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QUIESCENT OPTICAL SOLITONS IN MAGNETO–OPTIC WAVEGUIDES WITH KUDRYASHOV'S AND GENERALIZED NONLOCAL FORM OF SELF-PHASE MODULATION HAVING NONLINEAR CHROMATIC DISPERSION AND GENERALIZED TEMPORAL EVOLUTION

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Abstract. This paper retrieves quiescent optical solitons that merge from magneto-optic waveguides that maintain Kudryashov's form of self–phase modulation coupled with a generalized form of non–local nonlinearity. The model also comes with nonlinear chromatic dispersion and is considered with generalized temporal evolution. The enhanced direct algebraic method has made this retrieval possible. A full spectrum of solitons is thus recovered through the intermediary Jacobi's elliptic functions as well as Weierstrass' elliptic functions. The parameter constraints for the existence of such solitons are also enumerated.

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1. Introduction

The concept of quiescent optical solitons has been established for several decades, yet their practical applicability across a broad range of optoelectronic devices has only recently begun to gain momentum. Traditionally, optical solitons have been widely studied in optical fibers [8-12] and gap solitons in Bragg gratings [13,14]. However, the study of quiescent solitons—those that remain stationary in their propagation medium—has largely remained theoretical. The existence of these solitons in various physical systems has been recognized in previous studies [1-7]. Despite this, the theoretical models describing quiescent solitons are still in their infancy, with little experimental validation. The laws governing self-phase modulation (SPM), as proposed by Kudryashov, were introduced approximately half a decade ago. Consequently, these theoretical models have not yet been tested in a laboratory setting. No experimental demonstrations involving oscilloscopes or eye diagrams have been reported in the literature to date.

One of the key challenges in the study of quiescent solitons is the role of chromatic dispersion (CD). CD significantly influences pulse propagation in optical communication systems, leading to dispersion-induced pulse broadening. When CD is linear, solitons typically remain mobile. However, when CD is nonlinear, it can induce the formation of quiescent solitons, where the optical pulse becomes stationary. The current study investigates the emergence of such quiescent solitons in magneto-optic waveguides [6,15-22], incorporating a nonlinear CD component alongside Kudryashov's form of SPM. This work builds upon prior research [6], which presented preliminary results on this subject.

A crucial distinction between this study and previous works lies in the mathematical formulation of the problem. Earlier studies predominantly addressed mobile solitons under linear CD conditions. In contrast, the present work considers a nonlinear CD model, a critical factor for stabilizing quiescent solitons. Additionally, this paper explores the effects of generalized temporal evolution, where a tunable parameter (l) governs the temporal characteristics of the soliton dynamics. When $l=1$, the model reduces to the conventional form studied in laboratory experiments, referred to as linear temporal evolution. This generalization provides a broader understanding of soliton dynamics beyond the conventional frameworks.

The enhanced direct algebraic method is employed to retrieve the mathematical structures of quiescent solitons. This method has proven to be a robust analytical technique for solving nonlinear differential equations in soliton theory. The implementation of this approach allows for the recovery of a full spectrum of quiescent solitons within the proposed magneto-optic waveguide model. It is worth noting that while this study utilizes the enhanced direct algebraic method, other analytical techniques have also been successfully applied to related problems. These include the projective Riccati equation approach [7] and the robust Lie symmetry analysis [1,7], which have been instrumental in uncovering soliton solutions in different nonlinear optical systems.

The mathematical framework developed in this study involves intermediary elliptic functions, particularly Jacobi's elliptic functions (cnoidal waves) and Weierstrass' elliptic functions. These functions provide a natural extension of soliton solutions, where special cases arise when the modulus of ellipticity in Jacobi's functions approaches unity. In this limit, the solutions reduce to standard soliton profiles, which can be interpreted as fundamental quiescent solitons in the system. Elliptic functions enrich the analysis by offering a continuous transition from periodic wave structures to localized soliton solutions.

This paper explores the emergence of quiescent optical solitons in magneto-optic waveguides under the combined effects of nonlinear CD and Kudryashov's form of SPM. The study introduces generalized temporal evolution to provide a more comprehensive perspective on soliton dynamics. A diverse range of quiescent soliton solutions is retrieved by employing the enhanced direct algebraic method. The subsequent sections of this paper present the detailed mathematical modeling, solution methodology, and physical interpretations of the obtained soliton solutions.

2. Governing model

The dimensionless form of the nonlinear Schrödinger's equation (NLSE) with SPM structure from Kudryashov's law and dual form of generalized nonlocal nonlinearity having nonlinear CD and generalized linear temporal evolution is written as:

$$i\left(q^{l}\right)_{t}+a\left(|q|^{p}q^{l}\right)_{xx}+\left[b|q|^{n}+c|q|^{2n}+d|q|^{3n}+e|q|^{4n}+f\left(|q|^{n}\right)_{xx}+g\left(|q|^{2n}\right)_{xx}\right]q^{l}=0, \tag{1}$$

In Eq. (1), $q(x,t)$, is the complex-valued function representing the wave amplitude, while x and t are the independent variables that account for the spatial and temporal coordinates, respectively. The first term represents the generalized temporal evolution, with $i=\sqrt{-1}$ and l being the generalized temporal evolution parameter. The second term with coefficient a gives the CD, while the parameter p is the nonlinearity parameter. The remaining set of terms collectively represents the effect of SPM where the parameters b, c, d, and e come from Kudryashov's proposed structure of nonlinearity while f and g represent dual–forms of nonlinear forms of nonlocal nonlinearity (n). This model for an optical mono-mode fiber will now be reformed for a magneto-optic waveguide. This takes the form:

$$\begin{aligned}&i\left(q^{l}\right)_{t}+a_{1}\left(|q|^{p}q^{l}\right)_{xx}\\&+\begin{bmatrix}b_{1}|q|^{n}+c_{1}|q|^{2n}+d_{1}|q|^{3n}+e_{1}|q|^{4n}+f_{1}\left(|q|^{n}\right)_{xx}+g_{1}\left(|q|^{2n}\right)_{xx}\\+h_{1}|r|^{n}+l_{1}|r|^{2n}+s_{1}|r|^{3n}+n_{1}|r|^{4n}+p_{1}\left(|r|^{n}\right)_{xx}+q_{1}\left(|r|^{2n}\right)_{xx}\end{bmatrix}q^{l}=Q_{1}r^{l},\end{aligned} \tag{2}$$

and

$$\begin{aligned}&i\left(r^{l}\right)_{t}+a_{2}\left(|r|^{p}r^{l}\right)_{xx}\\&+\begin{bmatrix}b_{2}|r|^{n}+c_{2}|r|^{2n}+d_{2}|r|^{3n}+e_{2}|r|^{4n}+f_{2}\left(|r|^{n}\right)_{xx}+g_{2}\left(|r|^{2n}\right)_{xx}\\+h_{2}|q|^{n}+l_{2}|q|^{2n}+s_{2}|q|^{3n}+n_{2}|q|^{4n}+p_{2}\left(|q|^{n}\right)_{xx}+q_{2}\left(|q|^{2n}\right)_{xx}\end{bmatrix}r^{l}=Q_{2}q^{l}.\end{aligned} \tag{3}$$

In Eqs. (2) and (3), the parameters Q_j for (j=1,2) represent the magneto-optic parameters associated with the two wave components. Physically, these parameters correspond to the strengths of the external magnetic field along each of the two polarization components of the wave. This emanates from the fact that in 1845, Michael Faraday discovered that the polarization of a linearly polarized light beam is rotated upon propagating through a media that is placed in a magnetic field parallel to the propagation direction [8]. In the context of magneto-optic waveguides, the external magnetic field plays a crucial role in modulating the medium's optical properties, influencing the optical solitons' propagation dynamics. The presence of Q_1 and Q_2 ensures that the model accounts for the effects of magnetization along different directions, ultimately contributing to quiescent solitons' formation.

The wave amplitudes along these two components are denoted by q and r. These amplitude functions describe the evolution of the optical field in the presence of the nonlinear effects induced by SPM—the coupled nature of Eqs. (2) and (3) signify that the dynamics of each component influence the other, leading to a rich interplay of nonlinear effects.

A critical aspect of the nonlinear interactions in this system is governed by the set of coefficients h_j, l_j, s_j, n_j, p_j, q_j for j=1,2. These coefficients are fixed parameters that define the strength of the self-phase modulation (SPM) in the waveguide system. In nonlinear optics, SPM arises due to the intensity-dependent refractive index, leading to a self-induced phase shift in the propagating wave. The parameters h_j, l_j, s_j, n_j, p_j, and q_j specifically quantify different aspects of the nonlinear response of the medium. h_j represents a fundamental contribution to the SPM effect, typically associated with the leading-order nonlinear interaction term. l_j accounts for additional higher-order nonlinear effects that contribute to the refractive index modulation. s_j governs the strength of cubic nonlinearity, which is critical for forming of soliton solutions. n_j modulates the influence of higher-order nonlinear effects beyond cubic terms, which may be relevant in highly nonlinear regimes. p_j contributes to the nonlinear dispersion effects, influencing the pulse-shaping dynamics. q_j represents the interaction between SPM and other nonlocal nonlinearities, playing a crucial role in stabilizing of quiescent solitons. These parameters are crucial in determining the specific characteristics of the optical solitons in the magneto-optic waveguide. By carefully selecting their values, one can control the strength and nature of the nonlinear effects, leading to different soliton behaviors, including quiescent (stationary) solitons and their dynamic counterparts.

The coupled system described by Eqs. (2) and (3) will be analyzed using the enhanced direct algebraic approach. This method is a powerful analytical technique for solving nonlinear differential equations and is particularly well-suited for extracting soliton solutions. The approach involves transforming the coupled system into a solvable algebraic form, allowing for the systematic derivation of exact soliton solutions.

Through this mathematical framework, the emergence of quiescent optical solitons will be rigorously established. Unlike their mobile counterparts, these solitons remain stationary in the propagation medium due to the balance between nonlinear SPM and the magneto-optic effects introduced through Q_j. The results obtained from this analysis will be presented in the subsequent sections, providing detailed insights into the characteristics of the recovered soliton solutions.

3. Mathematical analysis

In order to analyze Eqs. (2) and (3) further, we assume that the wave profiles have the following phase–amplitude split–up:

$$
\begin{aligned}
r(x,t) &= \phi_2(x)e^{i\lambda t},\\
q(x,t) &= \phi_1(x)e^{i\lambda t},
\end{aligned}
\tag{4}
$$

where $\phi_j(x)$ (j=1,2) are real functions and λ is a constant that stands for the frequency. Inserting Eq. (4) into Eqs. (2) and (3) , we get:

$$
\begin{aligned}
&-\lambda l\phi_1^{l}(x)+a_1(p+l)(p+l-1)\phi_1^{p+l-2}(x)\phi_1'^{2}(x)+a_1(p+l)\phi_1^{p+l-1}(x)\phi_1''(x)\\
&+b_1\phi_1^{n+l}(x)+c_1\phi_1^{2n+l}(x)+d_1\phi_1^{3n+l}(x)+e_1\phi_1^{4n+l}(x)+f_1n(n-1)\phi_1^{n-2+l}(x)\phi_1'^{2}(x)\\
&+f_1n\phi_1^{n-1+l}(x)\phi_1''(x)+g_1(2n)(2n-1)\phi_1^{2n-2+l}(x)\phi_1'^{2}(x)+g_1(2n)\phi_1^{2n-1+l}(x)\phi_1''(x)\\
&+h_1\phi_2^{n+l}(x)+l_1\phi_2^{2n+l}(x)+s_1\phi_2^{3n+l}(x)+n_1\phi_2^{4n+l}(x)+p_1n(n-1)\phi_2^{n-2+l}(x)\phi_2'^{2}(x)\\
&+p_1n\phi_2^{n-1+l}(x)\phi_2''(x)+q_1(2n)(2n-1)\phi_2^{2n-2+l}(x)\phi_2'^{2}(x)+q_1(2n)\phi_2^{2n-1+l}(x)\phi_2''(x)\\
&=Q_1\phi_2^{l}(x),
\end{aligned}
\tag{5}
$$

and

$$\begin{aligned}&-\lambda l\phi_2^l(x)+a_2(p+l)(p+l-1)\phi_2^{p+l-2}(x)\phi_2'^2(x)+a_2(p+l)\phi_2^{p+l-1}(x)\phi_2''(x)\\&+b_2\phi_2^{n+l}(x)+c_2\phi_2^{2n+l}(x)+d_2\phi_2^{3n+l}(x)+e_2\phi_2^{4n+l}(x)+f_2n(n-1)\phi_2^{n-2+l}(x)\phi_2'^2(x)\\&+f_2n\phi_2^{n-1+l}(x)\phi_2''(x)+g_2(2n)(2n-1)\phi_2^{2n-2+l}(x)\phi_2'^2(x)+g_2(2n)\phi_2^{2n-1+l}(x)\phi_2''(x)\\&+h_2\phi_1^{n+l}(x)+l_2\phi_1^{2n+l}(x)+s_2\phi_1^{3n+l}(x)+n_2\phi_1^{4n+l}(x)+p_2n(n-1)\phi_1^{n-2+l}(x)\phi_1'^2(x)\\&+p_2n\phi_1^{n-1+l}(x)\phi_1''(x)+q_2(2n)(2n-1)\phi_1^{2n-2+l}(x)\phi_1'^2(x)+q_2(2n)\phi_1^{2n-1+l}(x)\phi_1''(x)=Q_2\phi_1^l(x).\end{aligned}\tag{6}$$

Now, for the sake of simplicity, let us put

$$\phi_2(x)=\chi\phi_1(x),\tag{7}$$

where χ is a nonzero constant and $\chi\neq1$. Eqs. (5) and (6) become:

$$\begin{aligned}&-\lambda l-Q_1\chi^l+a_1(p+l)(p+l-1)\phi_1^{p-2}(x)\phi_1'^2(x)+a_1(p+l)\phi_1^{p-1}(x)\phi_1''(x)\\&+[b_1+h_1\chi^{n+l}]\phi_1^n(x)+[c_1+l_1\chi^{2n+l}]\phi_1^{2n}(x)+[d_1+s_1\chi^{3n+l}]\phi_1^{3n}(x)\\&+[e_1+n_1\chi^{4n+l}]\phi_1^{4n}(x)+[f_1+p_1\chi^{n+l}]n(n-1)\phi_1^{n-2}(x)\phi_1'^2(x)\\&+[f_1+p_1\chi^{n+l}]n\phi_1^{n-1}(x)\phi_1''(x)+[g_1+q_1\chi^{2n+l}](2n)(2n-1)\phi_1^{2n-2}(x)\phi_1'^2(x)\\&+[g_1+q_1\chi^{2n+l}](2n)\phi_1^{2n-1}(x)\phi_1''(x)=0,\end{aligned}\tag{8}$$

and

$$\begin{aligned}&-\lambda l\chi^l-Q_2+a_2(p+l)(p+l-1)\chi^{p+l}\phi_1^{p-2}(x)\phi_1'^2(x)+a_2(p+l)\chi^{p+l}\phi_1^{p-1}(x)\phi_1''(x)\\&+[b_2\chi^{n+l}+h_2]\phi_1^n(x)+[c_2\chi^{2n+l}+l_2]\phi_1^{2n}(x)+[d_2\chi^{3n+l}+s_2]\phi_1^{3n}(x)\\&+[e_2\chi^{4n+l}+n_2]\phi_1^{4n}(x)+[f_2\chi^{n+l}+p_2]n(n-1)\phi_1^{n-2}(x)\phi_1'^2(x)\\&+[f_2\chi^{n+l}+p_2]n\phi_1^{n-1}(x)\phi_1''(x)+[g_2\chi^{2n+l}+q_2](2n)(2n-1)\phi_1^{2n-2}(x)\phi_1'^2(x)\\&+[g_2\chi^{2n+l}+q_2](2n)\phi_1^{2n-1}(x)\phi_1''(x)=0.\end{aligned}\tag{9}$$

Eqs. (8) and (9) are equivalent along with constraints conditions:

$$\begin{aligned}&\lambda l+Q_1\chi^l=\lambda l\chi^l+Q_2,\quad a_1=a_2\chi^{p+l},\quad b_1+h_1\chi^{n+l}=b_2\chi^{n+l}+h_2,\\&c_1+l_1\chi^{2n+l}=c_2\chi^{2n+l}+l_2,\quad d_1+s_1\chi^{3n+l}=d_2\chi^{3n+l}+s_2,\quad e_1+n_1\chi^{4n+l}=e_2\chi^{4n+l}+n_2,\\&f_1+p_1\chi^{n+l}=f_2\chi^{n+l}+p_2,\quad g_1+q_1\chi^{2n+l}=g_2\chi^{2n+l}+q_2.\end{aligned}\tag{10}$$

On solving Eq. (8), let $p=3n$ then Eq. (8) changes to

$$\begin{aligned}&-\lambda l-Q_1\chi^l+a_1(3n+l)(3n+l-1)\phi_1^{3n-2}(x)\phi_1'^2(x)+a_1(3n+l)\phi_1^{3n-1}(x)\phi_1''(x)\\&+[b_1+h_1\chi^{n+l}]\phi_1^n(x)+[c_1+l_1\chi^{2n+l}]\phi_1^{2n}(x)+[d_1+s_1\chi^{3n+l}]\phi_1^{3n}(x)\\&+[e_1+n_1\chi^{4n+l}]\phi_1^{4n}(x)+[f_1+p_1\chi^{n+l}]n(n-1)\phi_1^{n-2}(x)\phi_1'^2(x)\\&+[f_1+p_1\chi^{n+l}]n\phi_1^{n-1}(x)\phi_1''(x)+[g_1+q_1\chi^{2n+l}](2n)(2n-1)\phi_1^{2n-2}(x)\phi_1'^2(x)\\&+[g_1+q_1\chi^{2n+l}](2n)\phi_1^{2n-1}(x)\phi_1''(x)=0.\end{aligned}\tag{11}$$

Balancing $\phi_1^{3n-1}(x)\phi_1''(x)$ with $\phi_1^{4n}(x)$ in Eq.(11) gives $N=2/n$, $n\neq1$. Using the transformation

$$\phi_1(x)=V^{2/n}(x),\tag{12}$$

where $V(x)$ is a new function. On substituting (12) into Eq.(11). Then Eq.(11) becomes

$$\begin{aligned}&\Delta_1+\Delta_2V^4(x)V'^2(x)+\Delta_3V^5(x)V''(x)+\Delta_4V^2(x)+\Delta_5V^4(x)\\&+\Delta_6V^6(x)+\Delta_7V^8(x)+\Delta_8V'^2(x)+\Delta_9V(x)V''(x)+\Delta_{10}V^2(x)V'^2(x)\\&+\Delta_{11}V^3(x)V''(x)=0,\end{aligned}\tag{13}$$

where

$$\begin{aligned}
&\Delta_1=-\left(\lambda l+Q_1\chi^l\right),\quad \Delta_2=\frac{2a_1}{n}(3n+l)\left[\frac{2n^2+5n+2l-2}{n}\right],\\
&\Delta_3=\frac{2a_1}{n}(3n+l),\Delta_4=b_1+h_1\chi^{n+l},\quad \Delta_5=c_1+l_1\chi^{2n+l},\\
&\Delta_6=d_1+s_1\chi^{3n+l},\quad \Delta_7=e_1+n_1\chi^{4n+l},\\
&\Delta_8=2\left(1-\frac{4}{n}\right)\left[f_1+p_1\chi^{n+l}\right],\quad \Delta_9=2\left[f_1+p_1\chi^{n+l}\right],\\
&\Delta_{10}=4\left(\frac{4}{n}+1\right)\left(\frac{2}{n}-1\right)\left[g_1+q_1\chi^{2n+l}\right],\quad \Delta_{11}=4\left[g_1+q_1\chi^{2n+l}\right].
\end{aligned}\tag{14}$$

Next, we will construct the solitons of Eqs. (2) and (3) by implementing the integration scheme in the next section.

4. The enhanced direct algebraic approach

Based on the enhanced direct algebraic method [5], we presume that Eq. (13) has the formal solution:

$$V(x)=\alpha_0+\sum_{i=1}^{N}\left\{\alpha_j\, F^i(x)+\beta_j\, F^{-j}(x)\right\}\ ,\tag{15}$$

where $\alpha_0,\alpha_j,\beta_j$ $(j=1,...,N)$ are arbitrary constants, provided $\alpha_N^2+\beta_N^2\neq 0,$ while $F(x)$ is the solution of the equation:

$$F'^2(x)=\sum_{l=0}^{4}L_l\, F^l(x),\tag{16}$$

where L_j $(j=0,1,2,3,4)$ are constants, provided $L_4\neq 0$. Balancing $V^5(x)V''(x)$ and $V^8(x)$ in Eq. (13), we get the balance number N=1. Now, Eq. (13) has the formal solution:

$$V(x)=\alpha_0+\alpha_1F(x)+\frac{\beta_1}{F(x)}\ ,\tag{17}$$

where α_0,α_1 and β_1 are constants to be determined, provided $\alpha_1^2+\beta_1^2\neq 0$. Substituting Eq. (17) along with Eq. (16) into Eq. (13) and setting all the coefficients of $F^{j_1}(\xi)\left(F'(\xi)\right)^{j_2}$, $\left(j_1=-8,...,-1,0,1,2,...,8,\ j_2=0,1\right)$ to zero, we obtain the system of algebraic equations:

$$\begin{aligned}
&F^8(\xi):\quad \Delta_7\alpha_1^8+\alpha_1^6L_4\left(\Delta_2+2\Delta_3\right)=0,\\
&F^7(\xi):\quad 8\Delta_7\alpha_0\alpha_1^7+\alpha_1^6L_3\left(\Delta_2+\frac{3}{2}\Delta_3\right)+2\alpha_0\alpha_1^5L_4\left(2\Delta_2+5\Delta_3\right)=0,\\
&F^9(\xi):\quad \Delta_{10}\alpha_1^4L_4+\alpha_1^6L_2\left(\Delta_3+\Delta_2\right)+28\Delta_7\alpha_0^2\alpha_1^6+8\Delta_7\beta_1\alpha_1^7+5\alpha_1^5\beta_1L_4\left(\Delta_2+5\Delta_3\right)\\
&\qquad +2\alpha_0^2\alpha_1^4L_4\left(3\Delta_2+10\Delta_3\right)+\alpha_0\alpha_1^5L_3\left(4\Delta_2+\frac{15}{2}\Delta_3\right)+\alpha_1^6\Delta_6+2\alpha_1^4L_4\Delta_{11}=0,\\
&F^5(\xi):\quad \alpha_1^4L_3\Delta_{10}+\alpha_1^6L_1\left(\Delta_2+\frac{1}{2}\Delta_3\right)\Delta_5\alpha_1^2+6\Delta_6\alpha_0\alpha_1^5+56\Delta_7\alpha_0^3\alpha_1^5+\frac{3}{2}\alpha_1^4L_3\Delta_{11}\\
&\qquad +4\alpha_0\alpha_1^4\beta_1L_4\left(\Delta_2+10\Delta_3\right)+4\alpha_0^3\alpha_1^3L_4\left(\Delta_2+10\Delta_3\right)+4\alpha_0^3\alpha_1^3L_4\left(\Delta_2+5\Delta_3\right)\\
&\qquad +2\alpha_0\alpha_1^5L_4\left(4\Delta_2+5\Delta_3\right)+56\Delta_7\alpha_0^3\alpha_1^5+6\Delta_{11}\alpha_0\alpha_1^3L_4+\Delta_2\alpha_1^2L_2+\alpha_1^2L_2\Delta_9\\
&\qquad +20\Delta_8\beta_1\alpha_0\alpha_1^3+2\alpha_0\alpha_1^3L_4\Delta_{10}=0,
\end{aligned}\tag{18}$$

continued on the next page

$$
\begin{aligned}
F^4(\xi):\ & 2\alpha_1^4 L_4\beta_1^2(\Delta_2+20\Delta_3)+\alpha_0^3\alpha_1^3 L_3(15\Delta_3+4\Delta_2)+\alpha_0^2\alpha_1^4 L_3(6\Delta_2+10\Delta_3)\\
&+\alpha_0\alpha_1^5 L_1\left(2\Delta_2+\frac{5}{2}\Delta_3\right)+2\alpha_1^5 L_2\beta_1(\Delta_2+3\Delta_3)+4\Delta_7\alpha_0^3\alpha_1\\
&+\Delta_1\alpha_1^5+\Delta_5\alpha_1^4+\Delta_8 L_4\alpha_1^2+\alpha_0^4\alpha_1^2 L_4(10\Delta_3+\Delta_2)+6\Delta_{11}L_4\alpha_1^3\beta_1\\
&+2\Delta_{10}\alpha_0\alpha_1^3 L_3+3\Delta_{11}\alpha_0\alpha_1^2\left(\alpha_0 L_4+\frac{9}{2}\alpha_1\beta_1 L_3\right)=0,\\
F^3(\xi):\ & \Delta_3\alpha_0 L_4\beta_1^2-\Delta_3\alpha_1^2 L_1\beta_1-\Delta_3\alpha_1 L_3\beta_1^2+\Delta_3\alpha_0 L_0\alpha_1^2+\frac{1}{2}\alpha_0\alpha_1 L_1\Delta_9\\
&+2\alpha_1\beta_1 L_2\Delta_9-2\Delta_2\alpha_1 L_2\beta_1+\frac{5}{2}\alpha_1^2\beta_1 L_1\Delta_4+\frac{5}{2}\alpha_1\beta_1 L_3\Delta_4+\frac{1}{2}\alpha_1\alpha_0^2 L_1\Delta_4\\
&+6\Delta_6\alpha_0\alpha_1\beta_1+12\Delta_7\alpha_0^2\alpha_1\beta_1+\Delta_8\alpha_0^5+4\Delta_4\alpha_0\alpha_1\beta_1 L_2+\Delta_1\alpha_0+\alpha_0^2\Delta_5+\alpha_0^3\Delta_6\\
&+\Delta_2\alpha_1^2 L_0+\Delta_2\beta_1^2 L_4-2\Delta_3\alpha_0\alpha_1\beta_1 L_2+2\Delta_5\alpha_1\beta_1+6\Delta_7\alpha_1^2\beta_1^2+20\Delta_8\alpha_0^3\alpha_1\beta_1\\
&+30\Delta_8\alpha_0\alpha_1^2\beta_1^2+\frac{1}{2}\beta_1\alpha_0^2 L_3\Delta_4+\alpha_0^4\Delta_7+\frac{1}{2}\alpha_0\beta_1 L_3\Delta_9=0,\\
F^2(\xi):\ & 5\Delta_8\beta_1\left(\alpha_0^4+6\alpha_1\alpha_0^2\beta_1+2\alpha_1^2\beta_1^2\right)+\alpha_0\beta_1^2 L_3(\Delta_3+\Delta_4)+2\alpha_0\alpha_1 L_1\beta_1(-\Delta_3+2\Delta_4)\\
&+\Delta_8 L_4\beta_1^3+\alpha_1^2 L_0\beta_1(-\Delta_3+3\Delta_4)+\alpha_1 L_2\beta_1^2(-\Delta_3+3\Delta_4)+\Delta_9\alpha_0^2 L_2\beta_1\\
&+4\Delta_8\alpha_0\beta_1\left(\alpha_0^2+4\alpha_1\beta_1\right)+3\Delta_5\beta_1\left(\alpha_0^2+\alpha_1\beta_1\right)+\Delta_8\beta_1(-2\alpha_1 L_1+L_3\beta_1)\\
&+2\Delta_4\alpha_0\beta_1=0,\\
F(\xi):\ & \Delta_4\beta_1^2+\frac{3}{2}\beta_1 L_1+10\Delta_7\alpha_0\beta_1^2\left(\alpha_0^2+2\alpha_1\beta_1\right)-2\Delta_8 L_0\alpha_1\beta_1+\frac{3}{2}\Delta_3\alpha_0^2\beta_1 L_1\\
&+\alpha_0\beta_1^2 L_2(\Delta_2+2\Delta_3)+\alpha_1 L_1\beta_1^2\left(-\Delta_2+\frac{7}{2}\Delta_3\right)+2\Delta_6\beta_1^2\left(3\alpha_0^2+2\alpha_1\beta_1\right)\\
&+\beta_1^3 L_3\left(\Delta_2+\frac{1}{2}\Delta_3\right)+\Delta_8 L_2\beta_1^2+3\Delta_5\alpha_0\beta_1^2+2\alpha_0\alpha_1 L_0\beta_1(-\Delta_2+2\Delta_3)=0,\\
F^0(\xi):\ & 5\Delta_8\beta_1^3\left(2\alpha_0^2+\alpha_1\beta_1\right)+\alpha_0\beta_1^2 L_1(\Delta_2+\Delta_3)+\beta_1^3 L_2(\Delta_2+3\Delta_3)\\
&+\alpha_1\beta_1^2 L_0(-\Delta_2+4\Delta_3)+2\alpha_0\beta_1 L_0\Delta_9+4\alpha_0\beta_1^3\Delta_7+\Delta_6\beta_1^3\\
&+\Delta_8\beta_1^2 L_1+2\beta_1 L_0=0,\\
F^{-1}(\xi):\ & \Delta_7\beta_1^4+\beta_1^3 L_1\left(\Delta_2+\frac{3}{2}\Delta_3\right)+\beta_1^2 L_0(\Delta_2+2\Delta_3)+5\Delta_8\alpha_0\beta_1^2\\
&+\alpha_0\beta_1^2 L_0(\Delta_2+4\Delta_3)+\beta_1^3 L_3\left(\Delta_2+\frac{1}{2}\Delta_3\right)+\Delta_8 L_2\beta_1^2+3\Delta_5\alpha_0\beta_1^2\\
&+2\alpha_0\alpha_1 L_0\beta_1(-\Delta_2+2\Delta_3)=0,\\
F^{-2}(\xi):\ & \Delta_7\beta_1^4+\beta_1^3 L_1\left(\Delta_2+\frac{3}{2}\Delta_3\right)+\beta_1^2 L_0(\Delta_2+2\Delta_3)+5\Delta_8\alpha_0\beta_1^2\\
&+\alpha_0\beta_1^2 L_0(\Delta_2+4\Delta_3)+3\Delta_5\alpha_0\beta_1^2+2\alpha_0\alpha_1 L_0\beta_1(-\Delta_2+2\Delta_3)\\
&+\beta_1^3 L_3\left(\Delta_2+\frac{1}{2}\Delta_3\right)+\Delta_8 L_2\beta_1^2=0,\\
F^{-3}(\xi):\ & 5\Delta_8\beta_1^3\left(2\alpha_0^2+\alpha_1\beta_1\right)+\alpha_0\beta_1^2 L_1(\Delta_2+\Delta_3)+\beta_1^3 L_2(\Delta_2+3\Delta_3)\\
&+2\alpha_0\beta_1 L_0\Delta_9+4\alpha_0\beta_1^3\Delta_7+\Delta_6\beta_1^3+\Delta_8\beta_1^2 L_1+2\beta_1 L_0+\Delta_1\alpha_0\\
&+6\Delta_6\alpha_0\alpha_1\beta_1+12\Delta_7\alpha_0^2\alpha_1\beta_1+\Delta_8\alpha_0^5+4\Delta_4\alpha_0\alpha_1\beta_1 L_2\\
&+\alpha_0^2\Delta_5+\alpha_0^3\Delta_6+\alpha_1\beta_1^2 L_0(-\Delta_2+4\Delta_3)=0,
\end{aligned}
\tag{18}
$$

continued on the next page

$$
\begin{aligned}
F^{-4}(\xi):\ & \triangle_7\beta_1^4+\beta_1^3L_1\left(\triangle_2+\frac{3}{2}\triangle_3\right)+\beta_1^2L_0\left(\triangle_2+2\triangle_3\right)+\beta_1^3L_3\left(\triangle_2+\frac{1}{2}\triangle_3\right)\\
& +5\triangle_8\alpha_0\beta_1^2+\alpha_0\beta_1^2L_0\left(\triangle_2+4\triangle_3\right)+2\alpha_0\alpha_1L_0\beta_1\left(-\triangle_2+2\triangle_3\right)=0,\\
F^{-5}(\xi):\ & \beta_1^4L_3\triangle_{10}+\alpha_1^6L_1\left(\triangle_2+\frac{1}{2}\triangle_3\right)\triangle_5\beta_1^2+6\triangle_6\alpha_0\beta_1^5+56\triangle_7\alpha_0^3\beta_1^5\\
& +4\alpha_0\beta_1^4\alpha_1L_4\left(\triangle_2+10\triangle_3\right)+4\alpha_0^3\beta_1^3L_4\left(\triangle_2+10\triangle_3\right)\\
& +4\alpha_0^3\alpha_1^3L_4\left(\triangle_2+5\triangle_3\right)+2\alpha_0\beta_1^5L_4\left(4\triangle_2+5\triangle_3\right)+56\triangle_7\alpha_0^3\beta_1^5\\
& +6\triangle_{11}\alpha_0\beta_1^3L_4+\triangle_2\beta_1^2L_2+\beta_1^2L_2\triangle_9\\
& +20\triangle_8\alpha_1\alpha_0\beta_1^3+2\alpha_0\beta_1^3L_4\triangle_{10}+\frac{3}{2}\beta_1^4L_3\triangle_{11}=0, \qquad (18)\\
F^{-6}(\xi):\ & \triangle_{10}\beta_1^4L_0+\beta_1^6L_2\left(\triangle_3+\triangle_2\right)+28\triangle_7\alpha_0^2\beta_1^6+8\triangle_7\beta_1^7\alpha_1\\
& +5\alpha_1^5\beta_1L_4\left(\triangle_2+5\triangle_3\right)+2\alpha_0^2\beta_1^4L_4\left(3\triangle_2+10\triangle_3\right)+\alpha_1\beta_1^5L_0\left(4\triangle_2+5\triangle_3\right)\\
& +\beta_1^6\triangle_6+2\beta_1^4L_4\triangle_{11}\\
& +\alpha_0\beta_1^5L_1\left(4\triangle_2+\frac{15}{2}\triangle_3\right)=0,\\
F^{-7}(\xi):\ & 8\triangle_7\alpha_0\beta_1^7+\beta_1^6L_1\left(\triangle_2+\frac{3}{2}\triangle_3\right)+2\alpha_0\beta_1^5L_0\left(2\triangle_2+5\triangle_3\right)=0,\\
F^{-8}(\xi):\ & \triangle_7\beta_1^8+\beta_1^6L_0\left(\triangle_2+2\triangle_3\right)=0,
\end{aligned}
$$

Now, let us discuss the following cases for the algebraic system Eqs. (18), which can be solved using Maple to discover the unknown parameters in Eq. (13).

Case–1: If we set $L_0 = L_1 = L_3 = 0,$ in the algebraic system Eqs. (18) and by using the Maple, then we have the results

$$
\beta_1=0,\ \alpha_1=\sqrt{-\frac{L_4\triangle_3}{2\triangle_7}},\ L_2=-\frac{2\alpha_0^2L_4}{\alpha_1^2}, \qquad (19)
$$

with constraint conditions:

$$
\begin{aligned}
& \triangle_1=-\frac{3\alpha_0^4L_4\triangle_9}{2\alpha_1^2},\quad \triangle_2=-\frac{3\triangle_3}{2},\quad \triangle_4=\frac{\alpha_0^2L_4\left(4\triangle_9-3\triangle_{11}\alpha_0^2\right)}{2\alpha_1^2},\\
& \triangle_5=-\frac{L_4\left(3\triangle_3\alpha_0^4-4\triangle_{11}\alpha_0^2+\triangle_9\right)}{2\alpha_1^2},\quad \triangle_6=\frac{L_4\left(4\triangle_3\alpha_0^2-\triangle_{11}\right)}{2\alpha_1^2}, \qquad (20)\\
& \triangle_8=-\frac{3\triangle_9}{2},\quad \triangle_{10}=-\frac{3\triangle_{11}}{2},
\end{aligned}
$$

When $L_2>0$, $L_4<0$ and $\triangle_7\triangle_3>0$. Then Eqs. (2) and (3) have bright soliton solutions:

$$
q(x,t)=\left[\alpha_0\left(1+\sqrt{2}\,\mathrm{sech}\sqrt{-\frac{2\alpha_0^2L_4}{\alpha_1^2}}x\right)\right]^{\frac{2}{n}}e^{i\lambda t}, \qquad (21)
$$

and

$$
r(x,t)=\chi\left[\alpha_0\left(1+\sqrt{2}\,\mathrm{sech}\sqrt{-\frac{2\alpha_0^2L_4}{\alpha_1^2}}x\right)\right]^{\frac{2}{n}}e^{i\lambda t}, \qquad (22)
$$

provided $\alpha_0>0.$ The solutions (21)-(22) exist under the constraint conditions (20).

Case–2: If we set $L_0 = L_2^2/4L_4$, $L_1 = L_3 = 0$, in the algebraic system Eqs. (18), then we have the results:

$$\alpha_0 = \beta_1 = 0, \quad \alpha_1 = \sqrt{-\frac{L_4(\triangle_2+2\triangle_3)}{\triangle_7}}, \tag{23}$$

with constraint conditions:

$$\begin{aligned}
&\triangle_1 = -\frac{\alpha_1^2 L_2^2 \triangle_8}{4L_4}, \quad \triangle_4 = -\frac{L_2\left(\alpha_1^2 L_2 \triangle_{10} + 4\triangle_8 L_4 + 4\triangle_9 L_4\right)}{4L_4},\\
&\triangle_5 = -\frac{\alpha_1^4 L_2^2 \triangle_2 + 4\triangle_{10}\alpha_1^2 L_2 L_4 + 4\triangle_{11}\alpha_1^2 L_4 L_2 + 4\triangle_8 L_4^2 + 8\triangle_9 L_4^2}{4L_4\alpha_1^2},\\
&\triangle_6 = -\frac{\alpha_1^2 L_2 \triangle_2 + \alpha_1^2 L_2 \triangle_3 + \triangle_{10} L_4 + 2\triangle_{11} L_4}{\alpha_1^2}.
\end{aligned} \tag{24}$$

When $L_4 > 0, L_2 < 0, \triangle_7(\triangle_2+2\triangle_3) < 0$. Then Eqs. (2) and (3) have the dark soliton solutions:

$$q(x,t) = \left[\sqrt{\frac{L_2(\triangle_2+2\triangle_3)}{2\triangle_7}}\tanh\sqrt{-\frac{L_2}{2}}x\right]^{\frac{2}{n}} e^{i\lambda t}, \tag{25}$$

and

$$r(x,t) = \chi\left[\sqrt{\frac{L_2(\triangle_2+2\triangle_3)}{2\triangle_7}}\tanh\sqrt{-\frac{L_2}{2}}x\right]^{\frac{2}{n}} e^{i\lambda t}, \tag{26}$$

Also, Eqs. (2) and (3) have singular soliton solutions:

$$q(x,t) = \left[\sqrt{\frac{L_2(\triangle_2+2\triangle_3)}{2\triangle_7}}\coth\sqrt{-\frac{L_2}{2}}x\right]^{\frac{2}{n}} e^{i\lambda t}, \tag{27}$$

and

$$r(x,t) = \chi\left[\sqrt{\frac{L_2(\triangle_2+2\triangle_3)}{2\triangle_7}}\coth\sqrt{-\frac{L_2}{2}}x\right]^{\frac{2}{n}} e^{i\lambda t}. \tag{28}$$

The solutions (25)-(28) exist under the constraint conditions (24).

Case–3: If we set $L_1 = L_3 = 0$, in the algebraic system Eqs. (18), then we have the following:

(I) When $L_0 = \frac{m_1^2(1-m_1^2)L_2}{(2m_1^2-1)L_4}$, $0 < m_1 < 1$, we get

$$\alpha_1 = 0, \; L_4 = \frac{2\alpha_0^2 L_2 m_1^2(m_1^2-1)}{\beta_1^2(2m_1^2-1)^2}, \tag{29}$$

where m_1 is the modulus of Jacobi elliptic functions, with constraint conditions:

$$\begin{aligned}
&\triangle_1 = \frac{3\alpha_0^2 L_2 \triangle_9\left[8m_1^4 - 8m_1^2 + 1\right]}{4(2m_1^2-1)^2}, \quad \triangle_2 = -\frac{6\triangle_7\alpha_0^2}{L_2}, \quad \triangle_3 = \frac{4\triangle_7\alpha_0^2}{L_2},\\
&\triangle_4 = -\frac{L_2\left[-24m_1^4\alpha_0^2\triangle_{11} + 16m_1^4\triangle_9 + 24m_1^2\alpha_0^2\triangle_{11} - 16m_1^2\triangle_9 + 3\triangle_{11}\alpha_0^2 + 4\triangle_9\right]}{4(2m_1^2-1)^2},\\
&\triangle_5 = \frac{12\alpha_0^6\triangle_7\left(8m_1^4 - 8m_1^2 + 1\right) + 4\alpha_0^2\triangle_{11}L_2\left(-4m_1^4 + 4m_1^2 + 1\right) + \triangle_9 L_2\left(4m_1^4 - 4m_1^2 + 1\right)}{4\alpha_0^2(2m_1^2-1)^2},\\
&\triangle_6 = -\frac{16\triangle_7\alpha_0^4 - \triangle_{11}L_2}{\alpha_0^2}, \quad \triangle_8 = -\frac{3\triangle_9}{2}, \quad \triangle_{10} = -\frac{3\triangle_{11}}{2},
\end{aligned} \tag{30}$$

Now, Eqs. (2) and (3) have the Jacobi elliptic doubly periodic type soliton solutions:

$$q(x,t)=\left[\alpha_0\left(1+\frac{1}{\sqrt{-\frac{(2m_1^2-1)}{2(m_1^2-1)}}\,\mathrm{cn}\left(\sqrt{\frac{L_2}{2m_1^2-1}}x,m_1\right)}\right)\right]^{\frac{2}{n}}e^{i\lambda t}, \tag{31}$$

and

$$r(x,t)=\chi\left[\alpha_0\left(1+\frac{1}{\sqrt{-\frac{(2m_1^2-1)}{2(m_1^2-1)}}\,\mathrm{cn}\left(\sqrt{\frac{L_2}{2m_1^2-1}}x,m_1\right)}\right)\right]^{\frac{2}{n}}e^{i\lambda t}, \tag{32}$$

provided $L_4>0,(2m_1^2-1)L_2>0$, $\alpha_0>0$. The solutions (31), (32) exist under the constraint conditions (30).

(II) When $L_0=\frac{(1-m_1^2)L_2^2}{(2-m_1^2)^2L_4}$, $0<m_1<1$, we get

$$\alpha_1=0,\ L_4=\frac{2\alpha_0^2L_2(m_1^2-1)}{\beta_1^2(m_1^2-2)^2}, \tag{33}$$

with constraint conditions:

$$\begin{gathered}\vartriangle_1=\frac{3\alpha_0^2L_2\vartriangle_9m_1^4}{4(m_1^2-2)^2},\quad \vartriangle_2=-\frac{6\vartriangle_7\alpha_0^2}{L_2},\quad \vartriangle_3=\frac{4\vartriangle_7\alpha_0^2}{L_2},\\ \vartriangle_4=-\frac{L_2\left[-3m_1^4\alpha_0^2\vartriangle_{11}+4m_1^4\vartriangle_9-16m_1^2\vartriangle_9+16\vartriangle_9\right]}{4(m_1^2-2)^2},\\ \vartriangle_5=\frac{12m_1^4\alpha_0^6\vartriangle_7-4\alpha_0^2\vartriangle_{11}L_2(m_1^4+4m_1^2+1)+\vartriangle_9L_2(m_1^4-4m_1^2+4)}{4\alpha_0^2(m_1^2-2)^2},\\ \vartriangle_6=-\frac{16\vartriangle_7\alpha_0^4-\vartriangle_{11}L_2}{4\alpha_0^2},\quad \vartriangle_8=-\frac{3\vartriangle_9}{2},\quad \vartriangle_{10}=-\frac{3\vartriangle_{11}}{2},\end{gathered} \tag{34}$$

Now, Eqs. (2) and (3) have the Jacobi elliptic doubly periodic type soliton solutions:

$$q(x,t)=\left[\alpha_0\left(1+\frac{1}{\sqrt{-\frac{m_1^2(m_1^2-2)^2}{2(2-m_1^2)(m_1^2-1)}}\,\mathrm{dn}\left(\sqrt{\frac{L_2}{2-m_1^2}}x,m_1\right)}\right)\right]^{\frac{2}{n}}e^{i\lambda t}, \tag{35}$$

and

$$r(x,t)=\chi\left[\alpha_0\left(1+\frac{1}{\sqrt{-\frac{m_1^2(m_1^2-2)^2}{2(2-m_1^2)(m_1^2-1)}}\,\mathrm{dn}\left(\sqrt{\frac{L_2}{2-m_1^2}}x,m_1\right)}\right)\right]^{\frac{2}{n}}e^{i\lambda t}, \tag{36}$$

provided $L_4>0,(2-m_1^2)L_2>0$, $\alpha_0>0$. The solutions (35)-(36) is existed under the constraint conditions (34).

(III) When $L_0=\frac{m_1^2L_2^2}{(m_1^2+1)^2L_4}$, $0<m_1<1$, we get

$$\alpha_1 = 0, \quad ,L_2 = -\frac{\beta_1^2 L_4\left(m_1^4 + 2m_1^2 + 1\right)}{2\alpha_0^2 m_1^2}, \tag{37}$$

with constraint conditions:

$$\begin{aligned}
&\vartriangle_1 = \frac{3\beta_1^2 L_4 \vartriangle_9\left(m_1^4 - 2m_1^2 + 1\right)}{8m_1^2}, \quad \vartriangle_2 = -\frac{12\vartriangle_7 m_1^2 \alpha_0^4}{\beta_1^2 L_4\left(m_1^4 + 2m_1^2 + 1\right)}, \\
&\vartriangle_3 = -\frac{8\vartriangle_7 m_1^2 \alpha_0^4}{\beta_1^2 L_4\left(m_1^4 + 2m_1^2 + 1\right)}, \\
&\vartriangle_4 = \frac{\left[6m_1^2\alpha_0^6\left(4\alpha_0^2\vartriangle_7 + \vartriangle_6\right)\left(m_1^4 - 2m_1^2 + 1\right) + \beta_1^2 L_4 \vartriangle_9\left(m_1^8 + 4m_1^6 + 6m_1^4 + 4m_1^2 + 1\right)\right]}{2m_1^2\alpha_0^2\left(m_1^4 + 2m_1^2 + 1\right)}, \\
&\vartriangle_5 = -\frac{\left[\begin{array}{l} m_1^2\alpha_0^8\vartriangle_7\left(104m_1^4 + 304m_1^2 + 104\right) + 32m_1^2\alpha_0^6\vartriangle_7\left(m_1^4 + 2m_1^2 + 1\right) \\ +\beta_1^2 L_4\vartriangle_9\left(m_1^8 + 4m_1^6 + 6m_1^4 + 4m_1^2 + 1\right)\end{array}\right]}{8m_1^2\alpha_0^4\left(m_1^4 + 2m_1^2 + 1\right)}, \\
&\vartriangle_8 = -\frac{3\vartriangle_9}{2}, \quad \vartriangle_{10} = \frac{12m_1^2\alpha_0^4\left(4\vartriangle_7\alpha_0^2 + \vartriangle_6\right)}{\beta_1^2 L_4\left(m_1^4 + 2m_1^2 + 1\right)}, \quad \vartriangle_{11} = -\frac{8m_1^2\alpha_0^4\left(4\vartriangle_7\alpha_0^2 + \vartriangle_6\right)}{\beta_1^2 L_4\left(m_1^4 + 2m_1^2 + 1\right)}.
\end{aligned} \tag{38}$$

Now, Eqs. (2) and (3) have the Jacobi elliptic doubly periodic type soliton solutions:

$$q(x,t) = \left[\alpha_0\left(1 + \frac{\sqrt{2}}{\sqrt{\frac{m_1^4 + 2m_1^2 + 1}{\left(m_1^2 + 1\right)}}\,\mathrm{sn}\left(\sqrt{\frac{\beta_1^2 L_4\left(m_1^4 + 2m_1^2 + 1\right)}{2\alpha_0^2 m_1^2\left(m_1^2 + 1\right)}}x, m_1\right)}\right)\right]^{\frac{2}{n}} e^{i\lambda t}, \tag{39}$$

and

$$r(x,t) = \chi\left[\alpha_0\left(1 + \frac{\sqrt{2}}{\sqrt{\frac{m_1^4 + 2m_1^2 + 1}{\left(m_1^2 + 1\right)}}\,\mathrm{sn}\left(\sqrt{\frac{\beta_1^2 L_4\left(m_1^4 + 2m_1^2 + 1\right)}{2\alpha_0^2 m_1^2\left(m_1^2 + 1\right)}}x, m_1\right)}\right)\right]^{\frac{2}{n}} e^{i\lambda t}, \tag{40}$$

provided $\alpha_0 > 0$, $L_4 > 0, L_2 < 0$. In particular, when $m_1 \to 1^-$ in Eqs. (39) and (40), we have the singular soliton solutions:

$$q(x,t) = \left[\alpha_0\left(1 + \frac{1}{\tanh\left(\sqrt{\frac{\beta_1^2 L_4}{\alpha_0^2}}x\right)}\right)\right]^{\frac{2}{n}} e^{i\lambda t}, \tag{41}$$

and

$$r(x,t) = \chi\left[\alpha_0\left(1 + \frac{1}{\tanh\left(\sqrt{\frac{\beta_1^2 L_4}{\alpha_0^2}}x\right)}\right)\right]^{\frac{2}{n}} e^{i\lambda t}. \tag{42}$$

The solutions (41)-(42) exist under the constraint conditions (38).

Case–4: If we set $L_0 = L_1 = 0$, in the algebraic system Eqs. (18), then we have the results:

$$\beta_1 = 0,\ L_2 = -\frac{\alpha_0\left(4\alpha_0 L_4 - 3\alpha_1 L_3\right)}{2\alpha_1^2}, \tag{43}$$

with constraint conditions:

$$\begin{aligned}
&\triangle_1=-\frac{3\alpha_0^3\triangle_9\left(2\alpha_0L_4-\alpha_1L_3\right)}{4\alpha_1^2},\quad \triangle_2=-\frac{3\triangle_3}{2},\\
&\triangle_4=\frac{\alpha_0\left(-6\triangle_{11}\alpha_0^3L_4+3\triangle_{11}\alpha_0^2\alpha_1L_3+8\triangle_9\alpha_0L_4-3\triangle_9\alpha_1L_3\right)}{4\alpha_1^2},\\
&\triangle_5=-\frac{6\triangle_3\alpha_0^4L_4-3\triangle_3\alpha_0^3\alpha_1L_3-8\triangle_{11}\alpha_0^2L_4+3\triangle_{11}\alpha_0\alpha_1L_3+2\triangle_9L_4}{4\alpha_1^2},\\
&\triangle_6=\frac{8\triangle_3\alpha_0^2L_4-3\alpha_0\alpha_1L_3-2\triangle_{11}L_4}{4\alpha_1^2},\quad \triangle_7=-\frac{L_4\triangle_3}{2\alpha_1^2},\\
&\triangle_8=-\frac{3\triangle_9}{2},\quad \triangle_{10}=-\frac{3\triangle_{11}}{2},
\end{aligned} \tag{44}$$

Now, Eqs. (2) and (3) have the straddled soliton solutions as the following:

(I) when $L_2>0$, $L_4>0$ we get bright-dark solitons:

$$q(x,t)=\varepsilon\left[\alpha_0-\frac{\alpha_1L_2\operatorname{sech}^2\left(\frac{\sqrt{L_2}}{2}x\right)}{2\sqrt{L_2L_4}\tanh\left(\frac{\sqrt{L_2}}{2}x\right)+L_3}\right]^{\frac{2}{n}}e^{i\lambda t}, \tag{45}$$

and

$$r(x,t)=\chi\varepsilon\left[\alpha_0-\frac{\alpha_1L_2\operatorname{sech}^2\left(\frac{\sqrt{L_2}}{2}x\right)}{2\sqrt{L_2L_4}\tanh\left(\frac{\sqrt{L_2}}{2}x\right)+L_3}\right]^{\frac{2}{n}}e^{i\lambda t}, \tag{46}$$

also singular-singular solitons:

$$q(x,t)=\varepsilon\left[\alpha_0+\frac{\alpha_1L_2\operatorname{csch}^2\left(\frac{\sqrt{L_2}}{2}x\right)}{2\sqrt{L_2L_4}\coth\left(\frac{\sqrt{L_2}}{2}x\right)+L_3}\right]^{\frac{2}{n}}e^{i\lambda t}, \tag{47}$$

and

$$r(x,t)=\chi\varepsilon\left[\alpha_0+\frac{\alpha_1L_2\operatorname{csch}^2\left(\frac{\sqrt{L_2}}{2}x\right)}{2\sqrt{L_2L_4}\coth\left(\frac{\sqrt{L_2}}{2}x\right)+L_3}\right]^{\frac{2}{n}}e^{i\lambda t}. \tag{48}$$

where ε is a constant.

(II) when $L_2>0$ and $L_3\neq 0$

$$q(x,t)=\varepsilon\left[\alpha_0-\frac{\alpha_1L_2L_3\operatorname{sech}^2\left(\frac{\sqrt{L_2}}{2}x\right)}{L_3^2-L_2L_4\left[1-\tanh\left(\frac{\sqrt{L_2}}{2}x\right)\right]^2}\right]^{\frac{2}{n}}e^{i\lambda t}, \tag{49}$$

and

$$r(x,t)=\chi\varepsilon\left[\alpha_0-\frac{\alpha_1 L_2 L_3 \operatorname{sech}^2\left(\frac{\sqrt{L_2}}{2}x\right)}{L_3^2-L_2L_4\left[1-\tanh\left(\frac{\sqrt{L_2}}{2}x\right)\right]^2}\right]^{\frac{2}{n}} e^{i\lambda t}, \tag{50}$$

also,

$$q(x,t)=\varepsilon\left[\alpha_0+\frac{\alpha_1 L_2 L_3 \operatorname{csch}^2\left(\frac{\sqrt{L_2}}{2}x\right)}{L_3^2-L_2L_4\left[1-\coth\left(\frac{\sqrt{L_2}}{2}x\right)\right]^2}\right]^{\frac{2}{n}} e^{i\lambda t}, \tag{51}$$

and

$$r(x,t)=\chi\varepsilon\left[\alpha_0+\frac{\alpha_1 L_2 L_3 \operatorname{csch}^2\left(\frac{\sqrt{L_2}}{2}x\right)}{L_3^2-L_2L_4\left[1-\coth\left(\frac{\sqrt{L_2}}{2}x\right)\right]^2}\right]^{\frac{2}{n}} e^{i\lambda t}. \tag{52}$$

Here, Eqs. (49) and (50) describe bright-dark solitons, while (51) and (52) - singular-singular solitons. The solutions (45)-(52) exist under the constraint conditions (44).

Case–5: If we set $L_1 = L_3 = 0$, in the algebraic system Eqs. (18), then we have the results:

$$\alpha_0=\beta_1=0,\ \alpha_1=\sqrt{-\frac{L_4(\triangle_2+2\triangle_3)}{\triangle_7}}\ , \tag{53}$$

with constraint conditions:

$$\begin{gathered}\triangle_1=-\triangle_8\alpha_1^2L_0,\quad \triangle_4=-\triangle_{10}\alpha_1^2L_0-L_2(\triangle_8+\triangle_9),\\ \triangle_5=-\frac{\triangle_2\alpha_1^4L_0+\alpha_1^2L_2(\triangle_{10}+\triangle_{11})+L_4(\triangle_8+2\triangle_9)}{\alpha_1^2},\\ \triangle_6=-\frac{\alpha_1^2L_2(\triangle_2+\triangle_3)+L_4(\triangle_{10}+2\triangle_{11})}{\alpha_1^2}.\end{gathered} \tag{54}$$

Now, Eqs. (2) and (3) have the Weierstrass elliptic doubly periodic type solutions:

$$q(x,t)=\left[3\sqrt{-\frac{(\triangle_2+2\triangle_3)}{\triangle_7}}\left(\frac{\wp'[(x),g_2,g_3]}{6\wp[(x),g_2,g_3]+L_2}\right)\right]^{\frac{2}{n}} e^{i\lambda t}, \tag{55}$$

and

$$r(x,t)=\chi\left[3\sqrt{-\frac{(\triangle_2+2\triangle_3)}{\triangle_7}}\left(\frac{\wp'[(x),g_2,g_3]}{6\wp[(x),g_2,g_3]+L_2}\right)\right]^{\frac{2}{n}} e^{i\lambda t}, \tag{56}$$

where $L_4>0, \triangle_7(\triangle_2+2\triangle_3)<0$. Also, Eqs. (2) and (3) have

$$q(x,t)=\left[\sqrt{-\frac{L_4L_0(\triangle_2+2\triangle_3)}{9\triangle_7}}\left(\frac{6\wp[(x),g_2,g_3]+L_2}{\wp'[(x),g_2,g_3]}\right)\right]^{\frac{2}{n}} e^{i\lambda t}, \tag{57}$$

and

$$r(x,t)=\chi\left[\sqrt{-\frac{L_4L_0(\triangle_2+2\triangle_3)}{9\triangle_7}}\left(\frac{6\wp[(x),g_2,g_3]+L_2}{\wp'[(x),g_2,g_3]}\right)\right]^{\frac{2}{n}} e^{i\lambda t}, \tag{58}$$

where $L_0 > 0, L_4 > 0, \vartriangle_7(\vartriangle_2 + 2\vartriangle_3) < 0$. The solutions (55)-(58) are existed under constraint conditions (54). The soliton solutions given in Eqs. (55)–(58) exist only under specific constraint conditions defined by Eq. (54). These constraints impose necessary mathematical and physical conditions that must be satisfied for the solitons to be valid solutions within the magneto-optic waveguide system.

A crucial aspect of these constraints is the relationship between the magneto-optic parameters Q_1 and Q_2, which is explicitly governed by Eq. (10). This relationship defines how the two magneto-optic components interact and influence the soliton dynamics. The presence of Q_1 and Q_2 in the system indicates that the external magnetic field directly shapes the nonlinear optical response, affecting both the soliton stability and propagation characteristics.

Furthermore, in Eq. (14), the parameter $\vartriangle_1$ is explicitly dependent on Q_1. This dependency highlights that the properties of the nonlinear wave solutions are not only governed by the inherent material properties of the waveguide but are also externally tunable through the applied magnetic field. This tunability is a crucial feature in magneto-optic waveguides, allowing controlled manipulation of soliton characteristics.

For instance, the specific solutions (55)–(58) are valid under the constraint condition: $\vartriangle_1 = -\vartriangle_8 \alpha_1^2 L_0$ as given in equation (54). This condition establishes a direct link between $\vartriangle_1$, $\vartriangle_8$, α_1, and L_0. The presence of $\vartriangle_1$ in this expression indicates that the soliton existence conditions are intricately connected to the magneto-optic parameter Q_j. In other words, changes in Q_1 and Q_2 influence $\vartriangle_1$, which in turn determines whether the soliton solutions remain valid.

Physically, this means that by adjusting the external magnetic field strengths (which define Q_1 and Q_2), one can control the formation and properties of the quiescent optical solitons in the waveguide. This dependency on Q_j suggests that magneto-optic effects play a fundamental role in soliton generation, offering a mechanism to regulate nonlinear pulse dynamics externally.

In summary, the existence of soliton solutions (55)–(58) is inherently tied to the constraint conditions defined in Eq. (54), which link the magneto-optic parameters Q_1 and Q_2 through Eq. (10) and influence the dispersion term $\vartriangle_1$ in Eq. (14). This dependence underscores the key role of magneto-optic interactions in shaping the nonlinear wave dynamics in the system, making these solitons highly tunable based on external magnetic field adjustments.

5. Conclusions

The paper derived quiescent optical solitons for the first time in a magneto-optic waveguide that was considered with nonlinear CD and came with generalized temporal evolution. The SPM structure of the model stemmed from two sources, namely Kudryashov's nonlinear form of refractive index and dual forms of nonlocal nonlinearities. The intermediary cnoidal waves gave way to quiescent optical solitons when the modulus of ellipticity approached unity. The resulting solutions are a complete spectrum of quiescent optical solitons. The results are thus interesting, especially with their application to an optoelectronic device apart from optical fibers. These lead to a very promising future. Later, the study can be extended to additional optoelectronic devices such as fibers with Bragg gratings, fibers with differential group delay, dispersion–flattened fibers, optical couplers, and optical metamaterials, just to name a few. The results will be made available once they are recovered and aligned with the pre-existing ones [23-31].

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Анотація*. У цій статті отримуються розв'язки спокійних оптичних солітонів, що виникають у магніто-оптичних хвилеводах, які зберігають форму самофазової модуляції Кудряшова в поєднанні з узагальненою формою нелокальної нелінійності. Модель також враховує нелінійну хроматичну дисперсію та розглядається з урахуванням узагальненої часової еволюції. Завдяки вдосконаленому прямому алгебраїчному методу стало можливим отримання таких солітонів. Таким чином, повний спектр солітонів відновлюється за допомогою проміжних еліптичних функцій Якобі, а також еліптичних функцій Вейєрштраса. Також перераховані параметричні обмеження для існування таких солітонів.*

Ключові слова*: солітони, алгебраїчний метод. кноідальні хвилі*