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SUPERSYMMETRY APPROACH FOR DESCRIBING OPTICAL VORTEX GENERATION IN FIBERS

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Abstract. Optical vortices, characterized by a helical phase structure and the presence of orbital angular momentum, can be generated in optical fibers through the superposition of specific linearly polarized modes. This work investigates the formation of such vortex modes in fibers with a parabolic refractive index profile. By transforming the scalar wave equation into a form analogous to the Schrödinger equation, we apply methods from supersymmetric quantum mechanics to design refractive index profiles that support degenerate eigenmodes. These degenerate modes share identical propagation constants, enabling stable vortex generation. The study demonstrates that supersymmetric transformations provide a systematic approach for selecting and coupling modes capable of carrying orbital angular momentum, offering new possibilities for advanced light control in fiber-based optical systems.

Keywords: optical fiber, optical vortices, linearly polarised modes, supersymmetry
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1. Introduction

Optical vortices, characterized by phase singularities and carrying orbital angular momentum (OAM), represent a unique class of light beams with promising applications in optical communications, microscopy, and quantum information processing [1, 2]. In recent years, significant efforts have been devoted to generating, controlling, and manipulating vortex states in various media, including optical fibers, which offer a robust platform for multiplexed data transmission [3, 4]. Optical vortices are characterized by a helical phase front of the form $\exp(il\varphi)$, where l is the topological charge and φ the azimuthal angle. This unique structure enables them to carry OAM, distinguishing them from traditional Gaussian beams. Their generation can be achieved in both free space and optical fibers, each offering unique benefits and challenges. The generation of optical vortices differs significantly between free space and optical fiber in terms of mechanisms, control, and propagation environments. In free space, optical vortices are typically generated using external beam-shaping devices. These include spatial light modulators (SLMs), spiral phase plates, q-plates, and computer-generated holograms, all of which impose a desired azimuthal phase on an input beam [1,2]. The versatility of SLMs allows real-time modulation and switching between different OAM modes. Vortices in free space propagate unguided, and their beam profile diverges with distance and topological charge. While this makes them suitable for short-range applications, such as optical trapping, they are susceptible to environmental perturbations, including turbulence and scattering. In contrast, optical vortex generation in fibers involves exciting guided modes that intrinsically carry OAM. Specially designed fibers such as ring-core fibers or few-mode fibers support the propagation of such modes [3]. Techniques such as mode multiplexing, long-period gratings, and photonic lanterns are used to excite and maintain these vortex-carrying modes [4] selectively. In fibers, optical vortices benefit from the confinement provided by the waveguide, enabling long-distance transmission. However, challenges such as inter-modal coupling and mode dispersion can degrade the purity of OAM modes, necessitating precision fiber design and launch conditions.

One innovative approach to studying and optimizing such systems involves the application of supersymmetry (SUSY) — a theoretical framework developed initially in high-energy physics but increasingly adopted in optics [5, 6]. SUSY methods enable the design of pairs of isospectral optical structures, providing new tools for controlling the propagation and transformation of optical modes within fiber systems [7, 8].

This work investigates the application of supersymmetric techniques for analyzing and engineering optical vortex modes in optical fibers, with the goal of improving transmission performance and expanding the capabilities of fiber-based photonic devices.

2. Fiber modes and OAM states

To implement the SUSY approach to eigenmodes inside the optical fiber, an analogy with the Schrödinger equation is used [9]. It is usual to ignore polarization effects and neglect vector effects; then, the electric field in the fiber cross-section is given by the scalar wave equation

$$\nabla^2 E - \frac{n^2}{c^2}\frac{\partial^2 E}{\partial t^2} = 0, \tag{1}$$

where n is the refractive index. In a cylindrical geometry of an axially symmetric fiber with (r,φ,z) coordinates, we seek propagating solutions of a separable form

$$E \infty R(r) e^{il\varphi} e^{i(\beta z - \omega t)}, \tag{2}$$

where β is the propagation constant, ω is the frequency, $R(r)$ is the radial part, and l is the azimuthal mode number (integer). Substituting Eq. (2) into Eq. (1) and making a substitution $\psi = r^{1/2} R(r)$ for the dimensionless variable $y = r/\rho$, where ρ is the radius of the fiber, we get

$$-\frac{d^2\psi}{dy^2} + \frac{l^2 - 1/4}{y^2}\psi - \rho^2(n^2k^2 - \beta^2)\psi = 0, \quad k \equiv \frac{\omega}{c}, \tag{3}$$

Thus, this equation has the form of the Schrödinger equation. In order to tie the equation to a specific fiber geometry, it is advisable to introduce the following parameters [10]. First, we represent the radial dependence of the square of the refractive index in the form $n^2(y) = n_0^2[1 - 2\Delta f(y)]$, where $\Delta = (1 - n_{cl}^2 / n_0^2)/2$ contains n_0 that is the maximum refractive index in the fiber, and n_{cl} is the cladding index. The function $f(y)$ has the form of a potential well and satisfies the conditions $f(0) = 0$ and $f(1) = 1$. Next, following [10], we will define $V = \rho k n_0 \sqrt{2\Delta}$ (waveguide parameter), and $U = \rho\sqrt{k^2 n_0^2 - \beta^2}$ (normalized propagation constant). Using this notation, the equation can be rewritten as

$$-\frac{1}{2}\frac{d^2\psi}{dy^2}+\frac{1}{2}f_{eff}(y)\psi=\frac{U^2}{2}\psi,\quad f_{eff}(y)=V^2f(y)+\frac{(l^2-1/4)}{y^2}, \tag{4}$$

This equation is analogous to the one-dimensional Schrödinger equation, whose potential includes a term (the second term in f_{eff}) due to the circular nature of the fiber. This term is analogous to the "centrifugal potential" used in similar quantum mechanical considerations. The expression $U^2/2$ plays the role of energy, the value of which determines the propagation constant β. As with quantum mechanics, solutions of the wave Eq. (4) lead to a discrete set of bound modes and a continuum of radiation modes.

There exist three categories of allowed modes in optical fiber [10]. Core modes are modes guided by the core-cladding interface. Cladding modes are modes guided by the combination of core-cladding and cladding-buffer interfaces. These modes correspond to discrete values of the propagation constant β. Radiation modes form a continuum and are infinite in number. Thus, the total electromagnetic field is the sum over discrete plus the integral over continuous values of the propagation constant. In order to describe the mode structure in an optical fiber more precisely, the Eq. (1) is not enough [10]. A term proportional to $\left[\psi(\ln n^2)_y\right]_y$ comes into play, where the index is the derivative with respect to y. It was shown earlier [10] that using the weakly-guided assumption, it is possible to construct a system of approximate solutions only for transverse modes. For these approximate solutions, the transverse field is almost completely described by ψ. That is, the solutions are almost linearly polarized, and these modes are thus referred to as LP modes. In optical fibers, optical vortices cannot be directly supported by the scalar LP modes due to their intrinsic symmetry. However, through specific superpositions of LP modes, particularly higher-order degenerate pairs, it is possible to synthesize vector modes with helical phase fronts that effectively carry OAM [4].

LP modes are approximate solutions to the wave equation in weakly guiding step-index optical fibers, characterized by the notation LP_{lm}, where l is the azimuthal index and m the radial index. In general, modes with nearly identical $U^2/2$ also have identical group velocities and constitute one mode group, so that they can generate optical vortices. Thus, the first rule for selecting suitable candidates for this is related to degeneracy of the eigenvalues of the Eq. (4). In this context, for example, the mode group LP_{1m} consists of 4 vector modes - TE_{0m}, HE_{2m}^{even}, HE_{2m}^{odd} and TM_{0m}, and all other mode groups LP_{lm} ($l>0$) include combinations of true HE and EH modes [4].

For $l>0$ each LP mode is actually a degenerate pair composed of two orthogonal vector modes [4]. A typical combination is given by

$$\begin{pmatrix} LP_{11}^{a} \\ LP_{11}^{b} \end{pmatrix}=\begin{pmatrix} HE_{21}^{even}+TM_{01} \\ HE_{21}^{odd}+TE_{01} \end{pmatrix}. \tag{5}$$

To generate a vortex beam in a fiber, two degenerate LP modes with opposite angular momentum components are combined with a $\pi/2$ phase shift:

$$OAM_l=LP_{lm}^{a}+iLP_{lm}^{b}. \tag{6}$$

This results in a mode with a well-defined helical phase front and an OAM of $l\hbar$ per photon. Conventional step-index fibers do not maintain OAM states effectively over long distances. Therefore, ring-core fibers or specialty few-mode fibers with a larger index contrast or mode spacing are often required to preserve the LP superpositions. The generation of optical vortices in fibers via LP mode superpositions is a viable and scalable approach to harnessing OAM in guided-wave optics. By exploiting the degeneracy of higher-order LP modes and carefully engineering their phase relationships, it is possible to create stable vortex beams within optical fibers. While mode coupling and fiber imperfections pose challenges, advances in fiber design and mode control continue to improve the feasibility of OAM-based systems in practical applications.

Thus, for efficient generation of optical vortices in a fiber, it is necessary to use a tool that allows finding degenerate eigenmodes. In other words, it is necessary to determine a set of equalization potentials that have the same eigenvalues. Supersymmetry will help us with this.

3. Mode selection in graded-index optical fibers

The preservation and manipulation of vortex modes during propagation through optical fibers present unique challenges, primarily due to modal dispersion, intermodal coupling, and environmental perturbations. In this context, graded-index (GRIN) optical fibers emerge as a promising platform for the robust transmission of optical vortex beams. As we mentioned above, the optical vortex in the waveguide is constructed as a superposition of eigenmodes that have propagation constants with close or equal values. Considering the analogy of Eq. (4) with the Schrödinger equation, the most suitable tool for selecting degenerate eigenmodes is the supersymmetry method.

Supersymmetry was originally proposed in the context of high-energy physics as a symmetry between bosons and fermions. In Supersymmetric Quantum Mechanics (SUSY QM) [11], these concepts are adapted to non-relativistic quantum systems, typically in 1D. SUSY QM offers insights into quantum spectral problems, factorization methods, and solvable potentials.

The Hamiltonian for SUSY QM can be written in the form of two components, which are given by

$$H_{\pm}\psi_{\pm} = -\frac{1}{2}\frac{d^2\psi_{\pm}}{dy^2} + V_{\pm}(y)\psi_{\pm} = E\psi_{\pm}. \tag{7}$$

Let us define two first-order differential operators:

$$A = W + \frac{d}{dy}, \quad A^{+} = W - \frac{d}{dy}, \tag{8}$$

where $W(y)$ is called the superpotential. Then the partner Hamiltonians take the form:

$$H_{+} = \frac{1}{2}A^{+}A = -\frac{1}{2}\frac{d^2}{dy^2} + V_{+}(y), \quad H_{-} = \frac{1}{2}AA^{+} = -\frac{1}{2}\frac{d^2}{dy^2} + V_{-}(y), \tag{9}$$

with

$$V_{\pm}(y) = \frac{1}{2}\left(W^2 \mp W'\right), \tag{10}$$

where prime denotes the derivative with respect to y.

These two potentials are isospectral (except possibly for the ground state). Indeed, let the Hamiltonian H_+ have eigenfunctions ψ_{+m} with corresponding eigenvalues E_m: $H_+\psi_{+m} = E_m\psi_{+m}$. Then for $A\psi_{+m}$, we have

$$H_-(A\psi_{+m}) = \frac{1}{2}AA^+(A\psi_{+m}) = AH_+\psi_{+m} = E_m(A\psi_{+m}). \quad (11)$$

Thus, E_m is the energy spectra of H_- with eigenfunctions $\psi_{-m} = A\psi_{+m}$. However, $A\psi_{+0}$ is trivially zero since ψ_{+0} is the ground-state solution of H_+. We conclude that the spectra of H_+ and H_- are identical except for the ground state, $m = 0$, which is nondegenerate.

Thus, the solution is reduced to finding the ground state. After that, the SUSY partner is found and constructed from Eq. (11). As we saw earlier in Eq. (7), for any transverse distribution of the refractive index, the equation for the optical modes in a wave has the form of the Schrödinger equation, where f_{eff} plays the role of potential in Eq. (7). SUSY allows the construction of superpartner waveguides [7], in which one fiber supports a given fundamental mode, the second has a similar spectral behavior, but without a specific mode. In SUSY optics, a given optical potential (i.e., refractive index profile) has a SUSY partner, with a modified index profile but identical propagation constants (except for the fundamental mode). This means that light modes of one structure can be related to modes of its partner, excluding the lowest-order mode. This unique property can be used for mode filtering, conversion, and lossless coupling. Thus, the modes in both fibers can be used to construct an optical vortex, since they are degenerate. For example, consider a fiber with a refractive index profile, that supports the usual LP modes (LP_{01}, LP_{11},etc.). Applying a first-order SUSY transformation creates a new index profile, which lacks the LP_{01} mode but retains all higher-order modes with identical propagation constants.

As an example, consider a fiber with a refractive index profile $f(y) = y^2$. A parabolic refractive index fiber is a special type of GRIN optical fiber where the refractive index of the core decreases quadratically from the center toward the cladding. This design provides excellent control over modal dispersion and is widely used in multimode fiber systems. The parabolic refractive index profile leads to a wave equation that is mathematically analogous to the two-dimensional isotropic quantum harmonic oscillator [12]. Indeed, after changing the variable $x = V^{1/2}y$, Eq. (4) can be rewritten as

$$\left[-\frac{1}{2}\frac{d^2}{dx^2} + \frac{1}{2}\left(x^2 + \frac{(l^2 - 1/4)}{x^2}\right)\right]\psi = \frac{U^2}{2V}\psi, \quad (12)$$

This equation has a set of eigenfunctions and eigenvalues, $H\psi_{lm} = E_{lm}\psi_{lm}$, that define the mode spectrum of the GRIN optical fiber [10] as

$$\psi_{lm} = Nx^{|l|+\frac{1}{2}}e^{-\frac{x^2}{2}}L_m^{|l|}(x^2), \quad (13a)$$

and

$$E_{lm} = 2m + |l| + 1, \quad (13b)$$

where $L_m^{|l|}(x^2)$ is the associated Laguerre polynomial, m is the radial mode number (non-negative integer). In weakly guiding step-index optical fibers, the propagation constant, as follows from the definition of U, V and Eq. (13b), is given by

$$\beta_{lm} \cong kn_0\left(1 - 2\Delta\frac{2m + |l| + 1}{V}\right), \tag{14}$$

Thus, the propagation constants in a parabolic refractive index fiber form an equidistant sequence of eigenstates, Eq. (12). Each state is characterized by two numbers m and l. However, the propagation constants β_{lm} depend only on the combination $N = 2m + |l|$. Each value $N \geq 2$ can be realized by several combinations of values m and l, therefore, the corresponding modes with values $N \geq 2$ are degenerate. Thus, the superposition of these modes can form stable structures, since they have equal propagation constants. However, we can only speak of their existence if the phase of the superposition is proportional to $l\varphi$. Thus, if we take a pair (m,l), then in weakly guiding step-index optical fibers, Eq. (4) combinations (m,l) and $(m,l+2)$ come into the LP mode. In contrast, in parabolic refractive index fiber it is necessary to choose (m,l) and $(m-1,l+1)$.

In order to consider the SUSY generalization of Eq. (12), we define the superpotential as

$$W(x) = x - \frac{l - 1/2}{x}, \tag{15}$$

From Eq. (10), the potentials of the superpartners are given by

$$\begin{aligned} V_+(x) &= \frac{1}{2}x^2 + \frac{1}{2}\frac{(l^2 - 1/4)}{x^2} - (l+1), \\ V_-(x) &= \frac{1}{2}x^2 + \frac{1}{2}\frac{(l+1/2)(l+3/2)}{x^2} - l. \end{aligned} \tag{16}$$

It follows that states with l and $l+1$ are superpartners. However, the states described by Eq. (12) do not have such degeneracies. Instead, there are degeneracies corresponding to values $(m-1,l+1)$ differing by two units, $\beta_{l+1m-1} = \beta_{lm}$. This relationship becomes more natural if we consider that Eq. (12) is defined on a half-axis.

Let's consider the relationship between the superpartners in Eq. (9). It is easy to check that, using the superpotential (15) and the expression for the eigenvalue (13b), Eq. (12) can be rewritten as follows:

$$\left[-\frac{1}{2}\frac{d^2}{dx^2} + V_+\right]\psi_{lm} = 2m\psi_{lm}, \tag{17}$$

where ψ_{lm} is defined by Eq. (13a). Thus ψ_+ one corresponds to ψ_{lm}. Let us note that the operator on the left side of Eq. (17) does not depend on m. Based on Eqs. (9) and (16), for the second superpartner, we obtain

$$\left[-\frac{1}{2}\frac{d^2}{dx^2} + V_-\right]\psi_- = (2m+2)\psi_-, \tag{18}$$

In order for the eigenvalues of Eqs. (17) and (18) to coincide (be equal $2m$), it is necessary to determine $\psi_- = \psi_{l+1m-1}$. Since Eq. (11) implies that the supersymmetric partners have the same eigenvalues, it is easy to obtain a relation, up to a normalization constant, between the supersymmetric modes

$$A\psi_{lm} = \left[\frac{d}{dx} + x - \frac{l - 1/2}{x}\right]\psi_{lm} = -2Nx^{|l|+1+\frac{1}{2}}e^{-\frac{x^2}{2}}L_{m-1}^{|l|+1}(x^2) \approx \psi_{l+1m-1}. \tag{19}$$

In obtaining relation (19), the recurrence relation $\left(L_m^l\right)' = -L_{m+1}^{l-1}$ for generalized Laguerre polynomials was used along with Eq. (13a).

Thus, by using the SUSY transformation, it is possible to enable coupling between different modes in a fiber with a parabolic refractive index. SUSY modes are not obvious candidates for building vortex solutions in optical fibers, since they develop in different potentials, resulting in different propagation constants. However, if we analytically find a solution for one of the modes, then by successively applying the SUSY coupling, we can find the remaining solutions. This becomes especially important when the refractive index profile is not necessarily parabolic, but, for example, has a small perturbation. Then, we can use the exact solution and force the perturbation into the A operator, as done in Eq. (19). In other words, it is possible to excite two supersymmetric modes in a fiber with a parabolic refractive index profile. If the mode spectra coincide after the appropriate shift, then they have the same propagation constants. This circumstance can be used to select modes intended for constructing optical vortices.

4. Conclusions

This work shows that generating optical vortices in fibers can be effectively achieved through the controlled superposition of degenerate LP modes, especially in graded-index fibers with a parabolic refractive index profile. By using the mathematical analogy between the wave equation in optical fibers and the Schrödinger equation, we applied SUSY techniques to identify and engineer isospectral refractive index profiles that support such degenerate modes. Specifically, SUSY transformations enable the creation of partner mode sets with identical propagation constants (excluding the fundamental mode), allowing for the stable formation of vortex beams within the fiber. This method provides a systematic way to select and couple modes that preserve orbital angular momentum during propagation, even with structural or environmental disturbances. Our analysis confirms that SUSY-based mode selection is both theoretically sound and practically useful for enhancing OAM propagation in optical fibers. Future work will focus on experimental validation, integration into complex fiber systems, and exploring nonlinear and active media in SUSY-designed structures.

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Анотація. *Оптичні вихори, що характеризуються спіральною фазовою структурою та наявністю орбітального кутового моменту, можуть бути згенеровані в оптичних волокнах шляхом суперпозиції специфічних лінійно поляризованих мод. У цій роботі досліджується формування таких вихрових мод у волокнах з параболічним профілем показника заломлення. Перетворюючи скалярне хвильове рівняння у форму, аналогічну рівнянню Шредінгера, ми застосовуємо методи суперсиметричної квантової механіки для розробки профілів показника заломлення, які підтримують вироджені власні моди. Ці вироджені моди мають однакові константи поширення, що забезпечує стабільну генерацію вихорів. Дослідження демонструє, що суперсиметричні перетворення забезпечують системний підхід до вибору та зв'язку мод, здатних переносити орбітальний кутовий момент, пропонуючи нові можливості для вдосконаленого керування світлом у волоконно-оптичних системах.*

Ключові слова: *оптичне волокно, оптичні вихори, лінійно-поляризовані моди, суперсиметрія*