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Optical Analogy of the Stokes and Jones Parameters for Acoustic Waves in Anisotropic Media

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Abstract. The coherence matrix and the Stokes parameters for the acoustic waves have been obtained in the present work. It has been shown that the obtained Stokes parameters form a basis for the 8D Poincaré sphere. To analyze the properties of this sphere, it has been intersected by three 3D hyperplanes and one 2D plane. As a result, three 3D Poincaré spheres and one qutrit triangle have been obtained. The characteristics of these spheres and the triangle have been described in detail, and different types of acoustic waves have been distinguished. For all these polarization states, the corresponding Jones vectors have been obtained, and the Jones matrices for the simplest polarization acoustic elements are proposed.

Keywords: acoustic waves, Poincaré sphere, Stokes parameters, Jones vector, Jones matrix
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1. Introduction

Crystal acoustics, a subfield of crystal physics, explores the propagation of acoustic waves (AWs) within crystals. It closely relates to crystal optics but involves greater complexity due to elastic constants represented by a fourth-rank tensor, unlike the second-rank optical impermeability tensor. Key differences include the inability of AWs to travel in a vacuum and the absence of transverse acoustic waves in gases and liquids. The solid media support three types of acoustic eigenwaves (two transverse and one longitudinal), while optical waves are always transverse. AWs can be non-orthogonal, unlike electromagnetic waves, which are inherently orthogonal. AWs possess more significant oblique flow of acoustic energy compared to optical Poynting vectors. Crystals can have up to 16 acoustic axes, whereas only two may exist in crystal optics. Despite these differences, many phenomena are common to both fields. These include acoustic refraction and reflection, acoustic three-refringence (an analogue of birefringence in optics), acoustic activity as a spatial dispersion effect, etc. Both areas also consider effects related to the media's acoustic and optical properties.

Despite many similarities between crystal optics and crystal acoustics, several problems remain unaddressed or unresolved in the field of crystal acoustics. These include the Jones and Stokes approaches to describing the polarization state of acoustic waves. It should be noted that these methods are only effective in crystal acoustics when the obliquity of the acoustic energy flow is negligible, whereas in the acoustics of anisotropic materials, this parameter can reach tens of degrees. Nonetheless, acoustic polarimetric elements, such as a quarter-wave plate, have been shown to work [1], and therefore acoustic polarimetry can be regarded as a branch of the acoustics of anisotropic media. Since the AWs have three eigenstates of polarization, the coherence matrix must be a 3×3 matrix rather than a 2×2 matrix in optics for purely transverse electromagnetic waves. One has to note that the Stokes parameters can be derived directly from the coherence matrix. It is known from the literature that, in optics, the 3×3 coherence matrix has been considered for special cases, e.g., near-field and focused light, where the longitudinal component of the field becomes important [2-4]. As a result, for the fully polarized case of an electromagnetic wave, eight Stokes parameters have been introduced and proved to be enough for a complete description of the polarization state. Regarding AWs, as far as we know, there are few studies considering specific cases for Stokes parameters of oceanic waves [5] and diffuse random scattering of AWs [6]. In the first case, the authors suggested using four Stokes parameters, while in the second case, five parameters – one longitudinal and four specific to optics. However, to our knowledge, such an approach has not been developed for crystalline materials. Additionally, the propagation of AWs in crystals involves an extra parameter that is not specific to electromagnetic waves, as well as AWs in isotropic media, namely, the non-orthogonality of AWs' polarization. Consequently, this work aims to analyze the Stokes and Jones parameters in crystal acoustics.

2. Results and discussion

2.1. Stokes parameters and Poincaré sphere for AW

In optics, the Jones vector can be written as:

$$E=\begin{bmatrix} E_x e^{i\varphi_1} \\ E_y e^{i\varphi_2} \end{bmatrix} = e^{i\varphi_1}\begin{bmatrix} E_x \\ E_y e^{i\varphi} \end{bmatrix}, \quad (1)$$

where $\varphi=\varphi_2-\varphi_1$ is the phase difference between electric field components E_x and E_y, which are the complex amplitudes. Then the coherence matrix is written in the form:

$$J=\begin{bmatrix} \langle E_x E_x^* \rangle & \langle E_x E_y^* \rangle \\ \langle E_y E_x^* \rangle & \langle E_y E_y^* \rangle \end{bmatrix}. \quad (2)$$

For the fully polarized light this matrix can be simplified as:

$$J=\begin{bmatrix} J_{xx} & J_{xy} \\ J_{yx} & J_{yy} \end{bmatrix} = \begin{bmatrix} E_x^2 & E_x E_y e^{-i\varphi} \\ E_x E_y e^{i\varphi} & E_y^2 \end{bmatrix}. \quad (3)$$

The Stokes parameters are as follows:

$$S_0 = J_{xx}+J_{yy},\ S_1 = J_{xx}-J_{yy}\ S_2 = J_{xy}+J_{yx},\ S_3 = i(J_{xy}+J_{yx}). \quad (4)$$

These parameters are dependent on the angle of ellipticity (χ) and azimuth (ψ) of light as

$$\begin{aligned} S_0 &= I = E_x^2+E_y^2, \\ S_1 &= I\cos 2\psi \cos 2\chi = E_x^2-E_y^2, \\ S_2 &= I\sin 2\psi \cos 2\chi = 2E_x E_y \cos\varphi, \\ S_3 &= I\sin 2\chi = 2E_x E_y \sin\varphi. \end{aligned} \quad (5)$$

Here I, 2ψ and 2χ are the spherical coordinates. Four parameters given by Eqs. (5) constructs the Poincare sphere for 2D polarization state. Therefore, known Stokes parameters determinates the coordinates on the Poincare sphere and therefore the parameters of light ellipticity.

The transformation of Eq. (3) from 2D polarization state to the 3D one can be done with using of Gell-Mann matrices [7] as the generators of SU(3) group [2,3]. In this case, the Jones vector is written as:

$$E=\begin{bmatrix}E_x\\E_y\\E_z\end{bmatrix}=\begin{bmatrix}E_{0x}e^{i\varphi_x}\\E_{0y}e^{i\varphi_y}\\E_{0z}e^{i\varphi_z}\end{bmatrix}=e^{i\varphi_x}\begin{bmatrix}E_{0x}\\E_{0y}e^{i\varphi}\\E_{0z}e^{i\varphi'}\end{bmatrix}, \quad (6)$$

where $\varphi=\varphi_y-\varphi_x, \varphi'=\varphi_z-\varphi_x$. Then derived by us the 3D coherence matrix is:

$$J=\begin{bmatrix}\frac{1}{3}\Phi_0+\frac{1}{2}\Phi_3+\frac{1}{2\sqrt{3}}\Phi_8 & \frac{1}{2}\Phi_1-i\frac{1}{2}\Phi_2 & \frac{1}{2}\Phi_4-i\frac{1}{2}\Phi_5\\ \frac{1}{2}\Phi_1+i\frac{1}{2}\Phi_2 & \frac{1}{3}\Phi_0-\frac{1}{2}\Phi_3+\frac{1}{2\sqrt{3}}\Phi_8 & \frac{1}{2}\Phi_6-i\frac{1}{2}\Phi_7\\ \frac{1}{2}\Phi_4+i\frac{1}{2}\Phi_5 & \frac{1}{2}\Phi_6+i\frac{1}{2}\Phi_7 & \frac{1}{3}\Phi_0-\frac{1}{\sqrt{3}}\Phi_8\end{bmatrix}, \quad (7)$$

where Φ_i are the generalized Stokes parameters written as:

$$\begin{aligned}&\Phi_0=E_{0x}^2+E_{0y}^2+E_{0z}^2, \Phi_1=2E_{0x}E_{0y}\cos\varphi, \Phi_2=2E_{0x}E_{0y}\sin\varphi,\\ &\Phi_3=(E_{0x}^2-E_{0y}^2), \Phi_4=2E_{0x}E_{0z}\cos\varphi', \Phi_5=2E_{0x}E_{0z}\sin\varphi',\\ &\Phi_6=2E_{0y}E_{0z}\cos(\varphi'-\varphi), \Phi_7=2E_{0y}E_{0z}\sin(\varphi'-\varphi), \Phi_8=\frac{1}{\sqrt{3}}(E_{0x}^2+E_{0y}^2-2E_{0z}^2).\end{aligned} \quad (8)$$

It should be noted that the obtained matrix (7) agrees well with the same one presented in Ref. [2]. Therefore, we are dealing with an 8D Poincaré sphere with Φ_i as basic vectors.

However, as it has been noted above, the AWs propagated in the anisotropic media can acquire non-orthogonality, i.e., deviation of the displacement vector from the orthogonal (or longitudinal) position relative to the wavevector. In this case, the eigen vectors of the Christoffel tensor rotate with respect to the crystal physic coordinate system, which in most cases coincides with the crystallographic one or is based on it. The AWs velocities are determined on the basis of the Christoffel equation

$$C_{ijkl}m_jm_ku_l=\rho v^2u_i, \quad (9)$$

where $M_{il}=C_{ijkl}m_jm_k$ - is the second rank Christoffel tensor, and m_jm_k - are the components of the unit wavevector of the AWs. The angle of rotation of the eigenvectors of the Christoffel tensor is determined as

$$\tan 2\zeta=\frac{2M_{il}}{M_{ii}-M_{ll}}. \quad (10)$$

The angle ζ is the angle of non-orthogonality, i.e., the angle of deviation of the displacement vector from a pure orthogonal or longitudinal position.

Let us consider the laboratory coordinate system (Fig. 1a) in which the *Z* direction coincides with the direction of AW propagation and the displacement vector of pure longitudinal (PL) AW, while *X* and *Y* directions correspond to the directions of polarization of pure transverse (PT) waves. Since the AW possesses three types of polarization, namely two transverse and one longitudinal, the Jones vector should contain three components. The field of displacement of the plane, mono-frequency AW, propagated along the *Z* direction can be written as:

$$u(Z,t)=u_1e^{i(\Omega t-mZ+\delta_1)}+u_2e^{i(\Omega t-mZ+\delta_2)}+u_3e^{i(\Omega t-mZ+\delta_3)}, \qquad (11)$$

where u_i - are the amplitudes of AW polarization components, Ω - is the AW frequency, $m=2\pi/\Lambda$ - is the wavevector of AW, Λ - is the AW wavelength, t –is the time, and δ_i - are the phases of respective AW components. Eliminating the time dependence, the phasor of Eq. (11) can be presented in the form

$$u=e^{-imZ}\begin{bmatrix}u_1e^{i\delta_1}\\ u_2e^{i\delta_2}\\ u_3e^{i\delta_3}\end{bmatrix}=e^{-imZ}e^{i\delta_1}\begin{bmatrix}u_1\\ u_2e^{i\delta}\\ u_3e^{i\delta'}\end{bmatrix}, \qquad (12)$$

where $\delta=\delta_2-\delta_1$ and $\delta'=\delta_3-\delta_1$. The coherence matrix for pure AWs will be presented by the matrix similar to (7) with Stokes parameters:

$$\begin{aligned}&\Sigma_0=u_1^2+u_2^2+u_3^2, \quad \Sigma_1=2u_1u_2\cos\delta, \quad \Sigma_2=2u_1u_2\sin\delta, \quad \Sigma_3=(u_1^2-u_2^2),\\ &\Sigma_4=2u_1u_3\cos\delta', \quad \Sigma_5=2u_1u_3\sin\delta', \quad \Sigma_6=2u_2u_3\cos(\delta'-\delta),\\ &\Sigma_7=2u_2u_3\sin(\delta'-\delta), \quad \Sigma_8=\frac{1}{\sqrt{3}}(u_1^2+u_2^2-2u_3^2).\end{aligned} \qquad (13)$$

In the Eqs. (10) Σ_0 describes the total power, Σ_1 - correlation between u_1 and u_2 components of the displacement vector, Σ_2 - correlation between circular components in *XY* plane, Σ_3 - power difference for the waves polarized along *X* and *Y* axes, Σ_4 - correlation between u_1 and u_3 components of the displacement vector, Σ_5 - correlation between circular components in *XZ* plane, Σ_6 - correlation between u_2 and u_3 components of the displacement vector, Σ_7 - correlation between circular components in *YZ* plane, and Σ_8 - power difference between transverse and longitudinal components of AWs. Notice that for the canonical Poincaré sphere, the relation $S_1^2+S_2^2+S_3^2\le S_0^2$ is satisfied, while for the 8D sphere, this relation is changed to $\Sigma_1^2+\Sigma_2^2+...\Sigma_8^2\le\frac{4}{3}\Sigma_0^2$. Therefore, in the last case, one deals with a non-standard spherical volume. It should be noted that this approach has been employed in several studies, using the Bloch sphere to analyze qutrit properties [8,9] or the Poincaré sphere for optical waves with longitudinal projections of the transverse components [10,11]. To simplify the analysis of the 8D Poincaré sphere, it is reasonable to cut it with four 3D hyperplanes, e.g. $\Sigma_1\Sigma_2\Sigma_3$, $\Sigma_4\Sigma_5\Sigma_6$, $\Sigma_6\Sigma_7\Sigma_8$ and $\Sigma_1\Sigma_4\Sigma_6$. However, only the first cut produces a sphere identical to the canonical Poincaré one, which is characteristic of optics. The second and third cuts result in an ellipsoid, while the fourth yields an obese tetrahedron [9]. All these sections, except $\Sigma_1\Sigma_2\Sigma_3$, yield different numbers of pure states and are difficult to analyze. Thus, it seems to be reasonable to find the conditions giving one at least three different spheres similar to Poincaré's one. Such conditions can be found combining Σ_3 and Σ_8, i.e. $\Sigma_{13}=\frac{1}{2}\left(\Sigma_3+\sqrt{3}\Sigma_8\right)=u_1^2-u_3^2$ and $\Sigma_{23}=\frac{1}{2}\left(-\Sigma_3+\sqrt{3}\Sigma_8\right)=u_2^2-u_3^2$. In this case, one can obtain three spheres based on the Stokes parameters $\Sigma_1\Sigma_2\Sigma_3$, $\Sigma_4\Sigma_5\Sigma_{13}$ and $\Sigma_6\Sigma_7\Sigma_{23}$. These spheres represent the polarization of AW in the *XY*, *XZ*, and *YZ* planes, respectively. For the polarization in the *XY* plane, one can write the total power as $I_{xy}^2=\Sigma_1^2+\Sigma_2^2+\Sigma_3^2=\Sigma_{0xy}^2$ and represent the Stokes parameters via azimuth (ψ_{xy}) and ellipticity angle (χ_{xy}) as

$$\Sigma_1 = I_{xy} \sin 2\psi_{xy} \cos 2\chi_{xy}$$
$$\Sigma_2 = I_{xy} \sin 2\chi_{xy} \qquad (14)$$
$$\Sigma_3 = I_{xy} \cos 2\psi_{xy} \cos 2\chi_{xy}$$

The azimuth and ellipticity are defined as:

$$\psi_{xy} = \frac{1}{2} \mathrm{a}\tan \frac{\Sigma_1}{\Sigma_3} = \frac{1}{2} \mathrm{a}\tan\left(\frac{2u_1 u_2 \cos\delta}{u_1^2 - u_2^2}\right)$$

and (15)

$$\chi_{xy} = \frac{1}{2} \mathrm{a}\sin \frac{\Sigma_2}{\Sigma_{0xy}} = \frac{1}{2} \mathrm{a}\sin\left(\frac{2u_1 u_2 \cos\delta}{u_1^2 + u_2^2}\right),$$

respectively, where δ - is the phase difference between X and Y components (Fig. 1b). For the polarization in XZ plane the total power is $I_{xy}^2 = \Sigma_{xz}^2 + \Sigma_4^2 + \Sigma_5^2 = \left(u_1^2 + u_3^2\right)^2 = \Sigma_{0xz}^2$ and the Stokes parameters are as follows:

$$\Sigma_4 = I_{xz} \sin 2\psi_{xz} \cos 2\chi_{xz}$$
$$\Sigma_5 = I_{xz} \sin 2\chi_{xz} \quad . \qquad (16)$$
$$\Sigma_{xz} = I_{xz} \cos 2\psi_{xz} \cos 2\chi_{xz}$$

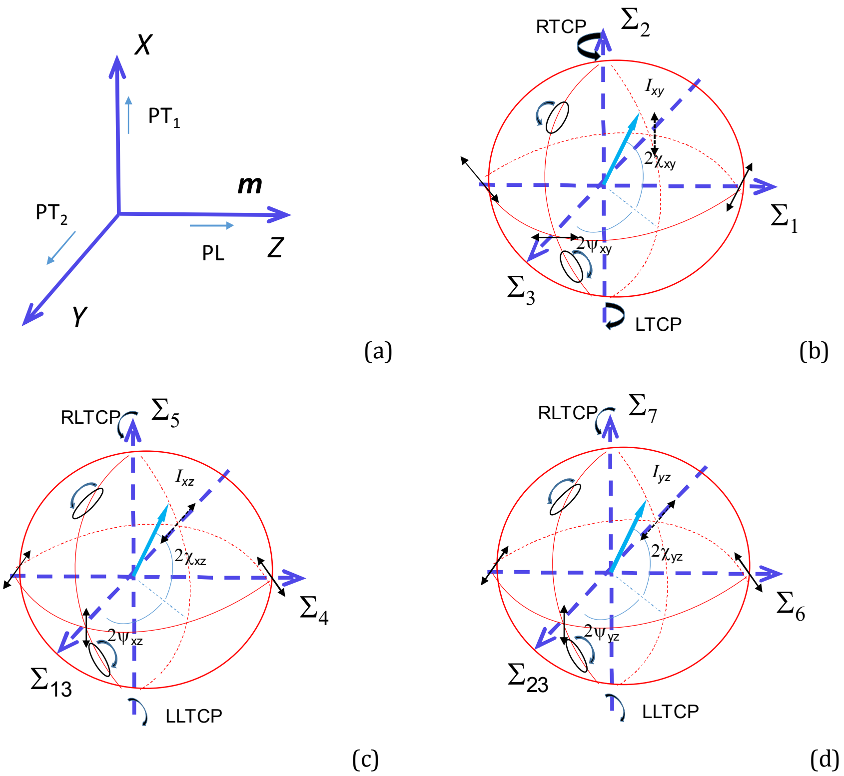

Fig. 1. Schematic view of laboratory coordinate system and direction of propagation and polarization of PT and PL AWs (a); $\Sigma_1\Sigma_2\Sigma_3$ (b), $\Sigma_4\Sigma_5\Sigma_{13}$ (c), and $\Sigma_6\Sigma_7\Sigma_{23}$ (d) Poincare spheres. RTCP and LTCP correspond to the right-handed and left-handed transverse circular polarization. RLTCP and LLTCP correspond to the right-handed longitudinally-transverse circular polarization and left-handed longitudinally-transverse circular polarization. The double arrows depict the direction of the linear polarization, while circular or elliptical arrows correspond to the direction of circular or elliptical polarization.

The azimuth (ψ_{xz}) and ellipticity angle (χ_{xz}) can be written as:

$$\psi_{xz} = \frac{1}{2}\mathrm{a}\tan\frac{\Sigma_4}{\Sigma_{xz}} = \frac{1}{2}\mathrm{a}\tan\left(\frac{2u_1u_3\cos\delta'}{u_1^2 - u_3^2}\right)$$
and
$$\chi_{xz} = \frac{1}{2}\mathrm{a}\sin\frac{\Sigma_5}{\Sigma_{0xz}} = \frac{1}{2}\mathrm{a}\sin\left(\frac{2u_1u_3\cos\delta'}{u_1^2 + u_3^2}\right), \qquad (17)$$

where δ' is the phase difference between X and Z components of AW (Fig. 1c). Finally, for the polarization that belong to the YZ plane (Fig. 1d) we have:

$$I_{yz}^2 = \Sigma_{yz}^2 + \Sigma_6^2 + \Sigma_7^2 = \left(u_2^2 + u_3^2\right)^2 = \Sigma_{0yz}^2$$
and
$$\Sigma_6 = I_{yz}\sin 2\psi_{yz}\cos 2\chi_{yz}, \quad \Sigma_7 = I_{yz}\sin 2\chi_{yz},$$
$$\Sigma_{yz} = I_{yz}\cos 2\psi_{yz}\cos 2\chi_{yz}, \qquad (18)$$

where azimuth and ellipticity are written as

$$\psi_{yz} = \frac{1}{2}\mathrm{a}\tan\frac{\Sigma_6}{\Sigma_{yz}} = \frac{1}{2}\mathrm{a}\tan\left(\frac{2u_2u_3\cos(\delta-\delta')}{u_2^2 - u_3^2}\right)$$
and
$$\chi_{yz} = \frac{1}{2}\mathrm{a}\sin\frac{\Sigma_7}{\Sigma_{0yz}} = \frac{1}{2}\mathrm{a}\sin\left(\frac{2u_2u_3\cos(\delta-\delta')}{u_2^2 + u_3^2}\right). \qquad (19)$$

Thus, one deals with three Poincaré spheres in which the equator corresponds to the linear polarization, while the poles correspond to the circular polarization. The $\Sigma_1\Sigma_2\Sigma_3$ sphere corresponds to the pure transverse polarizations of the AWs. This Poincaré sphere is the same as an optical one. On the other hand, the $\Sigma_4\Sigma_5\Sigma_{13}$ and $\Sigma_6\Sigma_7\Sigma_{23}$ spheres describe the QT and QL AWs. If χ_{xz} and χ_{yz} are equal to zero that correspond to the equator of these spheres the pure linear polarizations relates only to Σ_{13} and Σ_{23} axes. Besides, at the positive Σ_{13} and Σ_{23}, the polarization is purely longitudinal, whereas at the negative, it is purely transverse. On the $\pm\Sigma_4$ and $\pm\Sigma_6$ axes the QT (QL) polarizations lie, and the values of azimuth (χ_{xz} and χ_{yz}) are equal to the angle of nonorthogonality $\zeta = 45\,\mathrm{deg}$. At the shifting from the equator toward the poles, the ellipticity follows from zero to ±45 deg. At the same time, the ellipses of polarization lie in the XZ and YZ planes, and the ellipticity is longitudinally-transverse (Fig. 2a,b). And finally, at the reaching of poles, this type of elliptical polarization is transformed into the longitudinally-transverse circular polarization.

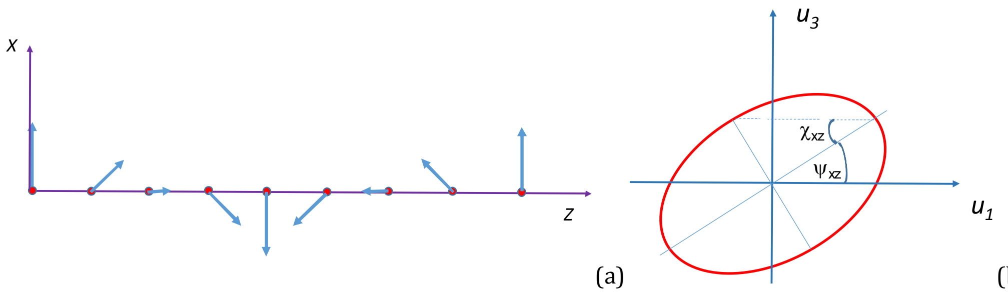

Fig. 2. Schematic view of the particles displacement under the action of longitudinally-transverse elliptical polarization of AW polarized in the XZ plane ($\psi_{xz} = 0$) (a) and its ellipse of polarization (b).

2.2. Qutrit triangle

In the above analysis, we have combined Σ_3 and Σ_8, having dispensed with the tool for analyzing the power difference between the two transverse and longitudinal components of AWs, i.e., between the basis states of phonon qutrits. For its restoration in our approach to AWs, let us cut the 8D Poincaré sphere along the 2D plane $\Sigma_3\Sigma_8$. As a result of such a cut, one obtains the equilateral qutrit triangle (Fig. 3). The vertices of this triangle correspond to three basis states of two transverse waves (phonons) and a longitudinal wave (phonon).

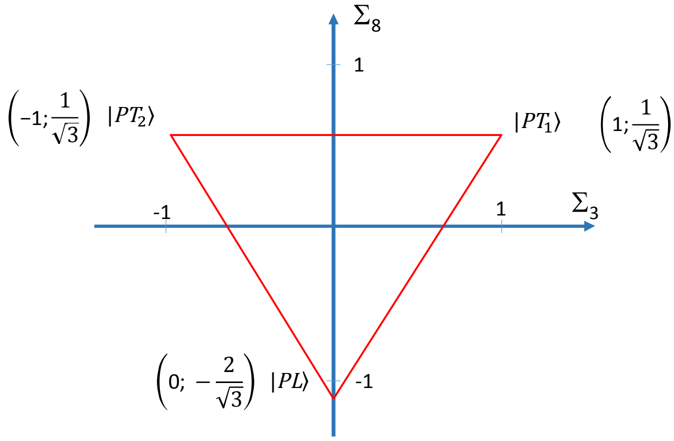

Fig. 3. A qutrit triangle that appears as a result of cutting the 8D Poincaré sphere by the 2D plane $\Sigma_3\Sigma_8$.

When the Σ_3 parameter is equal to zero, one reaches the energetic balance between transverse polarization components of the AWs. Otherwise, when $\Sigma_8 = 0$, the energetic balance between the sum of two transverse waves and the longitudinal wave is reached.

The interior region of this triangle can correspond to the mixed states. If the Stokes parameters Σ_1, Σ_2, Σ_4, Σ_5, Σ_6 and Σ_7 are equal to zero, the center of the qutrit triangle relates to a fully non-polarized acoustic field. However, in other cases, the interior region of the triangle corresponds to qutrit states. The qutrit wavefunction can be presented as

$$|\psi\rangle = \alpha|PT_1\rangle + \beta|PT_2\rangle + \gamma|PL\rangle, \quad |\alpha|^2 + |\beta|^2 + |\gamma|^2 = 1, \tag{20}$$

with the probability amplitudes $\alpha, \beta,$ and γ determined via Stokes parameters as:

$$\begin{aligned} \alpha &= \sqrt{\frac{1}{6}\left(2 + 3\Sigma_3 + \sqrt{3}\Sigma_8\right)}, \\ \beta &= \sqrt{\frac{1}{6}\left(2 - 3\Sigma_3 + \sqrt{3}\Sigma_8\right)}e^{i\delta} = \sqrt{\frac{1}{6}\left(2 - 3\Sigma_3 + \sqrt{3}\Sigma_8\right)}e^{ia\tan\frac{\Sigma_2}{\Sigma_1}}, \\ \gamma &= \sqrt{\frac{1}{3}\left(1 - \sqrt{3}\Sigma_8\right)}e^{i\delta'} = \sqrt{\frac{1}{3}\left(1 - \sqrt{3}\Sigma_8\right)}e^{ia\tan\frac{\Sigma_5}{\Sigma_4}}. \end{aligned} \tag{21}$$

When $\Sigma_2 = \Sigma_5 = \Sigma_7 = 0$, the center of the triangle $\Sigma_3 = \Sigma_8 = 0$, corresponds to the linear polarization of AW, with the displacement vector characterized by three equal directional cosines. In fact, this linearly polarized AW corresponds to the wave appearing in the conditions with three equal angles of non-orthogonality in the three coordinate planes. The wavefunction for such a state can be presented as

$$|\psi\rangle = \frac{1}{\sqrt{3}}\left(|PT_1\rangle + |PT_2\rangle + |PL\rangle\right). \tag{22}$$

When $\Sigma_2, \Sigma_5, \Sigma_7 \neq 0$, in the center of the triangle, one deals with circularly polarized AW, with the circle of displacement-vector rotation perpendicular to the direction of equal directional cosines. The respective wavefunctions are as follows:

$$|\psi\rangle = \frac{1}{\sqrt{3}}\left(|PT_1\rangle + e^{\pm i2\pi/3}|PT_2\rangle + e^{\pm i4\pi/3}|PL\rangle\right), \tag{23}$$

where the sign "–" and "+" correspond to the right and left handed circularly polarized state, respectively.

The lateral sides relate to the superposition of transverse-transverse PT_1-PT_2, as well as longitudinal-transverse PL-PT_1 and PL-PT_2 states. In fact, the states that lie on the lateral sides correspond to the AWs with a nonzero non-orthogonality angle. The superposition of the states yields the qubits:

$$|\psi_1\rangle = \frac{1}{\sqrt{2}}\left(|PT_1\rangle + |PT_2\rangle\right),$$
$$\text{when} \quad |\alpha| = |\beta| = 1/\sqrt{2}, \quad \gamma = 0, \quad \Sigma_8 = \frac{1}{\sqrt{3}}, \quad \Sigma_3 = 0, \quad \text{and} \quad \Sigma_2 = 0 \tag{24}$$

$$|\psi_2\rangle = \frac{1}{\sqrt{2}}\left(|PT_2\rangle + |PL\rangle\right),$$
$$\text{when} \quad |\gamma| = |\beta| = 1/\sqrt{2}, \quad \alpha = 0, \quad \Sigma_8 = -\frac{2+3\Sigma_3}{\sqrt{3}}, \quad \Sigma_3 = -1/2, \quad \text{and} \quad \Sigma_7 = 0 \tag{25}$$

$$|\psi_3\rangle = \frac{1}{\sqrt{2}}\left(|PL\rangle + |PT_1\rangle\right),$$
$$\text{when} \quad |\gamma| = |\alpha| = 1/\sqrt{2}, \quad \beta = 0, \quad \Sigma_8 = -\frac{2-3\Sigma_3}{\sqrt{3}}, \quad \Sigma_3 = 1/2 \quad \text{and} \quad \Sigma_5 = 0. \tag{26}$$

2.3. Jones vectors for AW

If we follow the above analysis, one can distinguish such types of AW polarizations: pure transverse and pure longitudinal linear polarizations; quasi-transverse and quasi-longitudinal linear polarizations; elliptical and circular transverse polarizations; and longitudinally-transverse elliptical and circular polarizations. Therefore, our next aim is to obtain the Jones vectors for all these AW polarization types.

Let us consider Eq. (12). As one can see, one deals with two phase differences instead of one, as it is peculiar for the two-component Jones vector. Therefore, for further analysis, we will consider cases in which the displacement vector belongs to one of the principal planes of the coordinate system. Otherwise, it is necessary to rotate the coordinate system to reach this condition. It should be noted that since three polarization vectors in acoustics are always orthogonal [12,13], it is possible to achieve the condition where two of them lie in the principal coordinate plane by rotating the coordinate system. Thus, in this case, e.g., for the displacement vector, which belongs to the *XZ* plane, we will use Eq. (12) in the form:

$$u = e^{-imZ} e^{i\delta_1} \begin{bmatrix} u_1 \\ 0 \\ u_3 e^{i\delta'} \end{bmatrix}, \tag{22}$$

or simply

$$u = \begin{bmatrix} u_1 \\ 0 \\ u_3 e^{i\delta'} \end{bmatrix}, \tag{23}$$

which represents the Jones vector for AWs, similar to that of electromagnetic waves [14].

Let us consider the PTs and PL AWs. PT_1 and PT_2 AWs are polarized parallel to the *X* and *Y* axes, respectively. The Jones vectors for these cases are collected in Table 1.

Table 1. Jones vectors for PTs and PL AWs.

Polarization state of AW	Jones vector	Eq. #
PT_1	$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$	(24)
PT_2	$\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$	(25)
PL	$\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$	(26)

As mentioned above in acoustics, one deals with longitudinal-transverse circular waves, which are characterized by transverse spin angular momentum (see e.g. [15] for surface oceanic waves). In this case, when the orthogonal components of the displacement vector are purely polarized, the Jones vector can be written as

$$\begin{bmatrix} 1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} e^{\pm i\pi/2} \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ \pm i \end{bmatrix}. \tag{27}$$

This Jones vector describes a longitudinal-transverse circularly polarized left-handed (sign "+") and right-handed (sign "–") AWs, consisting of the transverse (*X*-polarized) and longitudinal AWs with a phase shift of $\pm\pi/2$. The Jones vectors for circular waves consisting of transverse (*Y*-polarized) and longitudinal, as well as both transverse AWs, are as follows

$$\frac{1}{\sqrt{2}} \begin{bmatrix} 0 \\ 1 \\ \pm i \end{bmatrix} \text{ and } \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ \pm i \\ 0 \end{bmatrix}, \tag{28}$$

respectively. In the general case, the Jones vector for longitudinally-transverse elliptical polarization can be presented as

$$\begin{bmatrix} u_1 \\ 0 \\ \pm i u_3 \end{bmatrix}. \tag{29}$$

Let us consider quasi-transverse AW propagating in the *Z* direction with the displacement vector rotated at an angle of non-orthogonality ζ_2 relative to the *X* axis. Then two projections appear, namely on the *Z* and *X* axes, yielding the Jones vector

$$\begin{bmatrix} \cos\zeta_2 \\ 0 \\ e^{i\delta'}\sin\zeta_2 \end{bmatrix}. \tag{30}$$

If we remind that the center of a qutrit triangle at a certain condition may correspond to the linearly polarized state with the displacement vector being parallel to the direction with equal projections on the coordinate axes, the corresponding Jones vector can be written as:

$$\begin{bmatrix} \cos(\pi/2-\operatorname{atan}\cos\zeta_1) \\ \cos(\pi/2-\operatorname{atan}\cos\zeta_2) \\ \cos(\pi/2-\operatorname{atan}\cos\zeta_3) \end{bmatrix} = \frac{1}{\sqrt{3}}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \tag{31}$$

where ζ_1, ζ_2, and ζ_3 are the non-orthogonality angles in the *YZ*, *XZ*, and *XY* planes, respectively. One can consider the three-component Jones vector too, e.g., for the circularly polarized AW, for the case of energetic balance between three acoustic modes, the respective Jones vector is as follows:

$$V = \frac{1}{\sqrt{3}}\begin{bmatrix} 1 \\ e^{i2\pi/3} \\ e^{i4\pi/3} \end{bmatrix}. \tag{32}$$

In this case, the phases of the three components of the acoustic field are shifted by 120 deg, while the circle of displacement vector rotation is perpendicular to the direction of equal directional cosines. For the mentioned circularly polarized AW, the spin angular momentum can be determined as the imaginary part of the vector product $L = \operatorname{Im}[V \times V^*]$ [16] that yield

$$L = \begin{bmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix}, \tag{33}$$

and the length of this vector is equal to unity, describing pure circular polarization with the maximal spin angular momentum.

2.4. Jones matrices for acoustic elements

Let us introduce the simplest cases of Jones matrices for polarization acoustic elements, such as half-wave and quarter-wave plates, polarizers, etc. The peculiarity of Jones matrices in acoustics is that, instead of one matrix that is used in optics to describe the action of a certain optical element, in acoustics, one has to use three matrices since one deals with coupling between three eigen waves instead of two, as in optics. Besides, in acoustics, Jones matrices are 3×3 matrices since Jones vectors are three-component. Let us consider a few examples. The matrices for the linear polarizers, which can be represented e.g. by polarization prisms in analogy to, for example, Glan-Thompson prisms in optics, are as follows:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \text{ and } \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \tag{34}$$

for setting *X*, *Y*, and *Z* polarization, respectively. The matrices for the quarter-wave plate can be written as:

$$\begin{bmatrix}1 & 0 & 0\\0 & i & 0\\0 & 0 & 1\end{bmatrix}, \begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & i\end{bmatrix} \text{ and } \begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & i\end{bmatrix}, \quad (35)$$

for coupling between PT_1 and PT_2 AWs, PL and PT_1 AWs and PL, and PT_2 AWs, respectively. The quarter-wave plate in acoustics plays the same role as it does in optics. The difference is that, in acoustics, in addition to the creation of the ordinary transverse circularly-polarized AW, the action of the quarter-wave plate converts quasi-transverse (quasi-longitudinal) AW with an angle of non-orthogonality of 45 deg into a longitudinal-transverse circularly polarized AW and vice versa. The matrices of the half-wave plates are as follows:

$$\begin{bmatrix}1 & 0 & 0\\0 & -1 & 0\\0 & 0 & 1\end{bmatrix}, \begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & -1\end{bmatrix} \text{ and } \begin{bmatrix}-1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix}, \quad (36)$$

for coupling between PT_1 and PT_2 AWs, PL and PT_1 AWs and PL, and PT_2 AWs, respectively. These matrices convert polarization to the orthogonal one in *XY*, *XZ*, and *YZ* planes. The quarter- and a half-wave plates can be fabricated from anisotropic materials with a demanded phase difference between respective eigen waves. The matrices of the right and left circular polarizers can be presented, respectively, as:

$$\frac{1}{2}\begin{bmatrix}1 & -i & 0\\i & 1 & 0\\0 & 0 & 0\end{bmatrix}, \ \frac{1}{2}\begin{bmatrix}1 & i & 0\\-i & 1 & 0\\0 & 0 & 0\end{bmatrix}, \text{ for } XY \text{ plane}, \quad (37)$$

$$\frac{1}{2}\begin{bmatrix}1 & 0 & -i\\0 & 0 & 0\\i & 0 & 1\end{bmatrix}, \ \frac{1}{2}\begin{bmatrix}1 & 0 & i\\0 & 0 & 0\\-i & 0 & 1\end{bmatrix}, \text{ for } XZ \text{ plane}, \quad (38)$$

$$\frac{1}{2}\begin{bmatrix}0 & 0 & 0\\0 & 1 & -i\\0 & i & 1\end{bmatrix}, \ \frac{1}{2}\begin{bmatrix}0 & 0 & 0\\0 & 1 & i\\0 & -i & 1\end{bmatrix}, \text{ for } YZ \text{ plane}. \quad (39)$$

Such polarizers can convert unpolarized AW into circularly polarized AW and can be made from anisotropic materials, providing a $\pi/2$ phase difference between the components of AW.

3. Conclusions

In this work, we have obtained the coherence matrix and the Stokes parameters for the AW using a procedure analogous to that used to derive these characteristics for an optical wave with a longitudinal field component, which arises from projection onto the direction of the wavevector. The obtained Stokes parameters serve as the basis vectors of the 8D Poincaré sphere. To analyze the properties of this sphere, it has been intersected by three 3D hyperplanes and one 2D plane. As a result, three 3D spheres and one qutrit triangle have been obtained. The characteristics of these spheres and the triangle have been described in detail. As a result, one can distinguish the following types of AW polarizations: pure transverse and pure longitudinal linear polarizations; quasi-transverse and quasi-longitudinal linear polarizations; elliptical and circular transverse polarizations; and longitudinally-transverse elliptical and circular polarizations. For all these polarization states, the respective Jones vectors have been obtained. The Jones matrices for the simplest polarization acoustic elements have also been obtained.

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Authors contribution. Skab I. – coherence matrix and Stokes parameters derivation, Poincaré sphere and qutrit triangle obtaining, Kostyrko M. – Jones vectors and matrices obtaining, Vlokh R. – conceptualization, data analysis, writing of the paper draft, supervision.

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Анотація. *У цій роботі отримано матрицю когерентності та параметри Стокса для акустичних хвиль. Показано, що отримані параметри Стокса є базисними векторами 8D-сфери Пуанкаре. Для аналізу властивостей цієї сфери було зроблено її перетин трьома 3D-гіперплощинами та однією 2D-площиною. В результаті отримано три 3D-сфери Пуанкаре та один кутритний трикутник. Характеристики цих сфер та трикутника детально описані, а також виділено різні типи поляризацій акустичних хвиль. Для всіх цих станів поляризації отримано відповідні вектори Джонса та запропоновано матриці Джонса для найпростіших поляризаційних акустичних елементів.*

Ключові слова: *акустичні хвилі, сфера Пуанкаре, параметри Стокса, вектор Джонса, матриця Джонса*