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Optical Solitons for the Dispersive Concatenation Model with Kerr Law of Self–Phase Modulation by Lie Symmetry

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Abstract. This study delves into the realm of new optical solitons within the framework of the dispersive concatenation model, specifically focusing on Kerr law self-phase modulation. The research employs Lie Symmetry Analysis to transform the complex governing equations into ordinary differential equations (ODEs). These ODEs are then tackled using two distinct methodologies: the F-expansion method and a novel generalized method. Through these approaches, a broad spectrum of soliton solutions is successfully derived, showcasing the robustness and effectiveness of the proposed techniques. Additionally, the physical interpretations of these solutions are illustrated via 3D profile plots, offering profound insights into the intricate behavior of the solitons.

Keywords: optical solitons, new generalized method, concatenation model, Lie symmetry analysis, F-expansion method, power-law
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1. Introduction

In this paper, we explore a specialized concatenation model, initially proposed by Ankiewicz et al. in 2014 [1–3], which integrates several well-known equations from the field of photonics. Specifically, it combines the Lakshmanan-Porsezian-Daniel (LPD) model [4–6], the Sasa-Satsuma equation (SSE) [7–9], and the nonlinear Schrödinger equation (NLSE) [10–12]. This model has been extensively studied to retrieve soliton solutions, locate conservation laws, perform Painlevé analysis, and analyze various features such as magneto-optics, bifurcation analysis, and nonlinear chromatic dispersion (CD).

Further developments have led to studying the model in the context of birefringent fibers by integrating the Schrödinger-Hirota equation (SHE), the LPD model, and the fifth-order NLSE, encompassing fifth-order dispersion [13–15]. This enhanced model is commonly referred to as the dispersive concatenation model [14]. The current research focuses on locating soliton solutions within this model using two novel methods: the F-expansion method [16–18] and a new generalized method [19, 20].

Earlier methods for recovering soliton solutions for the dispersive concatenation model include the generalized sine-Gordon equation method, the projective Riccati equation approach, the Weierstrass-type Riccati equation expansion, and the enhanced Kudryashov method [11, 12, 14]. However, the F-expansion and new generalized methods stand out due to their effectiveness and simplicity. The Lie symmetry approach further enhances these methods by reducing the complexity of the partial differential equations (PDEs), making the integration process more convenient.

Key findings of this research include the successful retrieval of a wide range of optical solitons within the dispersive concatenation model. The F-expansion and new generalized methods facilitated this recovery, representing new and practical approaches to solving the problem. Applying the Lie symmetry approach to reduce ordinary differential equations (ODEs) significantly simplified the solution process. The study uncovers the essential parameter restrictions for soliton existence, naturally emerging through the process of soliton derivation.

These results notably advance the knowledge of optical solitons in the dispersive concatenation model and reveal potential applications in optical signal processing, nonlinear optics, optical communication, and the enhancement of soliton-based technologies. [11].

2. Governing equation

We will investigate the concatenation model in this paper, which can be described by the following expression:

$$\begin{aligned} & iq_t + aq_{xx} + b|q|^2 q - ic_1(\delta_1 q_{xxx} + \delta_2 |q|^2 q_x) \\ & + c_2\left(\delta_3 q_{xxxx} + \delta_4 |q|^2 q_{xx} + \delta_5 |q|^4 q + \delta_6 |q_x|^2 q + \delta_7 q_x^2 q^* + \delta_8 q_{xx}^* q^2\right) \\ & - ic_3\begin{pmatrix} \delta_9 q_{xxxxx} + \delta_{10}|q|^2 q_{xxx} + \delta_{11}|q|^4 q_x \\ + \delta_{12} q q_x q_{xx}^* + \delta_{13} q^* q_x q_{xx} + \delta_{14} q q_x^* q_{xx} + \delta_{15} q_x^2 q_x^* \end{pmatrix} = 0. \end{aligned} \tag{1}$$

Eq. (1) presents the dispersive concatenation model, incorporating the Kerr law of self-phase modulation (SPM). In this context, terms such as $q(x,t)$, $q_{xx\ldots}$, q^*, n, x, and t represent the wave profile, higher-order derivatives, conjugate terms, natural numbers, spatial variables, and temporal variables, respectively. The equation captures the intricate interplay between nonlinearity and dispersion, reflecting the rate of change of the pulse envelope, the impact of nonlinear CD (a), and the effects of nonlinear self-phase modulation (b).

The term c_1 encompasses the additional components of the SHE, which can be derived from the standard NLSE using the Lie transform [13, 15, 16]. The coefficients c_2 and c_3 are associated with the LPD model [9] and the fifth-order NLSE, respectively. The real-valued constant coefficients δ_1 and δ_2 originate from the SHE model, δ_3 to δ_8 are derived from the LPD model, and δ_9 to δ_{15} stem from the fifth-order NLSE. Adjusting these parameters and coefficients allows the equation to be tailored to different physical contexts. For instance, setting $c_2 = c_3 = 0$ converts the equation into the SSE, which accentuates higher-order dispersion effects. With both c_1 and c_2 set to zero, the equation simplifies to the fifth-order NLSE, concentrating on the essential features of soliton propagation. Removing c_1 and c_3 transforms the equation into the LPD model, focusing on dispersion and nonlinearity in soliton dynamics. When $c_1 = c_2 = c_3 = 0$, Eq. (1) simplifies to the standard NLSE.

In this paper, we initially obtain the reduced ordinary differential equation (ODE) of the specified system through the application of the Lie symmetry method. Subsequently, we reduced the system utilizing the F-expansion method and a new generalized approach to attain a wide array of exact solutions for the governing equation. Lastly, we illustrate several graphs of the solutions and provide their interpretations.

3. Lie symmetry analysis

By identifying transformations that leave PDEs invariant, the Lie symmetry method [13, 15, 16] provides an effective approach to solving them. The technique is extensively applied to find exact solutions for PDEs. This technique utilizes Lie group analysis to simplify the equations. The derivation of symmetry reductions for a system of equations starts with the characterization of $q(x,t)$. Subsequently, the system in Eq. (1) undergoes symmetry reduction through the application of Lie group analysis.

The function $q(x,t)$ is defined as follows:

$$q(x,t) = U(x,t)\mathrm{e}^{i(-kx+wt+\theta)}. \tag{2}$$

Here, θ, ω, and k correspond to the phase constant, frequency, and wave number of the soliton, respectively. Eq. (2) decomposes Eq. (1) into its real and imaginary components, as shown below

$$\begin{aligned}&(c_2\delta_3 - 5c_3\delta_9 k)U_{xxxx} + (10c_3\delta_9 k^3 + 6\delta_3 c_2 k^2 + 3\delta_1 c_1 k + a)U_{xx} - \big[c_2(\delta_4+\delta_8)\\ &+c_3k(3\delta_{10}+\delta_{12}+\delta_{13}-\delta_{14})\big]U^2U_{xx} + \big(c_3k(2\delta_{12}-2\delta_{13}-2\delta_{14}-\delta_{15}) + c_2(\delta_6+\delta_7)\big)UU_x^2\\ &+\big[k^3(c_2\delta_3 k + c_1\delta_1) - c_3\delta_9 k^5 + ak^2 + w\big]U + \big[c_3k^3(\delta_{10}+\delta_{12}+\delta_{13}-\delta_{14}-\delta_{15})\\ &+k\big(c_2k(\delta_4+\delta_7+\delta_8)+\delta_2c_1\big) + c_2\delta_6k^2 + b\big]U^3 + (c_2\delta_5 - c_3\delta_{11}k)U^5 \qquad = 0,\end{aligned} \tag{3}$$

$$\begin{aligned}&c_3\delta_9U_{xxxxx} + (10c_3\delta_9k^2 + c_3\delta_{10}U^2 + 4c_2\delta_3k + c_1\delta_1)U_{xxx} + c_3(\delta_{12}+\delta_{13}+\delta_{14})UU_xU_{xx}\\ &+(5c_3\delta_9k^3 + 4c_2\delta_3k^2 + 3c_1\delta_1k + 2a)kU_x + [2c_2k(\delta_4+\delta_7-\delta_8) + c_1\delta_2\\ &+c_3k^2(3\delta_{10}-\delta_{12}+3\delta_{13}-\delta_{14}-\delta_{15})]U^2U_x + c_3\delta_{15}U_x^3 + c_3\delta_{11}U^4U_x - U_t \qquad = 0.\end{aligned} \tag{4}$$

Assuming the Lie group of point transformations, as detailed below, is necessary to identify the symmetries of Eqs. (3-4):

$$\begin{aligned}x^* &= x + \epsilon\xi(x,t,U) + O(\epsilon^2),\\ t^* &= t + \epsilon\tau(x,t,U) + O(\epsilon^2),\\ U^* &= U + \epsilon\phi(x,t,U) + O(\epsilon^2),\end{aligned} \tag{5}$$

where ξ, τ and ϕ are the infinitesimal parameters [13, 15, 16]. The vector field for Eq. (5) is specified as:

$$V=\xi(x,t,U)\frac{\partial}{\partial x}+\tau(x,t,U)\frac{\partial}{\partial t}+\phi(x,t,U)\frac{\partial}{\partial U}. \tag{6}$$

The fifth prolongation formula [13, 15] of Eq. (6) for the Eqs. (3-4) is provided as:

$$\begin{aligned} pr^{(5)}V &= V+\phi^{t}\frac{\partial}{\partial U}+\phi^{x}\frac{\partial}{\partial U}+\phi^{xx}\frac{\partial}{\partial U_{xx}} \\ &+\phi^{xxx}\frac{\partial}{\partial U_{xxx}}+\phi^{xxxx}\frac{\partial}{\partial U_{xxxx}}+\phi^{xxxxx}\frac{\partial}{\partial U_{xxxxx}}. \end{aligned} \tag{7}$$

To determine the extended infinitesimals, denoted ϕ^t, ϕ^x, ϕ^{xx}, ϕ^{xxx}, ϕ^{xxxx}, and ϕ^{xxxxx}, we apply the invariance condition $pr^{(5)}V(\Delta)=0$ for $\Delta=0$ in the Eqs. (3-4). We derive a system of PDEs by substituting the infinitesimals and ensuring that the coefficients of the different derivative terms are zero. The resulting infinitesimals from solving this system are as follows:

$$\xi=C_1,\quad \tau=C_2,\quad \phi=0, \tag{8}$$

where C_1 and C_2 are arbitrary constants. Therefore, the infinitesimal generators span the Lie algebra of the Eqs. (3-4), specified as:

$$V_1=\frac{\partial}{\partial t},\quad V_2=\frac{\partial}{\partial x}. \tag{9}$$

The resultant characteristic equation, obtained by taking the linear combination of the vector field in Eq. (9), yields the following similarity variables when solved:

$$\begin{aligned} &q(x,t)=Q(\sigma)\mathrm{e}^{i\phi(x,t)}, \\ &\sigma(x,t)=h(x-\lambda t), \end{aligned} \tag{10}$$

where σ is identified as the wave variable, and $Q(\sigma)$ are the new dependent variable. Here, $\phi(x,t)$ indicates the phase part, $Q(\sigma)$ signifies the amplitude part, and λ and h represent the velocity and wave width of the soliton, respectively. Let us consider

$$\phi(x,t)=-kx+\omega t+\theta, \tag{11}$$

where θ, ω, and k correspond to the phase constant, frequency, and wave number of the soliton, respectively. Upon substituting Eq. (10) into Eq. (1), the real component is obtained as follows:

$$\begin{aligned} &(c_2\delta_3-5c_3\delta_9k)U_{\sigma\sigma\sigma\sigma}+((-3\delta_{10}-\delta_{12}-\delta_{13}+\delta_{14})c_3k-c_2(\delta_4+\delta_8))U^2U_{\sigma\sigma} \\ &+(c_3(-2\delta_{13}+2\delta_{12}-\delta_{15}-2\delta_{14})k+c_2(\delta_6+\delta_7))UU_{\sigma}^2+(10c_3k^3\delta_9-6c_2k^2\delta_3 \\ &-3c_1k\delta_1+a)U_{\sigma\sigma}-(c_3k\delta_{11}-c_2\delta_5)U^5-(c_3k^5\delta_9-c_2k^4\delta_3-c_1k^3\delta_1+ak^2+w)U \\ &-(-c_3(\delta_{12}+\delta_{13}-\delta_{14}-\delta_{15}+\delta_{10})k^3+c_2(\delta_7+\delta_8+\delta_4-\delta_6)k^2+c_1k\delta_2-b)U^3=0. \end{aligned} \tag{12}$$

The result for the imaginary component is:

$$\begin{aligned} &c_3\delta_9U_{\sigma\sigma\sigma\sigma\sigma}+(c_1\delta_1+4c_2\delta_3k-10c_3\delta_9k^2+c_3\delta_{10}U^2)U_{\sigma\sigma\sigma}+c_3(\delta_{12}+\delta_{13}+\delta_{14})UU_{\sigma}U_{\sigma\sigma} \\ &+c_3\delta_{11}U^4U_{\sigma}+c_3\delta_{15}U_{\sigma}^3+(2c_2k(\delta_4+\delta_7-\delta_8)-c_3k^2(3\delta_{13}-\delta_{12}-\delta_{14}-\delta_{15}+3\delta_{10}) \\ &+c_1\delta_2+5\delta_9k^4)U^2U_{\sigma}+(-4c_2k^3\delta_3-3c_1k^2\delta_1+2ak+\lambda)U_{\sigma}=0. \end{aligned} \tag{13}$$

By analyzing the imaginary component, the velocity is:

$$\lambda=-8c_2k^3\delta_3-2ak, \tag{14}$$

with the following additional parametric restrictions:

$$\begin{aligned} &\delta_9=\delta_{10}=\delta_{11}=\delta_{15} &&=0,\\ &2(\delta_4+\delta_7-\delta_8)c_2k+c_1\delta_2-4k^2c_3\delta_{13} &&=0,\\ &c_1\delta_1+4kc_3\delta_2 &&=0,\\ &\delta_{12}+\delta_{13}+\delta_{14} &&=0. \end{aligned} \tag{15}$$

By using the aforementioned restrictions, the Eqs. (12) and (13) are transformed into:

$$\begin{aligned} &c_2\delta_3U_{\sigma\sigma\sigma\sigma}+\left(\left(2c_3\delta_{14}k-c_2(\delta_8+\delta_4)\right)U^2-6c_2\delta_3k^2-3c_1\delta_1k+a\right)U_{\sigma\sigma}\\ &+\left(4c_3\delta_{12}k+c_2(\delta_6+\delta_7)\right)UU_{\sigma}^2+\left(c_2\delta_3k^4+c_1\delta_1k^3-ak^2-w\right)U\\ &+\left(-2c_3\delta_{14}k^3-c_2(\delta_4-\delta_6+\delta_7+\delta_8)k^2-c_1\delta_2k+b\right)U^3+c_2\delta_5U^5=0. \end{aligned} \tag{16}$$

4. Integration methodologies

4.1. A quick review of the F-expansion principle

A nonlinear PDE featuring two independent variables, x and t, is formulated as:

$$D(q,q_x,q_t,q_{xt},q_{xx},\ldots)=0. \tag{17}$$

Unknown functions D and q in the standard F-expansion principle include both linear and nonlinear higher-order derivatives. The following summary provides an outline of the main steps involved [7, 8].

Step 1: Through the application of the given traveling wave transformation, the nonlinear evolution equation (NLEE) (17) is simplified to a single-variable equation as described:

$$q(x,t)=Q(\sigma),\sigma=h(x-vt). \tag{18}$$

Eq.(18) transforms the NLEE described by Eq. (17) into a nonlinear ODE, involving wave velocity (v), wave variable (σ), and wave width (h), leading to the specified expression:

$$S(Q,-\mu vQ',\mu Q',\mu^2Q'',\ldots)=0. \tag{19}$$

Step 2: Let us consider the following form for the solution of Eq. (19):

$$Q(\sigma)=\sum_{i=0}^{N}\mu_iG^i(\sigma) \tag{20}$$

The equation in question involves real constants μ_i, a positive integer N, and a function $G^i(\sigma)$ that satisfies a first-order ODE. It is obtained by applying a balancing principle to the nonlinear and higher-order linear terms in Eq. (19), leading to a first-order ODE for $G(\sigma)$:

$$G'(\sigma)=\sqrt{XG^2(\sigma)+YG(\sigma)+Z}, \tag{21}$$

using the constants X, Y, and Z. The solutions of Eq. (21) are provided as follows:

$$\begin{aligned} &G(\sigma)=sn(\sigma)=\tanh(\sigma),\quad X=m^2,\quad Y=-(1+m^2),\quad Z=1,\quad m\to1,\\ &G(\sigma)=ns(\sigma)=\coth(\sigma),\quad X=1,\quad Y=-(1+m^2),\quad Z=m^2,\quad m\to1,\\ &G(\sigma)=sn(\sigma)=\tan(\sigma),\quad X=1-m^2,\quad Y=2-m^2,\quad Z=1,\quad m\to0,\\ &G(\sigma)=cs(\sigma)=\cot(\sigma),\quad X=1,\quad Y=2-m^2,\quad Z=1-m^2,\quad m\to0,\\ &G(\sigma)=cn(\sigma)=\mathrm{sech}(\sigma),\quad X=-m^2,\quad Y=2m^2-1,\quad Z=1-m^2,\quad m\to1,\\ &G(\sigma)=ds(\sigma)=\mathrm{csch}(\sigma),\quad X=1,\quad Y=2m^2-1,\quad Z=-m^2(1-m^2),\quad m\to1,\\ &G(\sigma)=nc(\sigma)=\csc(\sigma),\quad X=1,Y=-(1-m^2),R=m^2,m\to0, \end{aligned} \tag{22}$$

$$G(\sigma)=ns(\sigma)\pm ds(\sigma)=\coth(\sigma)\pm \operatorname{csch}(\sigma),$$
$$X=\frac{1}{4},\quad Y=\frac{(m^2-2)}{2},\quad Z=\frac{m^2}{4},\quad m\to 1,$$
$$G(\sigma)=sn(\sigma)\pm i\mathrm{cn}(\sigma)=\csc(\sigma)\pm\cot(\sigma),$$
$$X=\frac{1}{4},\quad Y=\frac{1-2m^2}{2},\quad Z=\frac{1}{4},\quad m\to 1, \tag{22 (continued)}$$
$$G(\sigma)=ns(\sigma)\pm cs(\sigma)=\csc(\sigma)\pm\cot(\sigma),$$
$$X=\frac{1}{4},\quad Y=\frac{1-2m^2}{2},\quad Z=\frac{1}{4},\quad m\to 0,$$
$$G(\sigma)=nc(\sigma)\pm ics(\sigma)=\sec(\sigma)\pm i\tan(\sigma),$$
$$X=\frac{1-m^2}{4},\quad Y=\frac{1+m^2}{2},\quad Z=\frac{1-m^2}{4},\quad m\to 0.$$

Step 3: This method involves placing Eqs. (20)-(21) into Eq. (19) to derive the strategic equations, which yield essential results when solved. Furthermore, inserting these significant outcomes along with Eq. (22) into Eq. (20) allows for the soliton solutions.

4.2. Overview of the new generalized method

Assume the following form for the nonlinear PDE:

$$\Xi(\eta,\eta_x,\eta_t,\eta_{xx},\eta_{xt},\eta_{tt},\ldots)=0, \tag{23}$$

where Ξ signifies a polynomial that incorporates the unknown function $\eta=\eta(x,t)$, its partial derivatives, and nonlinear terms.

The steps for the new generalized method [11, 12, 14] to solve these equations are as follows:

Step 1: Use the transformation to convert the PDE (23) into an ODE:

$$\eta(x,t)=\Lambda(\xi),\quad \xi=kx-vt, \tag{24}$$

where κ and v are constants to be defined. The substitution of Eq. (24) into the original PDE transforms it into a nonlinear ODE:

$$P(\Lambda,\Lambda',\Lambda'',\Lambda''',\ldots)=0. \tag{25}$$

Step 2: Suppose the solution of Eq. (25) is expressed as:

$$\Lambda(\xi)=\alpha_0+\sum_{i=0}^{N}\frac{\alpha_i+\beta_i\left(\psi'(\xi)\right)^i}{\left(\psi(\xi)\right)^i} \tag{26}$$

where $\alpha_0,\alpha_i,\beta_i$ (for $i=1,2,\ldots,N$) are constants to be determined subsequently, and the function $\psi(\xi)$ adheres to the relation:

$$\left[\psi'(\xi)\right]^2=\left[\psi(\xi)^2-\rho\right]\ln(d)^2, \tag{27}$$

in addition to:

$$\psi^{(n)}(\xi)=\begin{cases}\psi(\xi)\ln(d)^n, & n \text{ is even},\\ \psi'(\xi)\ln(d)^{n-1}, & n \text{ is odd},\end{cases}\quad n\ge 2.` \tag{28}$$

The form given below is considered as the solution for Eq. (27):

$$\psi(\xi)=k\ln(d)d^{\xi}+\frac{\rho}{4k\ln(d)d^{\xi}}, \tag{29}$$

where ρ and k represent arbitrary constants, also, d is a new independent variable.

Step 3: The value of the positive integer N in Eq. (26) can be determined using the balancing principle by balancing the nonlinear term with the highest-order derivative term in Eq. (25).

Step 4: To derive exact solutions for Eq. (23), substitute Eqs. (26) and (27), as well as their derivatives, into Eq. (25). This process generates a polynomial in $\frac{1}{\psi(\xi)}\left(\frac{\psi'(\xi)}{\psi(\xi)}\right)$. By collecting all terms with the same powers and setting them to zero, we determine the values of the unknown parameters $\kappa,\nu,\rho,k,\alpha_0,\alpha_i,\beta_i$ (for $i=1,2,\ldots,N$). Solving this system of algebraic equations will yield the parameter values required for the exact solutions to Eq. (23).

5. New optical solitons

5.1. Implementation of F-expansion method

Incorporating the transformation in Eq. (18) into Eq. (1), the real part emerges as:

$$\begin{aligned}&c_2h^4\delta_3U_{\sigma\sigma\sigma\sigma}+\left(\left(2c_3k\delta_{14}+c_2(\delta_4+\delta_8)\right)U^2-6c_2k^2\delta_3-3c_1k\delta_1+a\right)h^2U_{\sigma\sigma}\\&+\left(c_2(\delta_6+\delta_7)\right)h^2UU_\sigma^2+\left(4c_3k\delta_{12}+c_2k^4\delta_3+c_1k^3\delta_1-ak^2-w\right)U\\&+\left(-2c_3k^3\delta_{14}-c_2(\delta_4-\delta_6+\delta_8+\delta_7)k^2-c_1k\delta_2+b\right)U^3+c_2\delta_5U^5=0,\end{aligned} \tag{30}$$

with the wave velocity defined by:

$$v=-8c_2k^3\delta_3-2ak. \tag{31}$$

With the balancing principle applied from Eq. (30), we ascertain $N=1$. Thus, Eq. (20) becomes:

$$Q(\sigma)=a_0+a_1G(\sigma). \tag{32}$$

With Eq. (21) and Eq. (32) substituted into Eq. (30), a set of polynomials containing $G(\sigma)$ emerges. Setting all the coefficients to zero, we find:

$$\begin{aligned}R_1&=\begin{bmatrix}\left[\left(4c_3k\delta_{12}+c_2(\delta_6+\delta_7)\right)h^2s_1^2R+c_2\delta_5s_0^4+c_2k^4\delta_3+c_1k^3\delta_1\right.\\+\left(-2c_3k^3\delta_{14}-c_2(\delta_4-\delta_6+\delta_8+\delta_7)k^2-c_1k\delta_2+b\right)s_0^2-ak^2-w\end{bmatrix}s_0,\\R_2&=s_1\left(12PR+Q^2\right)c_2h^4\delta_3+\\&\quad\left[\left(2c_3k\delta_{14}+c_2(\delta_4+\delta_8)\right)s_0^2-6c_2k^2\delta_3-3c_1k\delta_1+a\right]h^2s_1Q\\&\quad+\begin{bmatrix}\left(4c_3k\delta_{12}+c_2(\delta_6+\delta_7)\right)h^2s_1^2R+c_2\delta_5s_0^4+c_2k^4\delta_3+c_1k^3\delta_1\\+\left(-2c_3k^3\delta_{14}-c_2(\delta_4-\delta_6+\delta_8+\delta_7)k^2-c_1k\delta_2+b\right)s_0^2-ak^2-w\end{bmatrix}s_1\\&\quad+\left[4c_2\delta_5s_0^3s_1+2\left(-2c_3k^3\delta_{14}-c_2(\delta_4-\delta_6+\delta_8+\delta_7)k^2-c_1k\delta_2+b\right)s_0s_1\right]s_0,\\R_3&=2\left(2c_3k\delta_{14}+c_2(\delta_4+\delta_8)\right)s_0s_1^2h^2Q\\&\quad+\left[4c_2\delta_5s_0^3s_1+2(-2c_3k^3\delta_{14}-c_2(\delta_4-\delta_6+\delta_8+\delta_7)k^2-c_1k\delta_2+b)s_0s_1\right]s_1\\&\quad+\begin{bmatrix}\left(4c_3k\delta_{12}+c_2(\delta_6+\delta_7)\right)h^2s_1^2Q+6c_2\delta_5s_0^2s_1^2\\+\left(-2c_3k^3\delta_{14}-c_2(\delta_4-\delta_6+\delta_8+\delta_7)k^2-c_1k\delta_2+b\right)s_1^2\end{bmatrix}s_0,\end{aligned} \tag{33}$$

$$\begin{aligned}R_4 &= 20s_1PQc_2h^4\delta_3 + 2\begin{pmatrix}\left(2c_3k\delta_{14} + c_2\left(\delta_4 + \delta_8\right)\right)s_0^2\\ -6c_2k^2\delta_3 - 3c_1k\delta_1 + a\end{pmatrix}h^2s_1P\\ &\quad + \left(2c_3k\delta_{14} + c_2\left(\delta_4 + \delta_8\right)\right)s_1^3h^2Q\\ &\quad + \begin{bmatrix}\left(4c_3k\delta_{12} + c_2\left(\delta_6 + \delta_7\right)\right)h^2s_1^2Q + 6c_2\delta_5s_0^2s_1^2\\ + \left(-2c_3k^3\delta_{14} - c_2\left(\delta_4 - \delta_6 + \delta_8 + \delta_7\right)k^2 - c_1k\delta_2 + b\right)s_1^2\end{bmatrix}s_1\\ &\quad + 4c_2\delta_5s_0^2s_1^3,\\ R_5 &= 4\left(2c_3k\delta_{14} + c_2\left(\delta_4 + \delta_8\right)\right)s_0s_1^2h^2P + 4c_2\delta_5s_0s_1^4\\ &\quad + \left(\left(4c_3k\delta_{12} + c_2\left(\delta_6 + \delta_7\right)\right)h^2s_1^2P + c_2\delta_5s_1^4\right)s_0,\\ G_6 &= 24s_1P^2c_2h^4\delta_3 + 2\left(2c_3k\delta_{14} + c_2\left(\delta_4 + \delta_8\right)\right)s_1^3h^2P\\ &\quad + \left(\left(4c_3k\delta_{12} + c_2\left(\delta_6 + \delta_7\right)\right)h^2s_1^2P + c_2\delta_5s_1^4\right)s_1.\end{aligned} \tag{33 (continued)}$$

We solve the system of algebraic equations and arrive at the solutions:

Case 1: $X = m^2$, $Y = -\left(1 + m^2\right)$, $Z = 1$, $m = 1$.

$$\begin{aligned}&s_0 = s_1,\quad \omega = \frac{20}{3}c_2h^2k^2\delta_3 - 64c_2h^4\delta_3 - \frac{2}{3}ak^2 - c_2k^4\delta_3, \delta_1 = \frac{20c_2h^2\delta_3 - 6c_2k^2\delta_3 + a}{3c_1k},\\ &\delta_5 = \frac{h^2\left(4c_3k\delta_{12}s_1^2 + 48c_2h^2\delta_3 + c_2\delta_6s_1^2 + c_2\delta_7s_1^2\right)}{3c_2s_1^4},\\ &\delta_2 = \frac{\begin{bmatrix}16c_3h^2k\delta_{12}s_1^2 + 8c_3k^3\delta_{12}s_1^2 + 240c_2h^4\delta_3 + 60c_2h^2k^2\delta_3 + 4c_2h^2\delta_6s_1^2 +\\ 4c_2h^2\delta_7s_1^2 + 5c_2k^2\delta_6s_1^2 - c_2k^2\delta_7s_1^2 + 3bs_1^2\end{bmatrix}}{3c_1ks_1^2},\\ &\delta_4 = -\frac{8c_3k\delta_{12}s_1^2 + 6c_3k\delta_{14}s_1^2 + 60c_2h^2\delta_3 + 2c_2\delta_6s_1^2 + 2c_2\delta_7s_1^2 + 3c_2\delta_8s_1^2}{3c_2s_1^2}.\end{aligned} \tag{34}$$

The dark soliton is found by substituting the above parameters into Eq. (18), which derives the solution for Eq. (30):

$$q\left(x,t\right) = s_1\left(\tanh\left(h\left(x - \nu t\right)\right) + 1\right)\mathrm{e}^{i\left(-kx + \omega t + \theta\right)}. \tag{35}$$

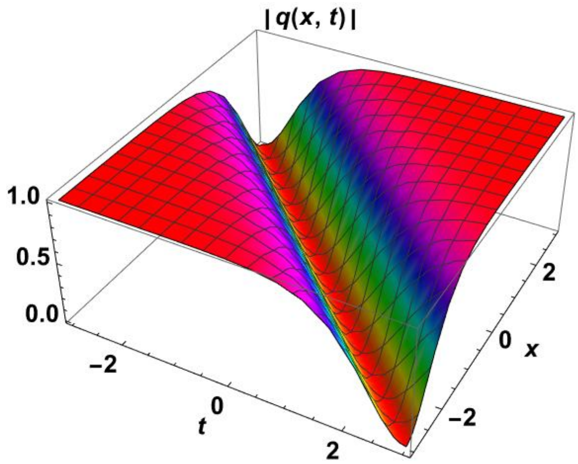

Fig. 1. Dark soliton's waveform, given by Eq. (35).

Case 2: $m = 1$, $Z = m^2$, $Y = -\left(1 + m^2\right)$, $X = 1$.

$$s_0 = s_1, \quad \omega = \frac{20}{3}c_2h^2k^2\delta_3 - 64c_2h^4\delta_3 - \frac{2}{3}ak^2 - c_2k^4\delta_3, \quad \delta_1 = \frac{20c_2h^2\delta_3 - 6c_2k^2\delta_3 + a}{3c_1k},$$
$$\delta_5 = \frac{h^2\left(4c_3k\delta_{12}s_1^2 + 48c_2h^2\delta_3 + c_2\delta_6 s_1^2 + c_2\delta_7 s_1^2\right)}{3c_2s_1^4},$$
$$\delta_2 = \frac{\begin{bmatrix}16c_3h^2k\delta_{12}s_1^2 + 8c_3k^3\delta_{12}s_1^2 + 240c_2h^4\delta_3 + 60c_2h^2k^2\delta_3 \\ +4c_2h^2\delta_6s_1^2 + 4c_2h^2\delta_7s_1^2 + 5c_2k^2\delta_6s_1^2 - c_2k^2\delta_7s_1^2 + 3bs_1^2\end{bmatrix}}{3c_1ks_1^2}, \tag{36}$$
$$\delta_4 = -\frac{8c_3k\delta_{12}s_1^2 + 6c_3k\delta_{14}s_1^2 + 60c_2h^2\delta_3 + 2c_2\delta_6s_1^2 + 2c_2\delta_7s_1^2 + 3c_2\delta_8s_1^2}{3c_2s_1^2}.$$

The singular soliton is derived by plugging the above parameters into Eq. (18), yielding the solution for Eq. (30):

$$q(x,t) = s_1\left(\coth\left(h(x - vt)\right) + 1\right)e^{i(-kx+\omega t+\theta)}. \tag{37}$$

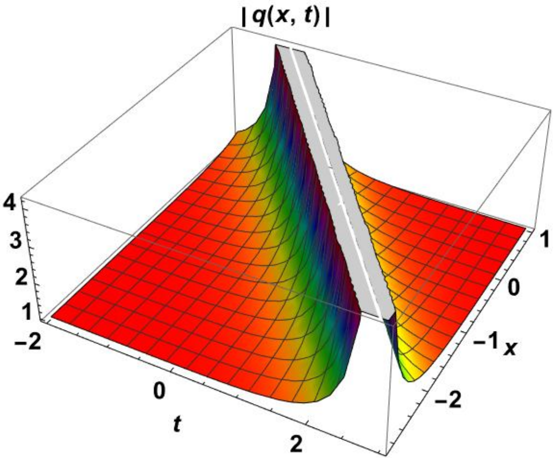

Fig. 2. Singular soliton's waveform given by Eq. (37).

Case 3: $m = 0$, $Z = 1$, $Y = 2 - m^2$, $X = 1 - m^2$.

$$\omega = -\frac{36c_2h^4\delta_3s_0^2 + 36c_2h^4\delta_3s_1^2 - 12c_2h^2k^2\delta_3s_0^2 - 8c_2h^2k^2\delta_3s_1^2 + 3c_2k^4\delta_3s_1^2 + 2ak^2s_1^2}{3s_1^2},$$
$$\delta_1 = \frac{12c_2h^2\delta_3s_0^2 + 8c_2h^2\delta_3s_1^2 - 6c_2k^2\delta_3s_1^2 + as_1^2}{3c_1ks_1^2}, \quad \delta_5 = \frac{12h^4\delta_3}{s_1^2\left(s_0^2 + s_1^2\right)},$$
$$\delta_2 = \frac{-4c_3k^3\delta_{12}s_0^2s_1^2 - 4c_3k^3\delta_{12}s_1^4 + 24c_2h^4\delta_3s_0^2 - 24c_2h^4\delta_3s_1^2}{c_2s_1^2\left(s_0^2 + s_1^2\right)}$$
$$+\frac{-60c_2h^2k^2\delta_3s_0^2 - 2c_2k^2\delta_7s_0^2s_1^2 - 2c_2k^2\delta_7s_1^4 + bs_0^2s_1^2 + bs_1^4}{s_1^2\left(s_0^2 + s_1^2\right)c_1k}, \tag{38}$$
$$\delta_4 = \frac{-2c_3k\delta_{14}s_0^2s_1^2 - 2c_3k\delta_{14}s_1^4 + 12c_2h^2\delta_3s_0^2 - 12c_2h^2\delta_3s_1^2 - c_2\delta_8s_0^2s_1^2 - c_2\delta_8s_1^4}{s_1^2\left(s_0^2 + s_1^2\right)c_1k}$$
$$\delta_6 = -\frac{4c_3k\delta_{12}s_0^2s_1^2 + 4c_3k\delta_{12}s_1^4 + 48c_2h^2\delta_3s_0^2 + 12c_2h^2\delta_3s_1^2 + c_2\delta_7s_0^2s_1^2 + c_2\delta_7s_1^4}{c_2s_1^2\left(s_0^2 + s_1^2\right)}.$$

The singular periodic solution emerges when deriving the solution of Eq. (30) from Eq. (18) with the parameters mentioned above:

$$q(x,t) = s_0 + s_1\tan\left(h(x - vt)\right)e^{i(-kx+\omega t+\theta)}. \tag{39}$$

Case 4: $X=-m^2, Y=2m^2-1, Z=1-m^2, m=1.$

$$s_0=0,$$
$$\omega=c_2h^4\delta_3-6c_2h^2k^2\delta_3+c_2k^4\delta_3-3c_1h^2k\delta_1+c_1k^3\delta_1+ah^2-ak^2,$$
$$\delta_2=-\frac{\left[\begin{array}{l}-4c_3h^2k\delta_{12}s_1^2-2c_3h^2k\delta_{14}s_1^2+2c_3k^3\delta_{14}s_1^2+20c_2h^4\delta_3-12c_2h^2k^2\delta_3\\-c_2h^2\delta_4s_1^2-c_2h^2\delta_6s_1^2-c_2h^2\delta_7s_1^2-c_2h^2\delta_8s_1^2+c_2k^2\delta_4s_1^2-c_2k^2\delta_6s_1^2\\+c_2k^2\delta_7s_1^2+c_2k^2\delta_8s_1^2-6c_1h^2k\delta_1+2ah^2-bs_1^2\end{array}\right]}{c_1ks_1^2}, \tag{40}$$
$$\delta_5=-\frac{h^2\left(-4c_3k\delta_{12}s_1^2-4c_3k\delta_{14}s_1^2+24c_2h^2\delta_3-2c_2\delta_4s_1^2-c_2\delta_6s_1^2-c_2\delta_7s_1^2-2c_2\delta_8s_1^2\right)}{c_2s_1^4}.$$

The bright soliton is obtained by applying the above parameters to Eq. (18) to derive the solution for Eq. (30):

$$q(x,t)=s_1\operatorname{sech}\left(h(x-vt)\right)e^{i(-kx+\omega t+\theta)}. \tag{41}$$

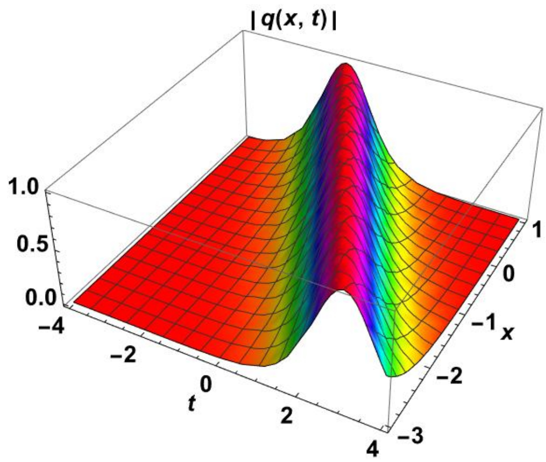

Fig. 3: Bright soliton's waveform [see Eq. (41)].

Case 5: $m=1$, $Z=(m^2)/4$, $Y=(m^2-2)/2$, $X=1/4$.

$$s_0=s_1,\quad \omega=-c_2k^4\delta_3+\frac{5}{3}c_2h^2k^2\delta_3-\frac{2}{3}ak^2-4c_2h^4\delta_3,\quad \delta_1=\frac{5c_2h^2\delta_3-6c_2k^2\delta_3+a}{3c_1k},$$
$$\delta_5=\frac{h^2\left(4c_3k\delta_{12}s_1^2+12c_2h^2\delta_3+c_2\delta_6s_1^2+c_2\delta_7s_1^2\right)}{12c_2s_1^4},$$
$$\delta_2=\frac{\left[\begin{array}{l}4c_3h^2k\delta_{12}s_1^2+8c_3k^3\delta_{12}s_1^2+15c_2h^4\delta_3+15c_2h^2k^2\delta_3+c_2h^2\delta_6s_1^2\\+c_2h^2\delta_7s_1^2+5c_2k^2\delta_6s_1^2-c_2k^2\delta_7s_1^2+3bs_1^2\end{array}\right]}{3c_1ks_1^2}, \tag{42}$$
$$\delta_4=-\frac{8c_3k\delta_{12}s_1^2+6c_3k\delta_{14}s_1^2+15c_2h^2\delta_3+2c_2\delta_6s_1^2+2c_2\delta_7s_1^2+3c_2\delta_8s_1^2}{3c_2s_1^2}.$$

To derive the solution for Eq. (30), one needs to insert the given parameters into Eq. (18), which then results in the straddled singular–singular soliton:

$$q(x,t)=s_1\left(1+\coth\left(h(x-vt)\right)+\operatorname{csch}\left(h(x-vt)\right)\right)e^{i(-kx+\omega t+\theta)}. \tag{43}$$

Case 6: $m=0$, $Z=\left(1-m^2\right)/4$, $Y=\left(1+m^2\right)/2$, $X=\left(1-m^2\right)/4$.

$$
\begin{gathered}
s_0=0,\quad \omega=c_3h^2ks_1^2\delta_{12}+c_2h^4\delta_3-3c_2h^2k^2\delta_3+\frac{1}{4}c_2h^2s_1^2\delta_6+\frac{1}{4}c_2h^2s_1^2\delta_7\\
+c_2k^4\delta_3-\frac{3}{2}c_1h^2k\delta_1+c_1k^3\delta_1+\frac{1}{2}ah^2-ak^2,\\
\delta_2=\frac{\begin{bmatrix}4c_3h^2ks_1^2\delta_{12}+2c_3h^2k\delta_{14}s_1^2-4c_3k^3\delta_{14}s_1^2+5c_2h^4\delta_3-6c_2h^2k^2\delta_3\\+c_2h^2\delta_4s_1^2+c_2h^2s_1^2\delta_6+c_2h^2s_1^2\delta_7+c_2h^2\delta_8s_1^2-2c_2k^2\delta_4s_1^2+2c_2k^2\delta_6s_1^2\\-2c_2k^2\delta_7s_1^2-2c_2k^2\delta_8s_1^2-3c_1h^2k\delta_1+ah^2+2bs_1^2\end{bmatrix}}{2c_1ks_1^2},\\
\delta_5=-\frac{h^2\left(4c_3k\delta_{12}s_1^2+4c_3k\delta_{14}s_1^2+6c_2h^2\delta_3+2c_2\delta_4s_1^2+c_2\delta_6s_1^2+c_2\delta_7s_1^2+2c_2\delta_8s_1^2\right)}{4c_2s_1^4}.
\end{gathered}
\tag{44}
$$

The solution to Eq. (30) can be found by applying the parameters from Eq. (18), leading to the singular periodic solution:

$$
q(x,t)=s_1\left(\tan\left(h(x-vt)\right)+\sec\left(h(x-vt)\right)\right)e^{i(-kx+\omega t+\theta)}. \tag{45}
$$

Case 7: $X=m^4-m^2, Y=2m^2-1, Z=1, m=0$.

$$
\begin{gathered}
s_0=0,\quad \delta_5=0,\ \omega=4c_3h^2ks_1^2\delta_{12}+c_2h^4\delta_3+6c_2h^2k^2\delta_3+c_2h^2s_1^2\delta_6,\\
+c_2h^2s_1^2\delta_7+c_2k^4\delta_3+3c_1h^2k\delta_1+c_1k^3\delta_1-ah^2-ak^2,\\
\delta_2=\frac{\begin{bmatrix}-4c_3h^2k\delta_{12}-2c_3h^2k\delta_{14}-2c_3k^3\delta_{14}-c_2h^2\delta_4-c_2h^2\delta_6\\-c_2h^2\delta_7-c_2h^2\delta_8-c_2k^2\delta_4+c_2k^2\delta_6-c_2k^2\delta_7-c_2k^2\delta_8+b\end{bmatrix}}{c_1k}.
\end{gathered}
\tag{46}
$$

By inserting the specified parameters into Eq. (18), the solution for Eq. (30) is derived, which gives us the singular periodic solution:

$$
q(x,t)=s_1\sin\left(h(x-vt)\right)e^{i(-kx+\omega t+\theta)}. \tag{47}
$$

The solutions in Eqs. (39), (45), and (47) differ in their physical behavior due to the distinct forms of their amplitude functions. Eq. (39) features a tangent function, which is singular at regular intervals. It represents a periodic wave with sharp intensity peaks over a constant background. Such profiles may correspond to localized wave trains exhibiting periodic blow-up. Eq. (45) combines tangent and secant terms, resulting in stronger and more asymmetric singularities. The solution indicates steeper intensity growth and may describe more abrupt nonlinear localization. Unlike the others, Eq. (47) is smooth and bounded. It models a regular periodic wave without singularities, corresponding to stable, repeating pulse trains in a nonlinear medium. Thus, Eqs. (39) and (45) describe singular periodic solitons with varying sharpness, while Eq. (47) gives a regular, non-singular periodic solution. These distinctions reflect different types of nonlinear wave behavior.

5.2. Implementation of the new generalized technique

By applying the technique from Section 3.2 to Eq. 24, we obtain the solutions to Eq. (1) in this section, which leads to the model reduction described by:

$$
\begin{gathered}
\kappa^4c_2\delta_3U_{\xi\xi\xi\xi}+\kappa^2\left(\left(2c_3\delta_{14}k+c_2\left(\delta_4+\delta_8\right)\right)U^2-6\delta_3k^2c_2-3\delta_1kc_1+a\right)U_{\xi\xi}\\
\kappa^2\left(4c_3\delta_{12}k+c_2\left(\delta_6+\delta_7\right)\right)UU_\xi^2+\left(c_2k^4\delta_3+c_1k^3\delta_1-ak^2-w\right)U\\
-\left(2c_3\delta_{14}k^3+c_2\left(\delta_4-\delta_6+\delta_7+\delta_8\right)k^2+c_1k\delta_2-b\right)U^3+U^5c_2\delta_5=0.
\end{gathered}
\tag{48}
$$

The positive integer N is derived using the balance principle. Balancing $U_{\xi\xi\xi\xi}$ with U^5 in Eq. (48) gives $N=1$. Therefore, we have:

$$U(\xi)=\alpha_0+\frac{\alpha_1+\beta_1\psi'(\xi)}{\psi(\xi)}, \tag{49}$$

as a solution obtained from Eq. (48). After substituting Eq. (49) into Eq. (48) and including Eqs. (27) and (28), the resulting polynomial is $\frac{1}{\psi(\xi)}\left(\frac{\psi'(\xi)}{\psi(\xi)}\right)$. Collecting all terms with the same powers and setting them to zero yields an over-determined system of algebraic equations. We solve the system and obtain the results given below:

Result 1:

$$\begin{gathered}\alpha_0=\frac{\sqrt{5}\sqrt{\frac{1}{\rho}}\alpha_1}{4},\quad \beta_1=0,\quad \omega=-\frac{5}{2}\delta_3c_2\kappa^4-\frac{31}{12}\kappa^2\delta_3k^2c_2-c_2k^4\delta_3-\frac{2}{3}ak^2,\\ \delta_2=\frac{60c_2k^2\kappa^2\rho\delta_3+8c_2\kappa^4\rho\delta_3-4c_3k^3\alpha_1^2\delta_{12}-2\alpha_1^2c_2k^2\delta_7+\alpha_1^2b}{\alpha_1^2c_1k},\\ \delta_4=-\frac{12\delta_3c_2\kappa^2\rho+2\alpha_1^2c_3\delta_{14}k+\alpha_1^2c_2\delta_8}{c_2\alpha_1^2},\quad \delta_5=0,\\ \delta_1=\frac{-24\delta_3k^2c_2-31c_2\kappa^2\delta_3+4a}{12c_1k},\quad \delta_6=\frac{48\delta_3c_2\kappa^2\rho-4\alpha_1^2c_3\delta_{12}k-\alpha_1^2c_2\delta_7}{c_2\alpha_1^2}.\end{gathered} \tag{50}$$

Incorporating the above result with Eqs. (49) and (29) into Eq. (2) allows us to derive the soliton solution of the concatenation model as:

$$q_1(x,t)=\alpha_1\left(\frac{\sqrt{5}\sqrt{\frac{1}{\rho}}}{4}+\frac{4k\ln(d)d^{\xi}}{4k^2\ln(d)^2d^{2\xi}+\rho}\right)e^{i(-\kappa x+\omega t+\zeta)}, \tag{51}$$

where κ and ω are provided by Eq. (50).

Result 2:

$$\begin{gathered}\alpha_0=0,\quad \alpha_1=2\sqrt{-\rho^2}\ln(d)\beta_1,\\ \omega=\frac{1}{18\kappa^4}\left[\begin{array}{l}\left(18\beta_1^4c_2\delta_5+\begin{pmatrix}-4c_3(\delta_{12}+7\delta_{14})k^3\\ +(17\delta_6-19\delta_7-14\delta_8-14\delta_4)c_2k^2\\ -18c_1k\delta_2+18b\end{pmatrix}\beta_1^2-12ak^2\right)\kappa^4\\ 20\begin{pmatrix}20\delta_5\beta_1^2c_2+12c_3(\delta_{12}-\delta_{14})k^3\\ +3(5\delta_6-3\delta_7-2\delta_8-2\delta_4)c_2k^2\\ -12c_1k\delta_2+12b\end{pmatrix}k^2\beta_1^2\kappa^2+12c_2k^4\beta_1^4\delta_5\end{array}\right],\\ \delta_1=\frac{1}{18c_1k\kappa^4}\left[\begin{array}{l}(((-\delta_6-\delta_7+4\delta_8+4\delta_4)c_2-4kc_3(\delta_{12}-2\delta_{14}))\beta_1^2+6a)\kappa^4\\ +2(10\delta_5\beta_1^2c_2+3(3\delta_6-\delta_7)k^2c_2+12c_3k^3\delta_{12}-6c_1k\delta_2+6b)\beta_1^2\kappa^2\\ +24c_2k^2\beta_1^4\delta_5\end{array}\right],\\ \delta_3=-\frac{\beta_1^2(4\kappa^2c_3\delta_{12}k+4\kappa^2c_3\delta_{14}k+2\kappa^2c_2\delta_4+\kappa^2c_2\delta_6+\kappa^2c_2\delta_7+2\kappa^2c_2\delta_8+4\delta_5\beta_1^2c_2)}{6c_2\kappa^4}.\end{gathered} \tag{52}$$

By inserting the above result together with Eqs. (49) and (29) into Eq. (2), we derive the exact solution of the concatenation model as:

$$q_1(x,t)=\frac{\beta_1\left(4k^2\ln(d)^2 d^{2\sigma}+4\sqrt{-\rho}\ln(d)kd^{\sigma}-\rho\right)\ln(d)}{4k^2\ln(d)^2 d^{2\sigma}+\rho}\mathrm{e}^{i(-\kappa x+\omega t+\zeta)}, \tag{53}$$

where κ and ω are outlined in Eq. (52). For instance, setting $d=\mathrm{e}$, $\rho=\pm 4k^2$, and all other parameters to unity in Eq. (53) gives us the complexiton solutions:

$$q_2^{(1)}(x,t)=2\beta_1\left(\tanh(kx-vt)+i\operatorname{sech}(kx-vt)\right)\mathrm{e}^{i(-\kappa x+\omega t+\zeta)}, \tag{54}$$

$$q_2^{(2)}(x,t)=2\beta_1\left(\coth(kx-vt)+i\operatorname{csch}(kx-vt)\right)\mathrm{e}^{i(-\kappa x+\omega t+\zeta)}. \tag{55}$$

Result 3:

$$\begin{gathered}\alpha_0=0,\quad \alpha_1=\sqrt{-\rho}\ln(d)\beta_1,\quad a=\frac{20\ln(d)^2 c_2\beta_1^4\delta_5+3\kappa^2\delta_1 kc_1}{\kappa^2},\\ \omega=-\frac{7c_2\kappa^2\ln(d)^4\beta_1^4\delta_5+20c_2k^2\beta_1^4\delta_5\ln(d)^2+2c_1k^3\delta_1\kappa^2}{\kappa^2},\quad \delta_3=0,\\ b=\frac{-8\ln(d)^2 c_2\beta_1^2\delta_5\kappa^2-c_2k^2\kappa^2\delta_6+c_2k^2\delta_7\kappa^2-8c_2k^2\beta_1^2\delta_5+c_1k\delta_2\kappa^2}{\kappa^2},\\ \delta_4=-\frac{2c_3k\kappa^2\delta_{14}+c_2\kappa^2\delta_8+8c_2\delta_5\beta_1^2}{c_2\kappa^2},\quad \delta_{12}=-\frac{\left(\kappa^2\delta_6+\kappa^2\delta_7-12\beta_1^2\delta_5\right)c_2}{4c_3k\kappa^2}.\end{gathered} \tag{56}$$

Applying the aforementioned result obtained along with Eqs. (49) and (29) into Eq. (2) provides the exact solution for the concatenation model as:

$$q_1(x,t)=\frac{\beta_1\left(4k^2\ln(d)^2 d^{2\xi}+4\sqrt{-\rho}\ln(d)kd^{\xi}-\rho\right)\ln(d)}{4k^2\ln(d)^2 d^{2\xi}+\rho}\mathrm{e}^{i(-\kappa x+\omega t+\zeta)}, \tag{57}$$

where ω and κ are defined by Eq. (56). The solution given by Eq. (57) exists under the conditions $4k^2\ln(d)^2 d^{2\xi}+\rho>0$ and $\rho<0$. For example, choosing $d=\mathrm{e}$, $\rho=-4k^2$, and setting the remaining parameters to unity in Eq. (57) results in the straddled dark–bright and singular–singular solitons:

$$q_2^{(1)}(x,t)=2\beta_1\left(\tanh(kx-vt)+\operatorname{sech}(kx-vt)\right)\mathrm{e}^{i(-\kappa x+\omega t+\zeta)}, \tag{58}$$

$$q_2^{(2)}(x,t)=2\beta_1\left(\coth(kx-vt)+\operatorname{csch}(kx-vt)\right)\mathrm{e}^{i(-\kappa x+\omega t+\zeta)}. \tag{59}$$

where ω and κ are detailed in Eq. (56), and v is expressed as $v=-\left(8c_2k^3\delta_3+2ak\right)\kappa$.

In Figs. 1-4, we address soliton solutions by setting the parameters: $s_1=1.3$, $a=0.35$, $k=1.5$, $c_2=1.2$, $\delta_3=0.4$, $h=0.1$, $\beta_1=0.4$, and $v=2.5$.

Result 4:

$$\begin{gathered}\omega=-\frac{2ak^2}{3},\quad \delta_1=\frac{a}{3c_1k},\quad \delta_2=\frac{-4c_3k^3\delta_{12}-2c_2k^2\delta_7+b}{c_1k},\quad \delta_3=0,\\ \delta_5=0,\quad \delta_4=-\frac{2c_3\delta_{14}k+c_2\delta_8}{c_2},\quad \delta_6=-\frac{4c_3\delta_{12}k+c_2\delta_7}{c_2}.\end{gathered} \tag{60}$$

Combining the above result with Eqs. (49) and (29) in Eq. (2), we get the soliton solution for the concatenation model as:

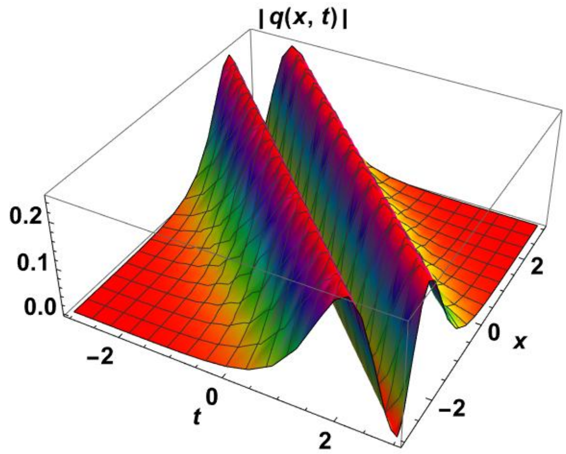

Fig. 4. Bright–dark soliton's waveform [see Eq. (58)].

$$q_1(x,t) = 4k^2\ln(d)^2 \frac{\left[\begin{array}{l}(\ln(d)\beta_1+\alpha_0)d^{2\xi}+ \\ +4\ln(d)d^{\xi}k\alpha_1-\rho(\ln(d)\beta_1-\alpha_0)\end{array}\right]}{4k^2\ln(d)^2 d^{2\xi}+\rho} \mathrm{e}^{i(-\kappa x+\omega t+\zeta)}, \tag{61}$$

where ω and κ are provided in Eq. (60). For the solution given by Eq. (61) to exist, the conditions $4k^2\ln(d)^2 d^{2\xi}+\rho>0$ and $\rho<0$ must be satisfied.

Result 5:

$$\begin{aligned}
&\alpha_0=0, \quad \alpha_1=0, \\
&\omega=-\frac{102\beta_1^4\ln(d)^4 c_2\delta_5\kappa^4+145\ln(d)^2 c_2k^2\kappa^2\beta_1^4\delta_5+3c_2k^4\beta_1^4\delta_5+72ak^2\kappa^4}{108\kappa^4}, \\
&\delta_1=\frac{-145\ln(d)^2 c_2\beta_1^4\delta_5\kappa^2-6c_2k^2\beta_1^4\delta_5+36\kappa^4 a}{108c_1k\kappa^4}, \\
&\delta_3=\frac{\beta_1^4\delta_5}{36\kappa^4}, \\
&\delta_2=\frac{35\ln(d)^2 c_2\beta_1^2\delta_5\kappa^2-72c_3k^3\kappa^2\delta_{12}-36c_2k^2\delta_7\kappa^2+195c_2k^2\beta_1^2\delta_5+18b\kappa^2}{18c_1k\kappa^2}, \\
&\delta_4=-\frac{12c_3k\kappa^2\delta_{14}+6c_2\kappa^2\delta_8+25c_2\delta_5\beta_1^2}{6c_2\kappa^2}, \quad \delta_6=-\frac{12c_3k\kappa^2\delta_{12}+3\kappa^2c_2\delta_7-20c_2\delta_5\beta_1^2}{3c_2\kappa^2}.
\end{aligned} \tag{62}$$

Inserting the above result, along with Eqs. (49) and (29), into Eq. (2) yields the exact solution of the concatenation model as:

$$q_1(x,t)=\frac{\beta_1\left(4k^2\ln(d)^2 d^{2\sigma}-\rho\right)\ln(d)}{4k^2\ln(d)^2 d^{2\sigma}+\rho}\mathrm{e}^{i(-\kappa x+\omega t+\zeta)}, \tag{63}$$

where $v=-(8c_2k^3\delta_3+2ak)\kappa$, with κ and ω derived from Eq. (21). For instance, choosing $d=\mathrm{e}$ and $\rho=\pm 4k^2$ in Eq. (63) results in the dark and singular solitons:

$$q_2^{(1)}(x,t)=\beta_1\tanh(kx-vt)\mathrm{e}^{i(-\kappa x+\omega t+\zeta)}, \tag{64}$$

$$q_2^{(2)}(x,t)=\beta_1\coth(kx-vt)\mathrm{e}^{i(-\kappa x+\omega t+\zeta)}. \tag{65}$$

Result 6:

$$
\begin{aligned}
&\alpha_0 = 0, \quad \beta_1 = 0,\\
&\omega = c_2 k^4 \delta_3 + c_1 k^3 \delta_1 + \kappa^2 a \ln(d)^2 - a k^2 + \delta_3 c_2 \kappa^4 \ln(d)^4\\
&\quad -3\kappa^2 \delta_1 k c_1 \ln(d)^2 - 6\kappa^2 \delta_3 k^2 c_2 \ln(d)^2,\\
&\delta_2 = \frac{1}{9 c_1 k \alpha_1^2}\begin{bmatrix} -2c_3(18\delta_{12} + 25\delta_{14})k^3 - c_2(25\delta_4 + 18\delta_7 + 25\delta_8)k^2 \\ -14\delta_{14} k c_3 \kappa^2 - 7c_2(\delta_4 + \delta_8)\kappa^2 + 9b \end{bmatrix}\alpha_1^2\\
&\quad -6\rho\left[-58\delta_3 k^2 c_2 - 10\kappa^2 c_2 \delta_3 - 9\delta_1 k c_1 + 3a\right]\kappa^2,\\
&\delta_5 = \frac{2\kappa^2 \rho \ln(d)^2 \left(12 c_2 \kappa^2 \rho \ln(d)^2 \delta_3 + 2c_3 k \alpha_1^2 \delta_{14} + c_2 \alpha_1^2 \delta_4 + c_2 \alpha_1^2 \delta_8\right)}{9 c_2 \alpha_1^4},\\
&\delta_6 = \frac{\begin{bmatrix} 240 c_2 \kappa^2 \rho \ln(d)^2 \delta_3 - 36 c_3 k \alpha_1^2 \delta_{12} \\ -32 c_3 k \alpha_1^2 \delta_{14} - 16 c_2 \alpha_1^2 \delta_4 - 9 c_2 \alpha_1^2 \delta_7 - 16 c_2 \alpha_1^2 \delta_8 \end{bmatrix}}{9 c_2 \alpha_1^2}.
\end{aligned}
\tag{66}
$$

Combining the above result with Eqs. (49) and (29) in Eq. (2), the concatenation model holds:

$$
q_1(x,t) = \frac{4\alpha_1 k \ln(d) d^{\sigma}}{4k^2 \ln(d)^2 d^{2\sigma} + \rho} \mathrm{e}^{i(-\kappa x + \omega t + \zeta)}, \tag{67}
$$

where ω and κ are defined in Eq. (21). By setting $d = \mathrm{e}$ and $\rho = \pm 4k^2$ in Eq. (67), the bright and singular solitons become:

$$
q_2^{(1)}(x,t) = \alpha_1 \operatorname{sech}(kx - vt) \mathrm{e}^{i(-\kappa x + \omega t + \zeta)}, \tag{68}
$$

$$
q_2^{(2)}(x,t) = \alpha_1 \operatorname{csch}(kx - vt) \mathrm{e}^{i(-\kappa x + \omega t + \zeta)}. \tag{69}
$$

Although both Eqs. (65) and (69) represent singular soliton solutions; they exhibit distinct physical characteristics due to the nature of the underlying hyperbolic functions. Eq. (65) involves the coth function, which has a pole at the origin and approaches finite values asymptotically. As a result, the amplitude profile corresponds to a non-vanishing singular soliton that exhibits a sharp, localized divergence at a finite point, while maintaining non-zero intensity in its tails. This structure may be interpreted as modeling intensity blow-up on a non-zero background, which could be relevant in systems where modulation instability or gain leads to localized energy concentration. In contrast, Eq. (69) involves the csch function, which also has a singularity at the origin but decays to zero at spatial infinity. Thus, the resulting soliton is localized in space and vanishes at infinity, representing a purely singular bright-type soliton. From a physical point of view, this structure might model collapsing pulses or sharp energy spikes in nonlinear media with localized blow-up and no background field. Therefore, Eq. (65) corresponds to a singular soliton on a non-zero pedestal, while Eq. (69) describes a strictly localized singular pulse. Their differences may have implications in nonlinear optics or plasma physics, particularly in the interpretation of energy localization, field singularities, and collapse dynamics.

Eqs. (54) and (55) represent complexiton-type soliton solutions, which are characterized by the presence of both real and imaginary hyperbolic components within their structure. These solutions are distinct from traditional solitons (e.g., bright or dark) because their amplitude profiles are inherently complex-valued, leading to nontrivial phase-amplitude coupling in the field. Specifically, Eq. (54) is constructed from a linear combination of the real tanh and imaginary sech functions, resulting in a solution of the form, which corresponds to a localized wave packet exhibiting both dispersive and oscillatory behavior. The real part governs the asymptotic background (via tanh), while the imaginary part localizes the energy (via sech), leading to a hybrid structure that may be interpreted as a nonlinear interference pattern. Similarly, Eq. (55) combines coth and i csch functions, which yields a singular complexiton, diverging at the origin but with complex-valued spatial modulation. These solutions do not decay uniformly or vanish at infinity; instead, they exhibit phase singularities and may describe nonlinear wave structures in gain/loss media or non-Hermitian systems. In physical terms, complexitons represent interacting modes where both amplitude and phase undergo coherent evolution, often seen in optical systems, Bose-Einstein condensates, or plasma waves with complex potentials or saturable nonlinearities. Unlike conventional solitons, they exhibit non-constant amplitude envelopes and are typically associated with non-integrable or non-conservative effects.

6. Conclusions

In this research paper, we explored a wide range of optical solitons derived from a power law by a concatenation model through the integration of three standard equations: the SSE, the LPD model, and the NLSE. We successfully derived bright, dark, periodic, bright-dark, singular-singular, and singular soliton solutions accompanied by relevant constraint conditions. The simplicity of our approach has allowed for the natural emergence of parameter relations and constraints. In essence, while we have achieved fundamental results in soliton solutions, a vast field remains for further exploration and refinement. The proposed future studies, ranging from analytical approaches to numerical algorithms, aim to enhance the versatility and applicability of the model, contributing to the ongoing advancements in the understanding and utilization of optical solitons in various optical systems [20].

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Анотація*. Це дослідження заглиблюється в сферу нових оптичних солітонів у рамках моделі дисперсійної конкатенації, особливо зосереджуючись на самофазовій модуляції за законом Керра. Дослідження використовує симетричний аналіз Лі для перетворення складних керуючих рівнянь у звичайні диференціальні рівняння (ODE). Потім ці ODE розглядаються за допомогою двох різних методологій: методу F-розширення та нового узагальненого методу. За допомогою цих підходів успішно отримано широкий спектр солітонних рішень, що демонструє надійність і ефективність запропонованих методів. Крім того, фізичні інтерпретації цих рішень проілюстровано за допомогою 3D-профілів, які пропонують глибоке розуміння складної поведінки солітонів.*

Ключові слова: *новий узагальнений метод, метод конкатенації, аналіз симетрії Лі, метод F-розкладу,степеневий закон, солітони*