, numerous definitions of fractional derivatives have been introduced, each tailored to particular analytical or modeling requirements. Prominent among these are the Caputo and Riemann-Liouville derivatives [3], which differ primarily in their treatment of initial conditions and physical interpretability, as well as the Grünwald-Letnikov approach, which provides a natural bridge to discrete and numerical formulations. Furthermore, alternative operators, including the Hadamard, Jumarie, and Katugampola derivatives [2, 26, 27], have been proposed to extend the classical framework by incorporating logarithmic kernels, handling non-smooth functions, or generalizing integral transforms. The availability of these diverse formulations offers significant flexibility in the modeling process, allowing researchers to select the most appropriate operator for accurately characterizing the underlying dynamics of nonlinear evolution equations and other complex systems. In recent years, Sousa et al. [28] proposed a new fractional operator, referred to as the M-truncated derivative (MTD), which is constructed as a natural extension of the classical derivative.

Definition. Let $f:[a,\infty)\to\mathbb{R}$ be a continuous function. Then the MTD of order $0<\alpha\leq 1$ is defined as follows:

$$\mathcal{M}_{k,t}^{\alpha,\beta} f(t)=\lim_{\varepsilon\to 0}\frac{f\left(tE_{k,\beta}\left(\varepsilon t^{-\alpha}\right)\right)-f(t)}{\varepsilon},$$

where $\boldsymbol{E}_{k,\beta}(t)=\sum_{k=0}^{j}\frac{t^{k}}{\Gamma(\beta k+1)}$, for $t\in\mathbb{C}$ and $\beta>0$

Theorem. Consider $u(t)$ and $v(t)$ are α -order differentiable; then the MTD satisfies the following features for any real constants a and b [28]:

(1) $\mathcal{M}_{k,t}^{\alpha,\beta}(au+bv)=a\mathcal{M}_{k,t}^{\alpha,\beta}(u)+b\mathcal{M}_{k,t}^{\alpha,\beta}(v),$

(2) $\mathcal{M}_{k,t}^{\alpha,\beta}(u\circ v)(t)=u'(v(t))\mathcal{M}_{k,t}^{\alpha,\beta}v(t),$

(3) $\mathcal{M}_{k,t}^{\alpha,\beta}(uv)=u\mathcal{M}_{k,t}^{\alpha,\beta}v+v\mathcal{M}_{k,t}^{\alpha,\beta}u,$

(4) $\mathcal{M}_{k,t}^{\alpha,\beta}(u)(t)=\frac{t^{1-\alpha}}{\Gamma(\beta+1)}\frac{du}{dt},$

(5) $\mathcal{M}_{k,t}^{\alpha,\beta}(t^{v})=\frac{v}{\Gamma(\beta+1)}t^{v-\alpha}.$

3. Traveling wave equation for FPQNLSE

To obtain the wave equation associated with the FPQNLSE (1), the following transformation is employed:

$$\mathcal{Z}(x,t)=\mathcal{Y}(\xi_\alpha)e^{i\phi_\alpha},\ \phi_\alpha=\frac{\Gamma(\beta+1)}{\alpha}\left[\phi_1 x^\alpha+\phi_2 t^\alpha\right] \text{ and } \xi_\alpha=\frac{\Gamma(\beta+1)}{\alpha}\left[\xi_1 x^\alpha+\xi_2 t^\alpha\right], \quad (2)$$

here the constants ϕ_1, ϕ_2, ξ_1, and ξ_2 are non-zero. We observe that

$$\mathcal{M}_{k,t}^{\alpha,\beta}\mathcal{Z}=\left[\xi_2\mathcal{Y}'+i\phi_2\mathcal{Y}\right]e^{i\phi_\alpha},\quad \mathcal{M}_{k,x}^{\alpha,\beta}\mathcal{Z}=\left(\xi_1\mathcal{Y}'+i\phi_1\mathcal{Y}\right)\exp\left(i\phi_\alpha\right),$$
$$\left|\mathcal{Z}\right|^2=\mathcal{Y}^2,\quad \mathcal{M}_{k,xx}^{2\alpha,\beta}\left(\left|\mathcal{Z}\right|^2\right)=2\xi_1^2[(\mathcal{Y}')^2+\mathcal{Y}\mathcal{Y}''],\tag{3}$$
$$\mathcal{M}_{k,xxxx}^{4\alpha,\beta}\mathcal{Z}=\left[\xi_1^4\mathcal{Y}''''+4i\phi_1\xi_1^3\mathcal{Y}'''-6\phi_1^2\xi_1^2\mathcal{Y}''-4i\xi_1\phi_1^3\mathcal{Y}'+\phi_1^4\mathcal{Y}\right]\exp\left(i\phi_\alpha\right).$$

Plugging Eq. (3) into Eq. (1), we have for the imaginary part

$$\left(\xi_2-\frac{\gamma_1}{6}\phi_1^3\xi_1\right)\mathcal{Y}'+\frac{\gamma_1}{6}\phi_1\xi_1^3\mathcal{Y}'''=0,\tag{4}$$

and for the real part

$$\frac{\gamma_1}{24}\xi_1^4\mathcal{Y}''''-\frac{\gamma_1}{4}\phi_1^2\xi_1^2\mathcal{Y}''+\left(\frac{\gamma_1}{24}\phi_1^4-\phi_2\right)\mathcal{Y}+\gamma_2\mathcal{Y}^3+2\gamma_3\xi_1^2[\mathcal{Y}(\mathcal{Y}')^2+\mathcal{Y}^2\mathcal{Y}'']=0,\tag{5}$$

From Eq. (4), we have the following constraint conditions

$$\xi_2=0 \ \text{ and } \ \phi_1=0.\tag{6}$$

By using Eq. (5), it becomes

$$\mathcal{Y}''''+\hbar_0\left[\mathcal{Y}(\mathcal{Y}')^2+\mathcal{Y}^2\mathcal{Y}''\right]-\hbar_1\mathcal{Y}+\hbar_2\mathcal{Y}^3=0,\tag{7}$$

where

$$\hbar_0=\frac{48\gamma_3}{\gamma_1\xi_1^2},\ \hbar_1=\frac{24\phi_2}{\gamma_1\xi_1^4},\ \text{ and }\ \hbar_2=\frac{24\gamma_2}{\gamma_1\xi_1^4}.\tag{8}$$

4. Exact solutions of the FPQNLSE

Here, we implement the mapping method mentioned in [29]. Considering the solutions of Eq. (7) as follows

$$\mathcal{Y}\left(\xi_\alpha\right)=\sum_{i=0}^{K}a_i\mathcal{P}^i\left(\xi_\alpha\right)\tag{9}$$

where a_i is an undefined constant and $\mathcal{P}'$ solves

$$\mathcal{P}'=\sqrt{\ell_1\mathcal{P}^4+\ell_2\mathcal{P}^2+\ell_3},\tag{10}$$

with ℓ_1,ℓ_2 and ℓ_3 are real constants.

To find the parameter K stated in Eq. (9), we equalize $\mathcal{Y}''''$ with $\mathcal{Y}^2\mathcal{Y}''$ in Eq. (7) as $2K+K+2=K+4\Rightarrow K=1$ Eq. (9), with $K=1$, becomes

$$\mathcal{Y}\left(\xi_\alpha\right)=a_0+a_1\mathcal{P}'\left(\xi_\alpha\right).\tag{11}$$

Differentiating Eq. (11) twice and using (10), we have

$$\mathcal{Y}''=a_1\left(\ell_2\mathcal{P}+2\ell_1\mathcal{P}^3\right).\tag{12}$$

Inserting Eqs. (11) and (12) into Eq. (7), we get

$$\begin{aligned}&\left[24a_1\ell_1^2+3\hbar_0a_1^3\ell_1\right]\mathcal{P}^5+\left[2\hbar_0a_0a_1^2\ell_1+\hbar_2a_1^3\right]\mathcal{P}^4\\&+\left[20a_1\ell_1\ell_2+3\hbar_0a_1^3\ell_2+a_0a_1^2\hbar_0\ell_1+3\hbar_2a_0a_1^2\right]\mathcal{P}^3+\left[3\hbar_0\ell_2a_0a_1^2+3\hbar_2a_0^2a_1\right]\mathcal{P}^2\\&+\left[12a_1\ell_1\ell_3+a_1\ell_2^2+3\hbar_0a_1^3\ell_3+\hbar_0a_0a_1^2\ell_2-a_1\hbar_1+3\hbar_2a_0a_1^2\right]\mathcal{P}\\&+\left[\hbar_0a_0a_1^2\ell_3-\hbar_1a_0+\hbar_2a_0^3\right]=0.\end{aligned}$$

Equating each coefficient of $\mathcal{P}^k$ to zero, we have a system of equations. By solving this system, we obtain

$$a_0=0,\ \ a_1=\pm\sqrt{\frac{-8\ell_1}{\hbar_0}},\ \ell_2=\frac{2\hbar_2}{\hbar_0}\ \ \text{and}\ \ \ell_3=\frac{\hbar_0^2\hbar_1-4\hbar_2^4}{4\hbar_0^2\ell_1}. \tag{13}$$

Hence, by using Eqs. (2), (11), and (13), the solution of Eq. (1) is

$$\mathcal{Z}(x,t)=\pm\sqrt{\frac{-8\ell_1}{\hbar_0}}\mathcal{P}(\xi_1 x)\exp\left(i\frac{\Gamma(\beta+1)\phi_2}{\alpha}t^\alpha\right). \tag{14}$$

Many cases based on ℓ_1 are:

Case 1. Assuming that $\ell_1=m^2,\ell_2=-(1+m^2)$ and $\ell_3=1$, it follows that $\mathcal{P}(\xi_\alpha)=sn(\xi_\alpha)$. Accordingly, by substituting into Eq. (14), the solution of the FPQNLSE (1) takes the form:

$$\mathcal{Z}(x,t)=\pm\sqrt{\frac{-8m^2}{\hbar_0}}\mathrm{sn}\left(\frac{\xi_1\Gamma(\beta+1)}{\alpha}x^\alpha\right)\exp\left(i\frac{\Gamma(\beta+1)\phi_2}{\alpha}t^\alpha\right),\ \text{for}\ \hbar_0<0. \tag{15}$$

If $m\to 1$, then Eq. (15) becomes

$$\mathcal{Z}(x,t)=\pm\sqrt{\frac{-8}{\hbar_0}}\tanh\left(\frac{\xi_1\Gamma(\beta+1)}{\alpha}x^\alpha\right)\exp\left(i\frac{\Gamma(\beta+1)\phi_2}{\alpha}t^\alpha\right),\ \text{for}\ \hbar_0<0. \tag{16}$$

Case 2. Let $\ell_1=1,\ell_2=2m^2-1$ and $\ell_3=-m^2(1-m^2)$. Then, one obtains $\mathcal{P}(\xi_\alpha)=ds(\xi_\alpha)$. Hence, by using Eq. (14), the corresponding solution of the FPQNLSE (1) is given by:

$$\mathcal{Z}(x,t)=\pm\sqrt{\frac{-8}{\hbar_0}}\mathrm{ds}\left(\frac{\xi_1\Gamma(\beta+1)}{\alpha}x^\alpha\right)\exp\left(i\frac{\Gamma(\beta+1)\phi_2}{\alpha}t^\alpha\right),\ \text{for}\ \hbar_0<0. \tag{17}$$

As $m\to 1$, Eq. (17) reduces to

$$\mathcal{Z}(x,t)=\pm\sqrt{\frac{-8}{\hbar_0}}\mathrm{csch}\left(\frac{\xi_1\Gamma(\beta+1)}{\alpha}x^\alpha\right)\exp\left(i\frac{\Gamma(\beta+1)\phi_2}{\alpha}t^\alpha\right),\ \text{for}\ \hbar_0<0. \tag{18}$$

If $m\to 0$, then Eq. (17) becomes

$$\mathcal{Z}(x,t)=\pm\sqrt{\frac{-8}{\hbar_0}}\csc\left(\frac{\xi_1\Gamma(\beta+1)}{\alpha}x^\alpha\right)\exp\left(i\frac{\Gamma(\beta+1)\phi_2}{\alpha}t^\alpha\right),\ \text{for}\ \hbar_0<0. \tag{19}$$

Case 3. Let $\ell_1=1,\ell_2=2-m^2$ and $\ell_3=(1-m^2)$. Then, we obtain $\mathcal{P}(\xi_\alpha)=cs(\xi_\alpha)$. Thus, by using Eq. (14), the corresponding solution of the FPQNLSE (1) is given by:

$$\mathcal{Z}(x,t)=\pm\sqrt{\frac{-8}{\hbar_0}}\mathrm{cs}\left(\frac{\xi_1\Gamma(\beta+1)}{\alpha}x^\alpha\right)\exp\left(i\frac{\Gamma(\beta+1)\phi_2}{\alpha}t^\alpha\right),\ \text{for}\ \hbar_0<0. \tag{20}$$

If $m\to 1$, then Eq. (20) tends to

$$\mathcal{Z}(x,t)=\pm\sqrt{\frac{-8}{\hbar_0}}\mathrm{csch}\left(\frac{\xi_1\Gamma(\beta+1)}{\alpha}x^\alpha\right)\exp\left(i\frac{\Gamma(\beta+1)\phi_2}{\alpha}t^\alpha\right),\ \text{for}\ \hbar_0<0. \tag{21}$$

As $m\to 0$, Eq. (20) turns into

$$\mathcal{Z}(x,t)=\pm\sqrt{\frac{-8}{\hbar_0}}\cot\left(\frac{\xi_1\Gamma(\beta+1)}{\alpha}x^\alpha\right)\exp\left(i\frac{\Gamma(\beta+1)\phi_2}{\alpha}t^\alpha\right),\ \text{for}\ \hbar_0<0. \tag{22}$$

Case 4. If $\ell_1=(1-m^2)^2/4,\ell_2=(1-m^2)^2/2$ and $\ell_3=1/4$, then $\mathcal{P}(\xi_\alpha)=\dfrac{\mathrm{sn}(\xi_\alpha)}{\mathrm{dn}(\xi_\alpha)+\mathrm{cn}(\xi_\alpha)}$.

Therefore, by using Eq. (14), the corresponding solution of the FPQNLSE (1) is given by:

$$\mathcal{Z}(x,t)=\pm\sqrt{\frac{-2(1-m^2)^2}{\hbar_0}}\left[\frac{\operatorname{sn}\left(\frac{\xi_1\Gamma(\beta+1)}{\alpha}x^\alpha\right)}{\operatorname{dn}\left(\frac{\xi_1\Gamma(\beta+1)}{\alpha}x^\alpha\right)+\operatorname{cn}\left(\frac{\xi_1\Gamma(\beta+1)}{\alpha}x^\alpha\right)}\right] \times\exp\left(i\frac{\Gamma(\beta+1)\phi_2}{\alpha}t^\alpha\right). \quad (23)$$

When $m\to 0$, Eq. (23) changes to

$$\mathcal{Z}(x,t)=\pm\sqrt{\frac{-2}{\hbar_0}}\left[\frac{\sin\left(\frac{\xi_1\Gamma(\beta+1)}{\alpha}x^\alpha\right)}{1+\cos\left(\frac{\xi_1\Gamma(\beta+1)}{\alpha}x^\alpha\right)}\right]\exp\left(i\frac{\Gamma(\beta+1)\phi_2}{\alpha}t^\alpha\right),\text{ for }\hbar_0<0. \quad (24)$$

Case 5. If $\ell_1=-m^2, \ell_2=2m^2-1$ and $\ell_3=1-m^2$, then $\mathcal{P}(\xi_\alpha)=\operatorname{cn}(\xi_\alpha)$. Therefore, the solution of FPQNLSE (1), utilizing Eq. (14), is

$$\mathcal{Z}(x,t)=\pm\sqrt{\frac{8m^2}{\hbar_0}}\operatorname{cn}\left(\frac{\xi_1\Gamma(\beta+1)}{\alpha}x^\alpha\right)\exp\left(i\frac{\Gamma(\beta+1)\phi_2}{\alpha}t^\alpha\right),\text{ for}\hbar_0>0. \quad (25)$$

When $m\to 1$, Eq. (25) changes to

$$\mathcal{Z}(x,t)=\pm\sqrt{\frac{8}{\hbar_0}}\operatorname{sech}\left(\frac{\xi_1\Gamma(\beta+1)}{\alpha}x^\alpha\right)\exp\left(i\frac{\Gamma(\beta+1)\phi_2}{\alpha}t^\alpha\right),\text{ for}\hbar_0>0. \quad (26)$$

Case 6. If $\ell_1=-1, \ell_2=2-m^2$ and $\ell_3=m^2-1$, then $\mathcal{P}(\xi_\alpha)=dn(\xi_\alpha)$. Accordingly, by substituting into Eq. (14), the solution of the FPQNLSE (1) can be expressed as follows:

$$\mathcal{Z}(x,t)=\pm\sqrt{\frac{8}{\hbar_0}}\operatorname{dn}\left(\frac{\xi_1\Gamma(\beta+1)}{\alpha}x^\alpha\right)\exp\left(i\frac{\Gamma(\beta+1)\phi_2}{\alpha}t^\alpha\right),\text{ for }\hbar_0>0. \quad (27)$$

If $m\to 1$, then Eq. (27) turns to Eq. (26).

Case 7. If $\ell_1=1, \ell_2=-m^2-1$ and $\ell_3=m^2$, then $\mathcal{P}(\xi_\alpha)=\operatorname{ns}(\xi_\alpha)=\frac{1}{\operatorname{sn}(\xi_\alpha)}$. Therefore, the solution of FPQNLSE (1), utilizing Eq. (14), is

$$\mathcal{Z}(x,t)=\pm\sqrt{\frac{-8}{\hbar_0}}\operatorname{ns}\left(\frac{\xi_1\Gamma(\beta+1)}{\alpha}x^\alpha\right)\exp\left(i\frac{\Gamma(\beta+1)\phi_2}{\alpha}t^\alpha\right),\text{ for }\hbar_0<0. \quad (28)$$

If $m\to 1$, then Eq. (28) tends to

$$\mathcal{Z}(x,t)=\pm\sqrt{\frac{-8}{\hbar_0}}\coth\left(\frac{\xi_1\Gamma(\beta+1)}{\alpha}x^\alpha\right)\exp\left(i\frac{\Gamma(\beta+1)\phi_2}{\alpha}t^\alpha\right),\text{ for }\hbar_0<0. \quad (29)$$

As $m\to 0$, then Eq. (28) changes to

$$\mathcal{Z}(x,t)=\pm\sqrt{\frac{-8}{\hbar_0}}\csc\left(\frac{\xi_1\Gamma(\beta+1)}{\alpha}x^\alpha\right)\exp\left(i\frac{\Gamma(\beta+1)\phi_2}{\alpha}t^\alpha\right),\text{ for }\hbar_0<0. \quad (30)$$

Case 8. Taking $\ell_1=1-m^2, \ell_2=2m^2-1$ and $\ell_3=-m^2$, we obtain $\mathcal{P}(\xi_\alpha)=\operatorname{nc}(\xi_\alpha)=1/\operatorname{cn}(\xi_\alpha)$. Accordingly, by substituting into Eq. (14), the solution of the FPQNLSE (1) can be expressed as follows:

$$\mathcal{Z}(x,t)=\pm\sqrt{\frac{-8(1-m^2)}{\hbar_0}}\operatorname{nc}\left(\frac{\xi_1\Gamma(\beta+1)}{\alpha}x^\alpha\right)\exp\left(i\frac{\Gamma(\beta+1)\phi_2}{\alpha}t^\alpha\right),\text{ for }\hbar_0<0. \quad (31)$$

When $m \to 0$, Eq. (31) turns to

$$\mathcal{Z}(x,t)=\pm\sqrt{\frac{-8}{\hbar_0}}\sec\left(\frac{\xi_1\Gamma(\beta+1)}{\alpha}x^{\alpha}\right)\exp\left(i\frac{\Gamma(\beta+1)\phi_2}{\alpha}t^{\alpha}\right),\ \text{for}\ \hbar_0<0. \tag{32}$$

Case 9. Let $\ell_1=\frac{1}{4}, \ell_2=\frac{1-2m^2}{2}$ and $\ell_3=\frac{1}{4}$. Then, one obtains $\mathcal{P}(\xi_\alpha)=\text{ns}(\xi_\alpha)+\text{cs}(\xi_\alpha)$. Hence, upon employing Eq. (14), the corresponding solution of the FPQNLSE (1) is given by:

$$\mathcal{Z}(x,t)=\pm\sqrt{\frac{-2}{\hbar_0}}\left[\text{ns}(\xi_1 x)+\text{cs}(\xi_1 x)\right]\exp\left(i\frac{\Gamma(\beta+1)\phi_2}{\alpha}t^{\alpha}\right),\ \text{for}\ \hbar_0<0. \tag{33}$$

As $m \to 1$, Eq. (33) simplifies to:

$$\begin{aligned}\mathcal{Z}(x,t)=\pm\sqrt{\frac{-2}{\hbar_0}}&\left[\text{sech}\left(\frac{\xi_1\Gamma(\beta+1)}{\alpha}x^{\alpha}\right)+\coth\left(\frac{\xi_1\Gamma(\beta+1)}{\alpha}x^{\alpha}\right)\right]\\ &\times\exp\left(i\frac{\Gamma(\beta+1)\phi_2}{\alpha}t^{\alpha}\right),\ \text{for}\ \hbar_0<0.\end{aligned} \tag{34}$$

If $m \to 0$, then Eq. (33) changes to

$$\begin{aligned}\mathcal{Z}(x,t)=\pm\sqrt{\frac{-2}{\hbar_0}}&\left[\csc\left(\frac{\xi_1\Gamma(\beta+1)}{\alpha}x^{\alpha}\right)+\cot\left(\frac{\xi_1\Gamma(\beta+1)}{\alpha}x^{\alpha}\right)\right]\\ &\times\exp\left(i\frac{\Gamma(\beta+1)\phi_2}{\alpha}t^{\alpha}\right),\ \text{for}\ \hbar_0<0.\end{aligned} \tag{35}$$

Case 10. Taking $\ell_1=(1-m^2)/4, \ell_2=(1-m^2)/2$ and $\ell_3=(1-m^2)/4$, we obtain $\mathcal{P}(\xi_\alpha)=cn(\xi_\alpha)/(1+sn(\xi_\alpha))$. Thus, by substituting into Eq. (14), the solution of the FPQNLSE (1) can be expressed as follows:

$$\begin{aligned}\mathcal{Z}(x,t)=\pm\sqrt{\frac{-2(1-m^2)}{\hbar_0}}&\left[\frac{\text{cn}\left(\frac{\xi_1\Gamma(\beta+1)}{\alpha}x^{\alpha}\right)}{1+\text{sn}\left(\frac{\xi_1\Gamma(\beta+1)}{\alpha}x^{\alpha}\right)}\right]\\ &\times\exp\left(i\frac{\Gamma(\beta+1)\phi_2}{\alpha}t^{\alpha}\right),\ \text{for}\ \hbar_0<0.\end{aligned} \tag{36}$$

As $m \to 0$, Eq. (36) turns to

$$\begin{aligned}\mathcal{Z}(x,t)=\pm\sqrt{\frac{-2}{\hbar_0}}&\left[\frac{\cos\left(\frac{\xi_1\Gamma(\beta+1)}{\alpha}x^{\alpha}\right)}{1+\sin\left(\frac{\xi_1\Gamma(\beta+1)}{\alpha}x^{\alpha}\right)}\right]\\ &\times\exp\left(i\frac{\Gamma(\beta+1)\phi_2}{\alpha}t^{\alpha}\right),\ \text{for}\ \hbar_0<0.\end{aligned} \tag{37}$$

Case 11. Let $\ell_1=1, \ell_2=0$ and $\ell_3=0$. Then, we get $\mathcal{P}(\xi_\alpha)=c/\xi_\alpha$. Therefore, by substituting into Eq. (14), the solution of the FPQNLSE (1) can be written as:

$$\begin{aligned}\mathcal{Z}(x,t)=\pm\sqrt{\frac{-8}{\hbar_0}}&\left[\frac{\text{c}\alpha}{\xi_1\Gamma(\beta+1)x^{\alpha}}\right]\\ &\times\exp\left(i\frac{\Gamma(\beta+1)\phi_2}{\alpha}t^{\alpha}\right),\ \text{for}\ \hbar_0<0.\end{aligned} \tag{38}$$

5. Physical Interpretation of the obtained solutions

In this section, we discuss the physical significance of the obtained analytical solutions of the FPQNLSE (1) and classify them according to their corresponding wave structures. This interpretation provides a clearer connection between the mathematical results and their relevance in physical systems, particularly in the context of nonlinear optics and pure-quartic dispersion media.

The Jacobi elliptic function solutions (e.g., sn, cn, dn) represent nonlinear periodic waves, often referred to as cnoidal wave structures. These solutions can be viewed as intermediate states between purely periodic waves and localized solitons. In particular, they describe periodic pulse trains in optical systems. An important feature of these solutions is that, in appropriate limiting cases (e.g., as the elliptic modulus approaches unity), they reduce to hyperbolic soliton solutions, thereby establishing a direct link between periodic and solitary wave regimes.

The hyperbolic function solutions represent localized wave structures and can be classified as soliton solutions. Depending on the parameter choices, these solutions correspond to either bright solitons, which describe localized pulse peaks propagating without distortion, or dark solitons, which represent localized intensity dips on a continuous background. Such structures are of fundamental importance in optical fiber communications, where stable pulse propagation is required.

The trigonometric function solutions correspond to periodic waveforms that describe oscillatory behavior in space and time. These solutions are associated with wave trains and modulation patterns that arise in nonlinear dispersive systems. Physically, they model situations where energy is distributed periodically rather than concentrated in a localized pulse.

The rational solutions are typically associated with singular or rogue-wave-like structures, characterized by strong localization in both space and time. These solutions may represent extreme events or transient high-amplitude pulses in nonlinear media. Their presence highlights the possibility of energy localization and instability phenomena in pure-quartic dispersive systems.

From a physical perspective, the obtained solutions are relevant to optical pulse propagation in nonlinear media with pure-quartic dispersion, where higher-order dispersion effects dominate. In such systems, different types of wave structures correspond to different propagation regimes, including stable pulse transmission (solitons), periodic modulation (trigonometric and elliptic waves), and localized extreme events (rational solutions).

Furthermore, incorporating the M-truncated fractional derivative provides additional flexibility in modeling the system dynamics. The fractional-order parameter modifies the effective dispersion and phase evolution, thereby influencing key characteristics such as amplitude, width, and propagation speed of the wave structures. As a result, the solutions provide insight into how fractional-order effects alter nonlinear wave behavior compared to classical models.

6. Influence of MTD

Here, we explore the impact of the MTD on the solutions of the FPQNLSE (1). To highlight this effect, a set of plots is presented for selected analytical solutions, including (16), (25), (26), and (30), as follows:

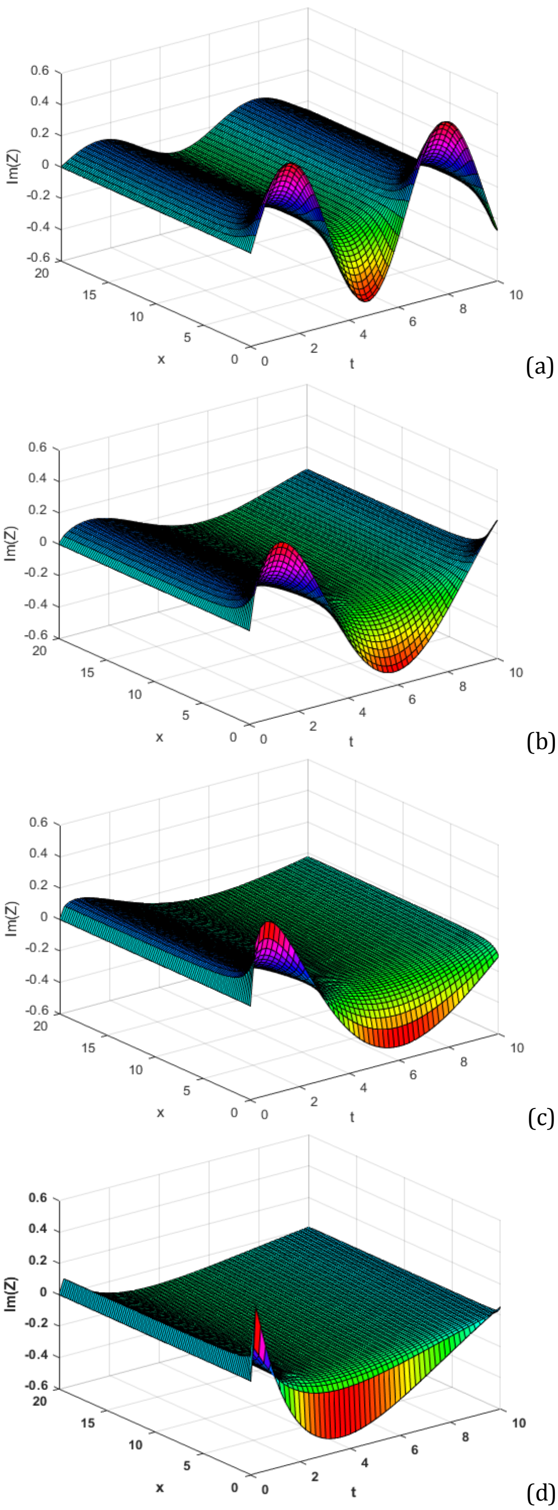


Fig. 1. Exhibits 3D-profile of $\mathrm{Im}(\mathcal{Z})$ where $\mathcal{Z}(x,t)=\frac{1}{\sqrt{6}}\tanh\left(\frac{\Gamma(\beta+1)}{\alpha}x^{\alpha}\right)\exp\left(i\frac{\Gamma(\beta+1)}{\alpha}t^{\alpha}\right)$ described in Eq. (16) with $\gamma_1=\gamma_2=\xi_1=\phi_1=1,\gamma_3=-1$. Hence, $\hbar_0=-48, \hbar_1=24$ and $\hbar_2=24$: (a) $\alpha=1$ and $\beta=0$, (b) $\alpha=0.7$ and $\beta=0.9$, (c) $\alpha=0.5$ and $\beta=0.9$, (d) $\alpha=0.3$ and $\beta=0.9$.

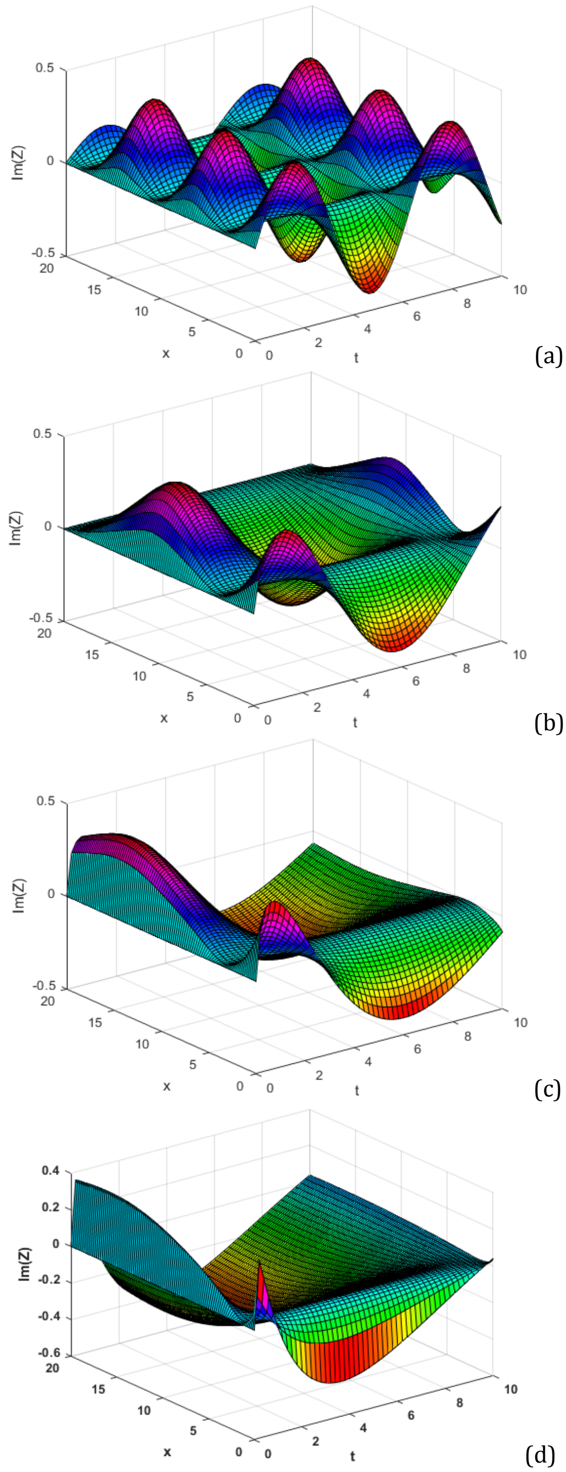
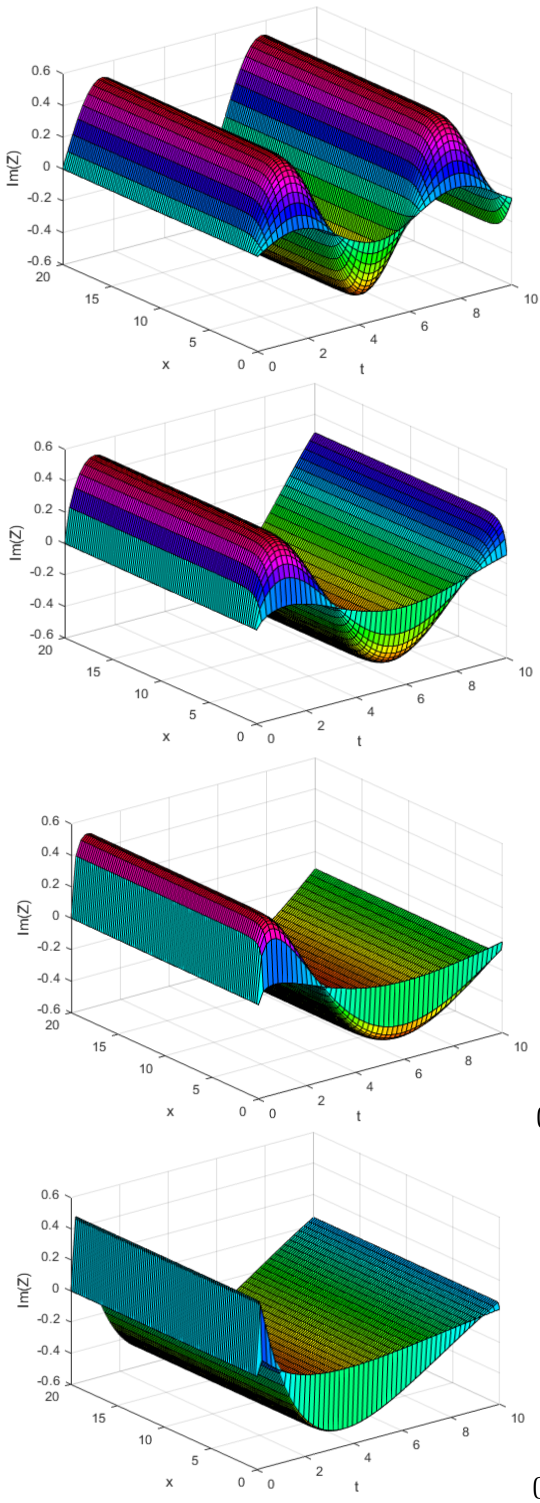


Fig. 2. Exhibits 3D-profile of Im($\mathcal{Z}$) where $\mathcal{Z}(x,t)=\frac{1}{\sqrt{24}}\mathrm{cn}\left(\frac{\Gamma(\beta+1)}{\alpha}x^{\alpha}\right)\exp\left(i\frac{\Gamma(\beta+1)}{\alpha}t^{\alpha}\right)$ described in Eq. (25) with $x\in[0,20]$, $t\in 0,10]$, $\gamma_1=\gamma_2=\gamma_3=\xi_1=\phi_1=1$ and $m=0.5$. Hence, $\hbar_0=48$, $\hbar_1=24$ and $\hbar_2=24$: (a) $\alpha=1$ and β =0, (b) $\alpha=0.7$ and β =0.9, (c) $\alpha=0.5$ and β =0.9, (d) $\alpha=0.3$ and β =0.9.



(a)

(b)

(c)

(d)

Fig. 3. Exhibits 3D-profile of $\mathrm{Im}(\mathcal{Z})$ where $\mathcal{Z}(x,t)=\frac{1}{\sqrt{6}}\left(\frac{1}{4}+\operatorname{sech}\left(\frac{\Gamma(\beta+1)}{\alpha}x^{\alpha}\right)\right)\exp\left(i\frac{\Gamma(\beta+1)}{\alpha}t^{\alpha}\right)$ described in Eq. (26) with $\gamma_1=\gamma_2=\gamma_3=\xi_1=\phi_1=1$. Hence, $\hbar_0=48$, $\hbar_1=24$ and $\hbar_2=24$: (a) $\alpha=1$ and β =0, (b) $\alpha=0.7$ and β =0.9, (c) $\alpha=0.5$ and β =0.9, (d) $\alpha=0.8$ and β =0.9.

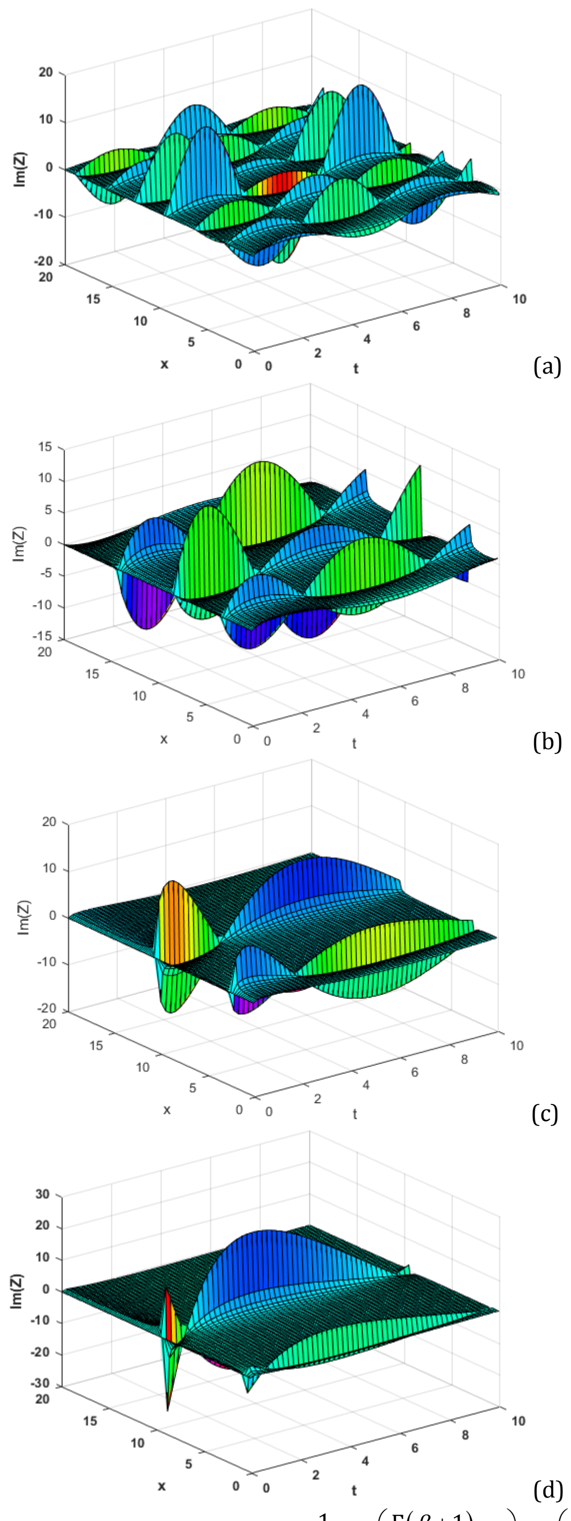


Fig. 4. Exhibits 3D-profile of Im($\mathcal{Z}$) where $\mathcal{Z}(x,t)=\frac{1}{\sqrt{6}}\csc\left(\frac{\Gamma(\beta+1)}{\alpha}x^{\alpha}\right)\exp\left(i\frac{\Gamma(\beta+1)}{\alpha}t^{\alpha}\right)$ described in Eq. (30) with $\gamma_1=\gamma_2=\gamma_3=\xi_1=\phi_1=1$. Hence, $\hbar_0=48$, $\hbar_1=24$ and $\hbar_2=24$: (a) $\alpha=1$ and β =0, (b) $\alpha=0.7$ and β =0.9, (c) $\alpha=0.5$ and β =0.9, (d) $\alpha=0.8$ and β =0.9.

The obtained solutions demonstrate that the fractional derivative parameter α plays a crucial role in controlling the dynamical behavior of the wave profile. For the classical case at α=1 (see Fig. 1-4a), the solution exhibits regular oscillatory propagation with stronger amplitude variations and well-defined periodic behavior, corresponding to standard local dynamics without memory effects. However, as the fractional order decreases to α=0.7, α=0.5, and α=0.3 (see Fig. 1- 4b-d), the oscillations gradually weaken, and the wave profile becomes smoother and less localized. This behavior indicates that the fractional derivative introduces significant memory and nonlocal effects into the system, leading to slower phase evolution and anomalous propagation characteristics. Physically, decreasing (α) enhances hereditary properties of the medium, suppresses energy transport, and increases dispersive effects, resuing in broader wave structures with reduced oscillatory intensity. These results confirm that the fractional-order parameter can effectively regulate localization, dispersion, and stability of nonlinear waves in complex media such as viscoelastic materials, plasma systems, nonlinear optical fibers, and anomalous diffusion environments.

7. Conclusion

In this work, the fractional pure-quartic nonlinear Schrödinger equation with the M-truncated fractional derivative was successfully investigated. Various classes of exact analytical solutions, including elliptic, rational, trigonometric, and hyperbolic solutions, were obtained by using the mapping method. The derived solutions reveal important characteristics of nonlinear wave propagation in fractional-dispersive media and provide useful insights into several physical phenomena arising in nonlinear optics, optical fiber communications, plasma physics, and related fields. Furthermore, MATLAB simulations were presented to analyze the dynamical behavior of the obtained solutions and to study the influence of the fractional-order parameter on wave propagation. The results demonstrate that fractional derivatives play a significant role in controlling localization, oscillation, and stability of nonlinear waves through memory and nonlocal effects. The proposed analytical approach is effective and can be extended to investigate other nonlinear fractional evolution equations arising in mathematical physics and engineering applications.

Authors' contribution. Sofian O. Obeidat: conceptualization, resources, software, data curation, writing-review editing. **Doaa Rizk**: methodology, investigation, funding acquisition, writing-original draft.

Declaration of competing interest. The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability. All data generated or analyzed during this study are included in this article.

Funding and acknowledgment: The researchers would like to thank the Deanship of Graduate Studies and Scientific Research at Qassim University (www.qu.edu.sa) for financial support (QU-APC-2026).

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Анотація. *Дробове чисто-квартичне нелінійне рівняння Шредінгера (FPQNLSE) відіграє важливу роль у моделюванні нелінійного поширення хвиль у складних дисперсійних середовищах, де квартична дисперсія домінує над традиційними ефектами другого порядку. Ця модель має значне застосування в нелінійній оптиці, оптичних волокнах, фізиці плазми та інших системах з властивостями пам'яті та нелокальними властивостями. У цій роботі отримано точні аналітичні розв'язки FPQNLSE з M-усіченою дробовою похідною за допомогою методу відображення. Отримано різні типи розв'язків, включаючи еліптичні, раціональні, тригонометричні та гіперболічні форми. Ці розв'язки надають важливе розуміння нелінійних явищ, таких як поширення оптичних імпульсів, передача сигналів та динаміка одиночних хвиль. Крім того, представлено моделювання в MATLAB для ілюстрації фізичної поведінки отриманих розв'язків та дослідження впливу параметра дробового порядку на поширення хвиль.*

Ключові слова: *рівняння Шредінгера, оптичні солітони, M-усічена похідна, точні розв'язки, моделювання, метод відображення*