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Optical Solitons with Arbitrary Intensity and Conservation Laws of the Perturbed Resonant Nonlinear Schrodinger's Equation

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Abstract. This paper addresses the perturbed resonant nonlinear Schrödinger's equation with power-law of self-phase modulation. The traveling wave hypothesis recovers the bright 1–soliton solutions to the model. The conservation laws are identified by the method of multipliers. The semi-inverse variational principle leads to the bright 1–soliton solution when the arbitrary intensity parameter of the perturbation terms differ from that of the unperturbed terms.

Key words: resonant nonlinear Schrödinger's equation, multipliers approach, semi–inverse variation
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1. Introduction

The dynamics of mobile soliton propagation through optical fibers and metamaterials is possible in the presence of a delicate balance between linear chromatic dispersion (CD) and self-phase modulation (SPM) effect. The slightest departure from this delicate balance would stall the solitons in the middle of an optical fiber, leading to wave collapse. Additionally, if the CD is rendered nonlinear, the quiescent solitons emerge. Several results have been reported in this context in various works published in a wide range of journals [1-10]. The current paper addresses the dynamics of optical solitons in the presence of linear CD and the power–law of SPM.

The governing model is the nonlinear Schrödinger equation (NLSE) in the presence of a resonant term and a few Hamiltonian-type perturbation terms. The unperturbed model appears in quantum fluids and quantum optics. After a quick and succinct introduction to the model, the details are exhibited in the rest of the paper.

The present paper aims to investigate the propagation dynamics of optical solitons in the presence of linear CD and a power-law self-phase modulation effect. This study explores how these factors influence soliton behavior by considering the NLSE with an additional resonant term and specific Hamiltonian-type perturbations. The unperturbed version of this model has been widely applied in quantum fluids and quantum optics. The analysis presented here provides further insight into the existence and characteristics of solitons under these conditions.

2. Governing model

The dimensionless form of the governing resonant NLSE with linear CD and power-law of SPM is structured as [7, 8, 10]:

$$iq_t + aq_{xx} + b|q|^{2n} q + c\left(\frac{|q|_{xx}}{|q|}\right)q = i\left[\lambda\left(|q|^{2n} q\right)_x + \sigma_1\left(|q|^{2n}\right)_x q + \sigma_2 |q|^{2n} q_x\right]. \tag{1}$$

Here in Eq. (1), $q(x,t)$ represents the wave amplitude and is a complex-valued function. The independent variables x and t represent spatial and temporal variables respectively. The first term is linear temporal evolution, with its coefficient being $i=\sqrt{-1}$. The second term with its coefficient being a is the linear CD. The third term with coefficient b is from power-law of SPM, with n being the power–law nonlinearity factor. The next term with coefficient c is from the resonant term, which appears in quantum fluids and quantum optics. The perturbation terms on the right hand side are from three sources. The coefficient of λ is with self-steepening effect while the remaining two terms with σ_j for $j=1,2$ are from self–frequency shift. The power-law parameter n is the power-law nonlinearity factor of SPM.

The governing equation is considered with arbitrary intensity. The traveling wave hypothesis reveals the bright 1–soliton solution of the perturbed NLSE along with the parameter constraints for the existence of such solitons. The conservation laws for the model are next recovered with the aid of the multipliers approach. Finally, when the intensity parameter of the power-law of SPM and the perturbation terms are different, the traveling wave hypothesis fails to recover the exact bright 1–soliton solution to the model. In this context, the semi-inverse variational principle (SVP) aids in the recovery of a bright 1-soliton solution to the model, although this solution is not exact [2–5].

This framed model, as described in Eq. (1), will now be addressed by the traveling wave hypothesis, and its conservation laws will be recovered using of the multipliers approach. Finally, the model will retrieve bright optical solitons with perturbation terms whose arbitrary intensity parameter m in Eq. (19) is different from n. In this case, the application of the semi-inverse variational principle (SVP) would come into play.

3. Approach of the traveling waves

In order to integrate Eq. (1) to search for its soliton solutions, the traveling wave hypothesis is picked:

$$q(x,t) = g(x-vt)e^{i(-\kappa x+\omega t+\theta_0)}. \tag{2}$$

Here in Eq.(2), the function g is the amplitude component of the traveling wave while from the phase component, κ is the soliton wave number, while ω is the frequency of the soliton, and finally, θ_0 is the phase constant. Also, v is speed of the soliton. Substituting Eq.(2) into Eq. (1) and splitting into real and imaginary parts, the imaginary component gives:

$$\left(v+2a\kappa\right)g'+\left\{\left(2n+1\right)\lambda+2n\sigma_1+\sigma_2\right\}g^{2n}g'=0, \tag{3}$$

while the real components yield:

$$\left(a+c\right)g''-\left(\omega+a\kappa^2\right)g+\left(b-\lambda\kappa-\sigma_2\kappa\right)g^{2n+1}=0. \tag{4}$$

From the imaginary part Eq. (3), one recovers the speed of the soliton as:

$$v=-2a\kappa, \tag{5}$$

and the parameter constraints:

$$\left(2n+1\right)\lambda+2n\sigma_1+\sigma_2=0. \tag{6}$$

From the real part equation given by Eq. (4), multiplying by g', and integrating twice while choosing the integration constant to be zero since the search is for solitons with the assumption of having no radiation, one recovers the bound state bright 1–soliton solution as:

$$q\left(x,t\right)=A\operatorname{sech}^{\frac{1}{n}}\left[B\left(x-vt\right)\right]e^{i\left(-\kappa x+\omega t+\theta_0\right)}, \tag{7}$$

where the amplitude A and the inverse width B of the soliton are given as:

$$A=\left[\frac{\left(n+1\right)\left(\omega+a\kappa^2\right)}{b-\left(\lambda+\sigma_2\right)\kappa}\right]^{\frac{1}{2n}}, \tag{8}$$

and

$$B=n\sqrt{\frac{\omega+a\kappa^2}{a+c}}. \tag{9}$$

A pair of additional parameter constraints that naturally emerge from the expressions for the amplitude and width of the soliton are:

$$\left(\omega+a\kappa^2\right)\left\{b-\left(\lambda+\sigma_2\right)\kappa\right\}>0, \tag{10}$$

and

$$\left(a+c\right)\left(\omega+a\kappa^2\right)>0, \tag{11}$$

respectively.

4. Conservation laws

The conservation laws for model (1) will be retrieved in this section using the 'multipliers approach.' It turns out that if system (1) is separated into real and imaginary parts with $q=u+iz$, the multipliers $Q=(Q_1,Q_2)$ for which the Euler operator that annihilates the resultant system is obtained by:

$$Q_1=(-u,z)\,,Q_2=(-z_x,u_x)\,,\ Q_3=(u_t,z_t)\,,$$

leading to, respectively, conserved "power." If, however,

$$\sigma_1=\lambda, \tag{12}$$

one can recover linear momentum and the Hamiltonian. The following densities subsequently emerge:

(i) Power (P):

$$T_1=\frac{1}{2}|q|^2, \tag{13}$$

(ii) Linear momentum (M):

$$T_2 = -\frac{b}{2(n+1)(u^2+z^2)^2}\begin{bmatrix} -nu^5 z_x + nu^4 z u_x - 2nu^3 z^2 z_x + 2nu^2 z^3 u_x \\ -nuz^4 z_x + nz^5 u_x - u^5 z_x + u^4 z u_x - 2u^3 z^2 z_x \\ +2u^2 z^3 u_x - uz^4 z_x + z^5 u_x \end{bmatrix} = \frac{b}{2} I(q^* q_x), \quad (14)$$

where $I(q^* q_x)$ represents the imaginary part of the quantity inside the parentheses and

(iii) Hamiltonian (H):

$$T_3 = \frac{1}{2(n+1)|q|^4}\begin{bmatrix} (n+1)a|q|^4 R(q^* q_{xx}) + b|q|^{2(n+3)} \\ +(n+1)c|q|^2\left\{|q|^2 R(q^* q_{xx}) + \left(I(q^* q_x)\right)^2\right\} \\ -(\lambda+\sigma_2)|q|^{2(n+2)} I(q^* q_x) \end{bmatrix}. \quad (15)$$

Using the 1–soliton solution to the model given by Eq. (7), the respective conserved quantities are:

$$P = \int_{-\infty}^{\infty} T_1 dx = \int_{-\infty}^{\infty} |q|^2 dx = \frac{A^2}{B}\frac{\Gamma\left(\frac{1}{n}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{1}{n}+\frac{1}{2}\right)}, \quad (16)$$

$$M = \int_{-\infty}^{\infty} T_2 dx = ib\int_{-\infty}^{\infty} (q^* q_x - q q_x^*) dx = \frac{b\kappa A^2}{B}\frac{\Gamma\left(\frac{1}{n}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{1}{n}+\frac{1}{2}\right)}, \quad (17)$$

and

$$H = \int_{-\infty}^{\infty} T_3 dx = \frac{A^2}{2n(n+1)(n+2)B}\begin{bmatrix} (a+c)(n+1)\{B^2 + n(n+2)\kappa^2\} \\ +\{b+(\lambda+\sigma_2)\kappa\} \ nA^{2n} \\ +2n(n+1)(n+2)c\kappa^2 \end{bmatrix}\frac{\Gamma\left(\frac{1}{n}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{1}{n}+\frac{1}{2}\right)}, \quad (18)$$

where $\Gamma(*)$ is gamma function.

5. Semi-inverse variation

This section will perform the integration of the resonant NLSE as given by Eq. (1) with the arbitrary intensity parameter for the perturbation terms being different from the power-law of nonlinear SPM structure. The governing model, therefore, reads:

$$iq_t + aq_{xx} + b|q|^{2n} q + c\left(\frac{|q|_{xx}}{|q|}\right)q = i\left[\lambda\left(|q|^{2m} q\right)_x + \sigma_1\left(|q|^{2m}\right)_x q + \sigma_2 |q|^{2m} q_x\right], \quad (19)$$

where the perturbation terms on the right-hand side now have m as its intensity parameter with $m \neq n$. Thus, with this feature, one cannot apply the traveling wave hypothesis or, as a matter of fact, any integration algorithm that can fetch an exact 1–soliton solution unless $m=n$.Therefore, this section would implement the semi-inverse variational principle (SVP) that would recover the bright 1–soliton solution when $m \neq n$. The retrieved solution would be analytical but not exact. The starting point is the traveling point hypothesis given by Eq. (2). Upon substituting Eq. (2) into Eq. (19) and splitting into real and imaginary components, the imaginary part yields:

$$(v+2a\kappa) + \{(2m+1)\lambda + 2m\sigma_1 + \sigma_2\} g^{2m} = 0, \quad (20)$$

while the real component yields:

$$(a+c)g'' - (\omega + a\kappa^2)g + bg^{2n+1} - (\lambda+\sigma_2)\kappa g^{2m+1} = 0. \quad (21)$$

From the imaginary part Eq. (20), one recovers the speed of the soliton as given by Eq. (5), while the parameter constraints come out as:

$$(2m+1)\lambda+2m\sigma_1+\sigma_2=0. \tag{22}$$

This conclusion follows from Eq. (20) since g' and 1 are linearly independent. Multiplying Eq. (21) by g' and integrating leads to:

$$(a+c)(g')^2-(\omega+a\kappa^2)g^2+\frac{b}{n+1}g^{2n+2}-\frac{(\lambda+\sigma_2)\kappa}{m+1}g^{2m+2}=K, \tag{23}$$

where K is the integration constant. The stationary integral is now defined as:

$$J=\int_{-\infty}^{\infty}\left[(a+c)(g')^2-(\omega+a\kappa^2)g^2+\frac{b}{n+1}g^{2n+2}-\frac{(\lambda+\sigma_2)\kappa}{m+1}g^{2m+2}\right]dx. \tag{24}$$

SVP states that the solution of the perturbed model given by Eq. (19) will be the same as that of its unperturbed version for $\lambda=\sigma_j=0$ for $j=1,2$, as given by Eq. (7). However, the amplitude A and the inverse width B of the perturbed soliton will differ, and their variations can be recovered from the solution of the coupled system [2–5]:

$$\frac{\partial J}{\partial A}=0, \tag{25}$$

and

$$\frac{\partial J}{\partial B}=0. \tag{26}$$

Substituting g from Eq. (7) into Eq. (24) and integrating simplifies it to:

$$\begin{aligned}J=&\left[\frac{(a+c)A^2B}{n(n+2)}-\frac{(\omega+a\kappa^2)A^2}{B}+\frac{2bA^{2n+2}}{(n+1)(n+2)B}\right]\\&\times\frac{\Gamma\left(\frac{1}{n}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{1}{n}+\frac{1}{2}\right)}-\frac{(\lambda+\sigma_2)\kappa A^{2m+2}}{(m+1)(m+2)B}\frac{\Gamma\left(\frac{1}{m}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{1}{m}+\frac{1}{2}\right)}.\end{aligned} \tag{27}$$

From Eq. (27), Eqs. (25) and (26) reduce to:

$$\begin{aligned}&\left[\frac{a+c}{n(n+2)}B^2-(\omega+a\kappa^2)+\frac{2b}{n+2}A^{2n}\right]\frac{\Gamma\left(\frac{1}{n}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{1}{n}+\frac{1}{2}\right)}\\&-\frac{(\lambda+\sigma_2)\kappa}{m+2}A^{2m}\frac{\Gamma\left(\frac{1}{m}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{1}{m}+\frac{1}{2}\right)}=0,\end{aligned} \tag{28}$$

and

$$\begin{aligned}&\left[\frac{(a+c)}{n(n+2)}B^2-(\omega+a\kappa^2)+\frac{2b}{(n+1)(n+2)}A^{2n}\right]\\&\times\frac{\Gamma\left(\frac{1}{n}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{1}{n}+\frac{1}{2}\right)}-\frac{(\lambda+\sigma_2)\kappa}{(m+1)(m+2)}A^{2m}\frac{\Gamma\left(\frac{1}{m}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{1}{m}+\frac{1}{2}\right)}=0.\end{aligned} \tag{29}$$

respectively. Uncoupling the two equations leads to the algebraic equation for the amplitude of the solitary wave as:

$$\left[2\left(\omega+a\kappa^{2}\right)-\frac{2b}{n+1}A^{2n}\right]\frac{\Gamma\left(\frac{1}{n}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{1}{n}+\frac{1}{2}\right)}+\frac{(\lambda+\sigma_{2})\kappa}{m+1}A^{2m}\frac{\Gamma\left(\frac{1}{m}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{1}{m}+\frac{1}{2}\right)}=0. \tag{30}$$

Eq. (30) is the algebraic equation for the amplitude of the bright soliton of the perturbed resonant NLSE, with the perturbation terms having a different intensity factor as compared to the unperturbed terms. This equation can be solved exactly for the quantity A for a few specific values of the parameters m and n. Subsequently, the width of the soliton can be recovered from Eqs. (28) or (29). This would lead to the analytical bright 1–soliton solution to Eq. (19) to be given by Eq. (7), where the solitons' amplitude and width are recoverable as discussed. The speed of the soliton is given by Eq. (5) with the parameter constraints as given by Eq. (22). It is to be noted that the bright 1–soliton solution, although analytical, is not exact.

Fig. 1 demonstrates the bright soliton (7) through the surface plot (a), contour plot (b), and 2D plot (c) by setting $n=0.6$, a=1, $\kappa=1$, $\omega=1$, c=1, $\sigma_2=0.2$, $\lambda=1$, and b=2.

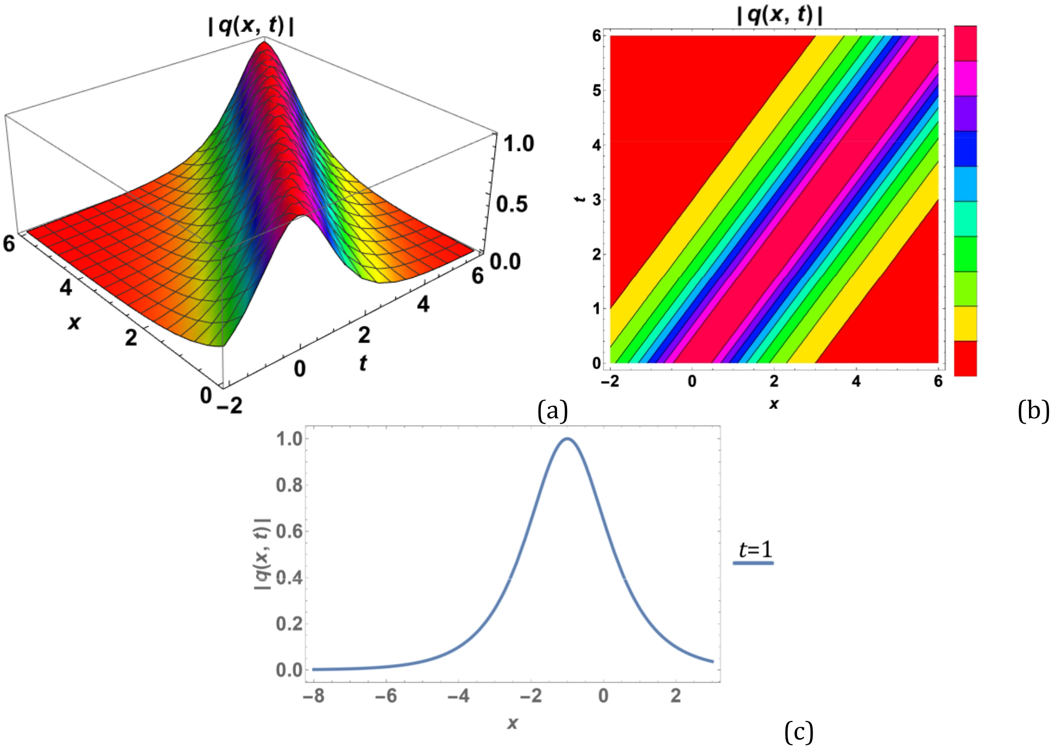

Fig. 1. Profile of a bright soliton $|q(x,1)|$: (a) surface plot, (b) contour plot, and (c) 2D plot..

6. Conclusions

This work recovered the bright 1–soliton solution to the perturbed resonant NLSE that was considered with power-law of SPM in the presence of perturbation terms. The traveling wave hypothesis has made this retrieval possible. The power-law nonlinearity parameter is taken to be the same as the arbitrary intensity parameter amongst the perturbation terms. The three conserved quantities were computed with the aid of the conserved densities, which were recovered from the multipliers approach. Finally, the analytical bright 1–soliton solution to the model was determined when the power-law nonlinearity parameter is not the same as the intensity parameter for the perturbation terms by the aid of SVP.

The paper's results hold a lot of promise for future investigations in this avenue. The model is to be addressed with fractional temporal evolution. Additionally, the model will be later studied with nonlinear chromatic dispersion along with linear and generalized temporal evolution by Lie symmetry, which would fetch quiescent optical solitons. Additional forms of SPM structure would be later taken up, leading to a wide range of further interesting results, which would be disseminated all across the board with time.

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Анотація. *У цій статті розглядається збурене резонансне нелінійне рівняння Шредінгера із степеневим законом самофазової модуляції. Гіпотеза біжучої хвилі дозволяє отримати яскраві одно-солітонні розв'язки цієї моделі. Дотримання законів збереження забезпечується за допомогою методу множників. За умови коли довільний параметр інтенсивності в збурених членах відрізняється від цього параметру у незбурених членах до яскравого одно-солітонного розв'язку приводить напівобернений варіаційний принцип.*

Ключові слова: *резонансне нелінійне рівняння Шредінгера, мультиплікаторний підхід, напівінверсна варіація*