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Three Efficient Schemes and Highly Dispersive Optical Solitons of Perturbed Fokas–Lenells Equation in Stochastic Form

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Abstract. We contemplate a highly dispersive stochastic perturbed Fokas–Lenells model for fiber Bragg gratings with spatiotemporal dispersion and generated white noise in the Itô meaning. To acquire soliton solutions, we pertain to an $\exp(-f(\xi))$-expansion method, a (G/G')-expansion technique and a simplest-equation method. 3D and 2D plots are built to make the wave-propagation behavior clear. The solitons of different kinds obtained by us like dark, singular, periodic, rational and combined solitons are compared with each other. The efficiency of our methods for the underlying model is evaluated.

Keywords: Fokas-Lenells model, $\exp(-f(\xi))$-expansion method, (G/G')-expansion method, the simplest-equation method, solitons
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1. Introduction

In recent years, optical solitons are one of the areas that attract the most attention in the field of telecommunications and nonlinear optics. As a result, a lot of researches have been performed on the subject (see, e.g., Refs. [1–24]). In fact, the main topic of these researches is the fibers that maintain polarization of light [25]. A Fokas–Lenells equation, a Lakshmanan–Porsezian–Daniel model, a Radhakrishnan–Kundu–Lakshmanan equation, a Schrödinger–Hirota equation, a Gerdjikov–Ivanov equation and some other equations are among the well-known models employed in the field [26–39].

Another area that still needs to be examined is birefringent fibers. An optical birefringence as a natural phenomenon encountered in optical fibers represents a serious limitation for the speed in high-speed fiber communication links. It can also cause incorrect data transmission. There is a delicate balance between a dynamic chromatic dispersion (CD) and a self-phase modulation in a known Manakov system and a Thirring model, which allows soliton propagation [26]. In brief, the CD implies that the group velocity, which is the propagation velocity of an optical signal, varies with the light wavelength. When the CD is low, irrational results can occur during the transmission of fiber-optic pulses [40]. Since the CD represents one of the main reasons hindering high-transmission speeds in the optical networks, various approaches have been suggested to deal with it. One of them is Bragg gratings with dispersive reflectivity. Purely quartic solitons, which are shape-maintaining pulses that appear in many optical materials with a dominant fourth-order dispersion, have been offered as the alternative approach [41, 42]. In the case of low CD, the CD is replaced with the fourth-order dispersion. The disadvantage of this model if the fact that only stationary optical solitons can be obtained analytically and analyzed numerically for the purely quartic nonlinear Schrödinger equation. Then a basic idea of cubic-quartic solitons has emerged, for which the CD is replaced by the third-order and fourth-order dispersions jointly.

In this context, sixth-, fifth-, fourth- and third-order dispersions, together with an inter-modal dispersion term, are considered in addition to the pre-existing CD. These dispersions make up highly dispersive solitons that provide a necessary delicate balance between the self-phase modulation and the CD for propelling solitons smoothly through the optical fibers at trans-continental and trans-oceanic distances [27, 43].

In this work we consider a stochastic perturbed Fokas–Lenells model for fiber Bragg gratings with a spatiotemporal dispersion and a generative white noise in the Itô meaning. It can be represented as follows [26]:

$$\begin{aligned} &iq_t + ia_{11}r_x + a_{12}r_{xx} + ia_{13}r_{xxx} + a_{14}r_{4,x} + ia_{15}r_{5,x} + a_{16}r_{6,x} + b_1 r_{xt} \\ &+\left(c_1|q|^2 + d_1|r|^2\right)\left(e_1 q + if_1 q_x\right) + qr^*\left(\gamma_1 r + i\eta_1 r_x\right) \\ &+\sigma\left(q - ib_1 r_x\right)\frac{dW}{dt} + i\alpha_1 q_x + \beta_1 r + \delta_1 q^* r^2 \\ &= i\left(\lambda_1\left(|q|^2 q\right)_x + v_1\left(|r|^2 q\right)_x + \mu_1\left(|q|^2\right)_x q + \theta_1\left(|r|^2\right)_x q\right), \end{aligned} \tag{1}$$

and

$$\begin{aligned} &ir_t + ia_{21}q_x + a_{22}q_{xx} + ia_{23}q_{xxx} + a_{24}q_{4,x} + ia_{25}q_{5,x} \\ &+a_{26}q_{6,x} + b_2 q_{xt} + \left(c_2|r|^2 + d_2|q|^2\right)\left(e_2 r + if_2 r_x\right) + rq^*\left(\gamma_2 q + i\eta_2 q_x\right) \\ &+\sigma\left(r - ib_2 q_x\right)\frac{dW}{dt} + i\alpha_2 r_x + \beta_2 q + \delta_2 r^* q^2 \\ &= i\left(\lambda_2\left(|r|^2 r\right)_x + v_2\left(|q|^2 r\right)_x + \mu_2\left(|r|^2\right)_x r + \theta_2\left(|q|^2\right)_x r\right), \end{aligned} \tag{2}$$

where $q(x,t)$ and $r(x,t)$ are complex-valued functions that describe the wave profiles, $i = \sqrt{-1}$, and q^* and r^* represent complex conjugates of q and r. Note that the first terms in the right-hand sides of Eqs. (1) and (2) stand for the linear temporal evolution. Table 1 introduces the parameters involved in Eqs. (1) and (2). The model discussed in this study, which is given by Eqs. (1) and (2), has been examined for the first time by Zayed et al. [26]. They have applied two different methods, an addendum Kudryashov's method and a unified Riccati-equation expansion method to find the explicit solutions. As a consequence, these authors have acquired a bright soliton, straddled solitary solutions, singular solitons and a dark soliton. In the present work, we will discuss Eqs. (1) and (2) with the motivations of the study [26].

The article is organized as follows. In Section 2, a mathematical analysis of the model is made and its description is presented as an ordinary differential equation (ODE) with the complex-wave transform. Section 3 introduces an $\exp(-f(\xi))$-expansion method, a (G/G')-expansion technique and a simplest-equation method, which represent our main approaches. Section 4 is devoted to applying the three above analytical schemes to a reduced ordinary differential equation. Here we obtain the soliton solutions of our main model. In Section 5 we give the physical structure of the solutions and their graphical representation. Finally, conclusions are drawn in Section 6.

Table 1. Description of the parameters involved in Eqs. (1) and (2).

Parameter	Description
a_{lk} $(l=1,2; k=1,...,6)$	Constant coefficients of CD and third-, fourth-, fifth- and sixth-order dispersions
b_l $(l=1,2)$	Spatiotemporal dispersion terms (constant coefficients)
c_l and e_l $(l=1,2)$	Self-phase modulation terms (constant coefficients)
d_l coupled with e_l $(l=1,2)$	Cross-phase modulation terms (constant coefficients)
f_l $(l=1,2)$	Nonlinear dispersion terms (constant coefficients)
γ_l $(l=1,2)$	Four-wave mixing terms (constant coefficients)
η_l $(l=1,2)$	Constant coefficients of cross-phase modulation and four-wave mixing terms
σ	A constant coefficient of noise strength
$W(t)$	A standard Wiener process
$dW(t)/dt$	A white noise
$\alpha_l, \beta_l, \delta_l$ $(l=1,2)$	Constant coefficients of inter-modal dispersion-detuning effect and four-wave mixing
$\lambda_l, \mu_l, \nu_l, \theta_l$ $(l=1,2)$	Self-sleeping and nonlinear dispersion terms (constant coefficients)

2. Mathematical analysis

We use the following transformation involving the noise coefficients and the $W(t)$ terms in order to convert Eqs. (1) and (2) into an ODE with the variable $\xi = x - vt$ [26]:

$$q(x,t) = u_1(\xi)\exp\left[i\left(-\kappa x + \omega t + \sigma W(t) - \sigma^2 t\right)\right], \tag{3}$$

$$r(x,t) = u_2(\xi)\exp\left[i\left(-\kappa x + \omega t + \sigma W(t) - \sigma^2 t\right)\right], \tag{4}$$

where v implies the soliton velocity, κ the wavevector, and ω is the frequency. The real functions u_1 and u_2 represent the amplitudes of the wave transformations. After substituting Eqs. (3) and (4) into Eqs. (1) and (2) and separating their real and imaginary parts, we take into account the following condition for the systems to be compatible:

$$u_2 = \chi u_1, \ \chi \neq 0, 1. \tag{5}$$

Considering the real and imaginary parts rewritten with taking Eq. (5) into account, one obtains the soliton velocity,

$$\begin{aligned} v &= \frac{\alpha_1 \chi + \chi^2\left(a_{11} - 3a_{13}\kappa^2 + 4a_{14}\kappa^3 + 5a_{15}\kappa^4 - a_{16}\kappa^5 - 2a_{12}\kappa + b_1\omega - b_1\sigma^2\right)}{\chi - b_1\kappa} \\ &= \frac{\alpha_2 \chi + \chi^2\left(a_{21} - 3a_{23}\kappa^2 + 4a_{24}\kappa^3 + 5a_{25}\kappa^4 - a_{26}\kappa^5 - 2a_{22}\kappa + b_2\omega - b_2\sigma^2\right)}{\chi - b_2\kappa}, \end{aligned} \tag{6}$$

the soliton wavevector,

$$\kappa = \frac{a_{15}}{6a_{16}} = \frac{a_{25}}{6a_{26}}, \tag{7}$$

and the constraint conditions,

$$\begin{aligned} &20a_{16}\kappa^3 - 10a_{15}\kappa^2 - 4a_{14}\kappa + a_{13}\kappa = 0, \quad 20a_{26}\kappa^3 - 10a_{25}\kappa^2 - 4a_{24}\kappa + a_{23}\kappa = 0, \\ &c_1 f_1 - 3\lambda_1 - 2\mu_1 + \left(d_1 f_1 + \eta_1 - 3\nu_1 - 2\theta_1\right)\chi^2 = 0, \\ &\chi\left(c_2 f_2 - 3\lambda_2 - 2\mu_2\right) + \left(d_2 f_2 + \eta_2 - 3\nu_2 - 2\theta_2\right)\chi^2 = 0. \end{aligned} \tag{8}$$

Then the ODE

$$u_1^{(6)} + \Delta_1 u_1^{(4)} + \Delta_2 u_1'' + \Delta_3 u_1 + \Delta_4 u_1^3 = 0\,, \tag{9}$$

can be obtained, with

$$\begin{aligned}
\Delta_1 &= \frac{5a_{15}\kappa + a_{14} - 15a_{16}\kappa^2}{a_{16}}, \\
\Delta_2 &= \left(15a_{16}\kappa^4 - 10a_{15}\kappa^3 - 6a_{14}\kappa^2 + 3a_{13}\kappa + a_{12} - \mathcal{V}b_1\right)a_{16}{}^{-1}, \\
\Delta_3 &= \begin{pmatrix} b_1\kappa w - b_1\kappa\sigma^2 + a_{14}\kappa^4 + a_{15}\kappa^5 + a_{11}\kappa - a_{13}\kappa^3 \\ -a_{16}\kappa^6 - a_{12}\kappa^2 + \beta_1 + \dfrac{\alpha_1\kappa + \sigma^2 - w}{\chi} \end{pmatrix} a_{16}{}^{-1}, \\
\Delta_4 &= \left(\chi\left(d_1 f_1\kappa + d_1 e_1 + \kappa\eta_1 - \kappa v_1 + \delta_1 + \gamma_1\right) + \frac{c_1 f_1\kappa + c_1 e_1 - \kappa\lambda_1}{\chi}\right)a_{16}{}^{-1}.
\end{aligned} \tag{10}$$

Note that a detailed mathematical analysis of Eqs. (1) and (2) has been made by Zayed et al. [26].

3. Our methods

Below we describe the algorithms of our methods used to study the exact solutions of the underlying equation. Let us consider a nonlinear equation of the form

$$F(q, q_x, q_t, q_{xx}, q_{tt}, ...) = 0, \tag{11}$$

where $q = q(x,t)$ is an unknown function and F denotes a polynomial in $q = q(x,t)$ and its partial derivatives, where the highest-order derivatives and the nonlinear terms are involved. To reduce the nonlinear equation given by Eq. (11), we choose the traveling-wave transformation:

$$q(x,t) = q(\xi)\,, \tag{12}$$

with $\xi = x - vt$. Using the traveling-wave variable in Eq. (12), one can reduce Eq. (11) to the ODE:

$$F(q, q', q'', ...) = 0, \tag{13}$$

with $q' = \dfrac{dq}{d\xi}, q'' = \dfrac{d^2q}{d\xi^2}, ...$

3.1. $\exp(-f(\xi))$ *-expansion method*

Here we give a detailed explanation of the $\exp(-f(\xi))$-expansion method [45]. Suppose that the solution of Eq. (13) can be expressed by a polynomial in $\exp(-f(\xi))$ as follows:

$$q = \sum_{i=0}^{N} A_i exp(-f(\xi))^i\,, \tag{14}$$

with $f(\xi)$ satisfying the equation

$$f'(\xi) = exp(-f(\xi)) + \tau_1 exp(f(\xi)) + \tau_2. \tag{15}$$

Here $A_i, i = 0, ..., N, \tau_1, \tau_2$ are constants.

Eq. (15) has the solutions described as follows.

i. $\tau_2^2 - 4\tau_1 > 0, \tau_1 \neq 0$:

$$f(\xi) = \ln\left(\left(-\tau_2 - \sqrt{\tau_2{}^2 - 4\,\tau_1}\,\tanh\left(\frac{\sqrt{\tau_2{}^2 - 4\,\tau_1}}{2}(\xi + C)\right)\right)\Big/ 2\tau_1\right). \tag{16}$$

ii. $\tau_2^2-4\tau_1<0,\tau_1\neq 0$:

$$f(\xi)=\ln\left(-\tau_2+\sqrt{4\,\tau_1-\tau_2{}^2}\tan\left(\frac{\sqrt{4\,\tau_1-\tau_2{}^2}}{2}(\xi+C)\right)\Big/2\tau_1\right). \tag{17}$$

iii. $\tau_2^2-4\tau_1>0,\tau_1=0,\tau_2\neq 0$:

$$f(\xi)=-\ln\left(\tau_2/e^{\tau_2(\xi+C)}-1\right). \tag{18}$$

iv. $\tau_2^2-4\tau_1=0,\tau_1\neq 0,\tau_2\neq 0$:

$$f(\xi)=\ln\left(-\left[2\,\tau_2(\xi+C)+4\right]/\left[\tau_2{}^2(\xi+C)\right]\right). \tag{19}$$

v. $\tau_2^2-4\tau_1=0,\tau_1=\tau_2=0$:

$$f(\xi)=\ln(\xi+C). \tag{20}$$

The positive integer N can be found using a balancing principle. After substituting Eq. (14) into Eq. (13), using the ODE given by Eq. (15) and then collecting together all the terms with the same orders of $\exp(-f(\xi))$, one can convert the left-hand side of Eq. (13) into a new polynomial in $\exp(-f(\xi))$. Setting each of the coefficients of this polynomial to zero yields in a system of algebraic equations for $A_i, i=0,...,N, \tau_1,\tau_2$. Solving the system of these equations and substituting A_i, $i=0,...,N$, one can get the exact solutions of reduced Eq. (13).

3.2. A brief overview of the (G'/G)-expansion method

The principal aspects of the well-known (G'/G)-expansion method can be described as follows [47–49]. Suppose that the solution of Eq. (13) can be expressed by a polynomial in (G'/G) as

$$q=\sum_{i=0}^{N}A_i\left(\frac{G'}{G}\right)^i, \tag{21}$$

where $G=G(\xi)$ satisfies the second order linear ODE in the form

$$G''+BG'+CG=0,. \tag{22}$$

while $A_1,A_2,...,A_N,B,C$ are constants to be determined later ($A_N\neq 0$). Using the general solutions of Eq. (22), we have

$$\frac{G'}{G}=\begin{cases}-\dfrac{B}{2}+\dfrac{\sqrt{B^2-4C}}{2}\dfrac{\left(C_1\sinh\left(\frac{\sqrt{B^2-4C}}{2}\xi\right)+C_2\cosh\left(\frac{\sqrt{B^2-4C}}{2}\xi\right)\right)}{C_1\cosh\left(\frac{\sqrt{B^2-4C}}{2}\xi\right)+C_2\sinh\left(\frac{\sqrt{B^2-4C}}{2}\xi\right)}, & B^2-4C>0\\[2ex] -\dfrac{B}{2}+\dfrac{\sqrt{-B^2+4C}}{2}\dfrac{\left(-C_1\sin\left(\frac{\sqrt{-B^2+4C}}{2}\xi\right)+C_2\cos\left(\frac{\sqrt{-B^2+4C}}{2}\xi\right)\right)}{C_1\cos\left(\frac{\sqrt{-B^2+4C}}{2}\xi\right)+C_2\sin\left(\frac{\sqrt{-B^2+4C}}{2}\xi\right)}, & B^2-4C<0\\[2ex] \dfrac{C_2}{C_1+C_2\xi}-\dfrac{B}{2}, & B^2-4C=0.\end{cases}, \tag{23}$$

The parameter N in Eq. (21) is a positive constant. It can be determined by balancing the nonlinear term with the linear term of the highest order in Eq. (13). By substituting Eq. (21) into Eq. (13) and using the second-order terms in the linear ODE Eq. (22), we obtain an algebraic equation in the powers of (G'/G). Since all the coefficients involved in $(G'/G)^i$ must be identically equal to zero, we arrive at a system of algebraic equations for $A_1, A_2,..., A_N, B, C$. In this stage, one can solve this system, using computer-algebra systems such as Maple. After solving the system and inserting $A_1, A_2,..., A_N$ from the general solutions of Eq. (22) into Eq. (21), one obtains the traveling-wave solutions of Eq. (11).

3.3. A simplest-equation method

In this subsection, we outline the basic steps of the simplest-equation method suggested by Kudryashov [44, 46]. Suppose that Eq. (13) has the solutions of the form

$$q(\xi)=\sum_{i=0}^{M}k_i\left(w(\xi)\right)^i, \tag{24}$$

where $w(\xi)$ satisfies the well-known Bernoulli and Riccati equations. Balancing the highest-order derivative term with the nonlinear term, one arrives at a system of algebraic equations for the arbitrary constants $k_0, k_1, \ldots, k_M$. The forms of the solutions obtained using auxiliary equations are given below.

For the Bernoulli equation we have

$$w'(\xi)=aw(\xi)+bw(\xi)^2, \tag{25}$$

where a and b are arbitrary constants. The solution is represented as

$$w(\xi)=\frac{a\left(\cosh\left(a(\xi+C)\right)+\sinh\left(a(\xi+C)\right)\right)}{1-b\cosh\left(a(\xi+C)\right)-b\sinh\left(a(\xi+C)\right)}. \tag{26}$$

For the Riccati equation we have

$$w'(\xi)=Bw(\xi)+Aw(\xi)^2+D, \tag{27}$$

where A, B and D are arbitrary constants. The solutions are represented as

$$w(\xi)=-\frac{B+\Theta\tanh\left(\frac{\Theta}{2}(\xi+C)\right)}{2A} \tag{28}$$

and

$$w(\xi)=-\frac{B+\Theta\tanh\left(\frac{\Theta}{2}\xi\right)}{2A}+\frac{sech\left(\frac{\Theta}{2}\xi\right)}{C\cosh\left(\frac{\Theta}{2}\xi\right)-\frac{2A}{\Theta}\sinh\left(\frac{\Theta}{2}\xi\right)}, \tag{29}$$

where the condition $\Theta^2=B^2-4AD>0$ holds true and C is a constant of integration.

4. Applications of our methods

In this Section, the methods suggested above are implemented to Eq. (9) in order to retrieve the optical-soliton solutions.

4.1. Application of the $\exp(-f(\xi))$ ***-expansion method***

Using the balancing principle for the terms $u_1^{(6)}$ and u_1^3 in Eq. (9), we obtain that $N=3$. Therefore, the trial solution can be considered as

$$q = A_0 + A_1 e^{-f(\xi)} + A_2 e^{-2f(\xi)} + A_3 e^{-3f(\xi)}. \tag{30}$$

Substituting Eqs. (30) and (15) into Eq. (9) and then equating each coefficient associated with $\exp(-f(\xi))$ to zero, we get a set of algebraic equations. Solving this system, we obtain the following two parameter sets.

Set 1: $$\left\{\begin{gathered}\Delta_1 = 332\tau_1 - 83\tau_2{}^2, \Delta_2 = 946(4\tau_1 - \tau_2{}^2)^2, \Delta_3 = -1260(4\tau_1 - \tau_2{}^2)^3, \\ A_0 = 6\sqrt{-35\Delta_4{}^{-1}}\tau_2(6\tau_1 - \tau_2{}^2), A_1 = 72\sqrt{-35\Delta_4{}^{-1}}\tau_1, \\ A_2 = 36\sqrt{-35\Delta_4{}^{-1}}\tau_2, A_3 = 24\sqrt{-35\Delta_4{}^{-1}}. \end{gathered}\right\} \tag{31}$$

Inserting these values in Eq. (30), we acquire the solutions as follows.

If $\tau_2^2 - 4\tau_1 > 0, \tau_1 \neq 0$, the singular solutions are given by

$$\begin{aligned} q_1(x,t) = \Bigg(& 6\sqrt{-35\Delta_4{}^{-1}}\tau_2(6\tau_1 - \tau_2{}^2) \\ & + \frac{144\sqrt{-35\Delta_4{}^{-1}}\tau_1{}^2}{-\tau_2 - \sqrt{\tau_2{}^2 - 4\tau_1}\tanh\left(1/2\sqrt{\tau_2{}^2 - 4\tau_1}(x - vt + C)\right)} \\ & + \frac{144\sqrt{-35\Delta_4{}^{-1}}\tau_2\tau_1{}^2}{\left(-\tau_2 - \sqrt{\tau_2{}^2 - 4\tau_1}\tanh\left(1/2\sqrt{\tau_2{}^2 - 4\tau_1}(x - vt + C)\right)\right)^2} \\ & + \frac{192\sqrt{-35\Delta_4{}^{-1}}\tau_1{}^3}{\left(-\tau_2 - \sqrt{\tau_2{}^2 - 4\tau_1}\tanh\left(1/2\sqrt{\tau_2{}^2 - 4\tau_1}(x - vt + C)\right)\right)^3}\Bigg) e^{i(-\kappa x + \omega t + \sigma W(t) - \sigma^2 t)}, \end{aligned} \tag{32}$$

and

$$r_1(x,t) = \chi q_1(x,t). \tag{33}$$

If $\tau_2^2 - 4\tau_1 < 0, \tau_1 \neq 0$, the singular periodic solutions are given by

$$\begin{aligned} q_2(x,t) = \Bigg(& 6\sqrt{-35\Delta_4{}^{-1}}\tau_2(6\tau_1 - \tau_2{}^2) \\ & + \frac{144\sqrt{-35\Delta_4{}^{-1}}\tau_1{}^2}{-\tau_2 + \sqrt{4\tau_1 - \tau_2{}^2}\tan\left(1/2\sqrt{4\tau_1 - \tau_2{}^2}(x - vt + C)\right)} \\ & + \frac{144\sqrt{-35\Delta_4{}^{-1}}\tau_2\tau_1{}^2}{\left(-\tau_2 + \sqrt{4\tau_1 - \tau_2{}^2}\tan\left(1/2\sqrt{4\tau_1 - \tau_2{}^2}(x - vt + C)\right)\right)^2} \\ & + \frac{192\sqrt{-35\Delta_4{}^{-1}}\tau_1{}^3}{\left(-\tau_2 + \sqrt{4\tau_1 - \tau_2{}^2}\tan\left(1/2\sqrt{4\tau_1 - \tau_2{}^2}(x - vt + C)\right)\right)^3}\Bigg) e^{i(-\kappa x + wt + \sigma W(t) - \sigma^2 t)}, \end{aligned} \tag{34}$$

and

$$r_2(x,t) = \chi q_2(x,t). \tag{35}$$

If $\tau_2^2 - 4\tau_1 > 0, \tau_1 = 0, \tau_2 \neq 0$, one arrives at the bright-singular combo optical-soliton solution:

$$q_3(x,t) = \left(-6\sqrt{-35\Delta_4{}^{-1}}\tau_2{}^3 + \frac{36\sqrt{-35\Delta_4{}^{-1}}\tau_2{}^3}{\left(e^{\tau_2(x - vt + C)} - 1\right)^2} + \frac{24\sqrt{-35\Delta_4{}^{-1}}\tau_2{}^3}{\left(e^{\tau_2(x - vt + C)} - 1\right)^3} \right), \tag{36}$$
$$\times e^{i(-\kappa x + \omega t + \sigma W(t) - \sigma^2 t)}$$

and

$$r_3(x,t)=\chi q_3(x,t). \tag{37}$$

If $\tau_2^2-4\tau_1=0,\tau_1\neq 0,\tau_2\neq 0,$ one has the rational solutions:

$$\begin{aligned} q_4(x,t)=\Bigg(&3\sqrt{-35\Delta_4{}^{-1}}\tau_2{}^3-\frac{18\sqrt{-35\Delta_4{}^{-1}}\tau_2{}^4\left(x-vt+C\right)}{\left(2\tau_2\left(x-vt+C\right)+4\right)}\\ &+\frac{36\sqrt{-35\Delta_4{}^{-1}}\tau_2{}^5\left(x-vt+C\right)^2}{\left(2\tau_2\left(x-vt+C\right)+4\right)^2}\\ &-\frac{24\sqrt{-35\Delta_4{}^{-1}}\tau_2{}^6\left(x-vt+C\right)^3}{\left(2\tau_2\left(x-vt+C\right)+4\right)^3}\Bigg)e^{i\left(-\kappa x+wt+\sigma W(t)-\sigma^2 t\right)}, \end{aligned} \tag{38}$$

and

$$r_4(x,t)=\chi q_4(x,t). \tag{39}$$

If $\tau_2^2-4\tau_1=0,\tau_1=\tau_2=0,$ the other rational solutions follow:

$$q_5(x,t)=\frac{24\sqrt{-35\Delta_4{}^{-1}}}{\left(x-vt+C\right)^3}\,e^{i\left(-\kappa x+\omega t+\sigma W(t)-\sigma^2 t\right)} \tag{40}$$

and

$$r_5(x,t)=\chi q_5(x,t). \tag{41}$$

$$\textbf{Set 2:}\ \left\{\begin{array}{c} \Delta_1=\dfrac{83}{2686}\left(4\tau_1-\tau_2{}^2\right)\left(-293+3i\sqrt{2399}\right),\\ \Delta_2=-\dfrac{1}{3607298}\left(-44697365+1880673i\sqrt{2399}\right)\left(4\tau_1-\tau_2{}^2\right)^2,\\ \Delta_3=\dfrac{630}{1803649}\left(-32129+879i\sqrt{2399}\right)\left(4\tau_1-\tau_2{}^2\right)^3,\\ A_0=\dfrac{3}{1343}i\left(-4200i\tau_1+12\tau_1\sqrt{2399}-293i\tau_2{}^2-3\tau_2{}^2\sqrt{2399}\right)\sqrt{-35\Delta_4{}^{-1}}\tau_2,\\ A_1=\dfrac{18}{1343}i\left(-993i\tau_2{}^2-1400i\tau_1+4\tau_1\sqrt{2399}-\tau_2{}^2\sqrt{2399}\right)\sqrt{-35\Delta_4{}^{-1}},\\ A_2=36\sqrt{-35\Delta_4{}^{-1}}\tau_2,A_3=24\sqrt{-35\Delta_4{}^{-1}}. \end{array}\right\} \tag{42}$$

Inserting these values in Eq. (30), we acquire the solutions as follows.
If $\tau_2^2-4\tau_1>0,\tau_1\neq 0$, we have

$$\begin{aligned} q_6(x,t)=\Bigg(&\frac{3}{1343}i\left(-4200i\tau_1+12\tau_1\sqrt{2399}-293i\tau_2{}^2-3\tau_2{}^2\sqrt{2399}\right)\sqrt{-35\Delta_4{}^{-1}}\tau_2\\ &+\frac{36}{1343}\frac{i\left(-993i\tau_2{}^2-1400i\tau_1+4\tau_1\sqrt{2399}-\tau_2{}^2\sqrt{2399}\right)\sqrt{-35\Delta_4{}^{-1}}\tau_1}{\left(-\tau_2-\sqrt{\tau_2{}^2-4\tau_1}\tanh\left(1/2\sqrt{\tau_2{}^2-4\tau_1}\left(x-\mathcal{V}t+C\right)\right)\right)}\\ &+\frac{144\sqrt{-35\Delta_4{}^{-1}}\tau_2\tau_1{}^2}{\left(-\tau_2-\sqrt{\tau_2{}^2-4\tau_1}\tanh\left(1/2\sqrt{\tau_2{}^2-4\tau_1}\left(x-\mathcal{V}t+C\right)\right)\right)^2}\\ &+\frac{192\sqrt{-35\Delta_4{}^{-1}}\tau_1{}^3}{\left(-\tau_2-\sqrt{\tau_2{}^2-4\tau_1}\tanh\left(1/2\sqrt{\tau_2{}^2-4\tau_1}\left(x-\mathcal{V}t+C\right)\right)\right)^3}\Bigg)\ e^{i\left(-\kappa x+wt+\sigma W(t)-\sigma^2 t\right)}, \end{aligned} \tag{43}$$

and

$$r_6(x,t)=\chi q_6(x,t). \quad (44)$$

Complexitons are denoted by these solutions, subject to the conditions given by $\tau_2^2-4\tau_1>0$ and $\Delta_4<0$. These solutions contain singularities of unifications of both exponential and trigonometric function waves that possess novel style distinct travelling wave speeds.

If $\tau_2^2-4\tau_1<0,\tau_1\neq 0$, we obtain

$$\begin{aligned} q_7(x,t)=&\left(\frac{3}{1343}i\left(-4200i\tau_1+12\tau_1\sqrt{2399}-293i\tau_2{}^2-3\tau_2{}^2\sqrt{2399}\right)\sqrt{-35\Delta_4{}^{-1}}\tau_2\right.\\ &+\frac{36}{1343}\frac{i\left(-993i\tau_2{}^2-1400i\tau_1+4\tau_1\sqrt{2399}-\tau_2{}^2\sqrt{2399}\right)\sqrt{-35\Delta_4{}^{-1}}\tau_1}{\left(-\tau_2+\sqrt{4\tau_1-\tau_2{}^2}\tan\left(1/2\sqrt{4\tau_1-\tau_2{}^2}\left(x-vt+C\right)\right)\right)}\\ &+\frac{144\sqrt{-35\Delta_4{}^{-1}}\tau_2\tau_1{}^2}{\left(-\tau_2+\sqrt{4\tau_1-\tau_2{}^2}\tan\left(1/2\sqrt{4\tau_1-\tau_2{}^2}\left(x-vt+C\right)\right)\right)^2}\\ &\left.+\frac{192\sqrt{-35\Delta_4{}^{-1}}\tau_1{}^3}{\left(-\tau_2+\sqrt{4\tau_1-\tau_2{}^2}\tan\left(1/2\sqrt{4\tau_1-\tau_2{}^2}\left(x-vt+C\right)\right)\right)^3}\right)\times e^{i\left(-\kappa x+\omega t+\sigma W(t)-\sigma^2 t\right)}, \end{aligned} \quad (45)$$

and

$$r_7(x,t)=\chi q_7(x,t). \quad (46)$$

These solutions stand for the complexitons solutions based on the conditions $\tau_2^2-4\tau_1<0$ and $\Delta_4<0$.

If $\tau_2^2-4\tau_1>0,\tau_1=0,\tau_2\neq 0,$ we have

$$\begin{aligned} q_8(x,t)=&\left(\frac{3}{1343}i\left(-293i\tau_2{}^2-3\tau_2{}^2\sqrt{2399}\right)\sqrt{-35\Delta_4{}^{-1}}\tau_2\right.\\ &+\frac{18}{1343}i\left(-993i\tau_2{}^2-\tau_2{}^2\sqrt{2399}\right)\\ &\left.\times\frac{\sqrt{-35\Delta_4{}^{-1}}\tau_2}{\left(e^{\tau_2(x-vt+C)}-1\right)}+\frac{36\sqrt{-35\Delta_4{}^{-1}}\tau_2{}^3}{\left(e^{\tau_2(x-vt+C)}-1\right)^2}+\frac{24\sqrt{-35\,\Delta_4{}^{-1}}\tau_2{}^3}{\left(e^{\tau_2(x-vt+C)}-1\right)^3}\right)\\ &\times e^{i\left(-\kappa x+\omega t+\sigma W(t)-\sigma^2 t\right)}, \end{aligned} \quad (47)$$

and

$$r_8(x,t)=\chi q_8(x,t). \quad (48)$$

The solutions obtained are indicative of complexitons, given that the condition $\Delta_4<0$ is satisfied.

If $\tau_2^2-4\tau_1=0,\tau_1\neq 0,\tau_2\neq 0,$ the rational solutions are obtained:

$$\begin{aligned} q_9(x,t)=&\left(3\sqrt{-35\Delta_4{}^{-1}}\tau_2{}^3-\frac{18\sqrt{-35\Delta_4{}^{-1}}\tau_2{}^4\left(x-vt+C\right)}{\left(2\tau_2\left(x-vt+C\right)+4\right)}\right.\\ &\left.+\frac{36\sqrt{-35\Delta_4{}^{-1}}\tau_2{}^5\left(x-vt+C\right)^2}{\left(2\tau_2\left(x-vt+C\right)+4\right)^2}-\frac{24\sqrt{-35\Delta_4{}^{-1}}\tau_2{}^6\left(x-vt+C\right)^3}{\left(2\tau_2\left(x-vt+C\right)+4\right)^3}\right)\\ &\times e^{i\left(-\kappa x+\omega t+\sigma W(t)-\sigma^2 t\right)}, \end{aligned} \quad (49)$$

and

$$r_9(x,t)=\chi q_9(x,t). \tag{50}$$

If $\tau_2^2-4\tau_1=0, \tau_1=\tau_2=0$, the other rational solutions can be found:

$$q_{10}(x,t)=\frac{24\sqrt{-35\Delta_4{}^{-1}}}{\left(x-vt+C\right)^3}\, e^{i\left(-\kappa x+\omega t+\sigma W(t)-\sigma^2 t\right)}, \tag{51}$$

and

$$r_{10}(x,t)=\chi q_{10}(x,t). \tag{52}$$

4.2. Implementations of the (G'/G)-expansion method

Using the homogeneous balance principle in Eq. (9), we find N = 3. Then Eq. (21) degenerates to the following form:

$$q=A_0+A_1\left(\frac{G'}{G}\right)+A_2\left(\frac{G'}{G}\right)^2+A_3\left(\frac{G'}{G}\right)^3. \tag{53}$$

Substituting Eqs. (53) and (22) into Eq. (9), collecting the coefficients of $(G'/G)^i$ and putting them to be zero, we obtain an algebraic system. Solving this system by Maple, we derive the following sets of parameters.

Set 1: $$\left\{\begin{array}{l} A_0=-6\sqrt{-35\Delta_4{}^{-1}}B\left(-6C+B^2\right),\, A_1=72\sqrt{-35\Delta_4{}^{-1}}C,\, A_2=36\sqrt{-35\Delta_4{}^{-1}}B, \\ A_3=24\sqrt{-35\Delta_4{}^{-1}},\, \Delta_1=-83B^2+332C,\, \Delta_2=946\left(B^2-4C\right)^2,\, \Delta_3=1260\left(B^2-4C\right)^3. \end{array}\right\} \tag{54}$$

Inserting Eq. (54) into Eq. (53), one can obtain the hyperbolic, trigonometric and rational-function solutions.

When $B^2-4C>0$, the solitary-wave solutions are as follows:

$$\begin{aligned} q_{11}(x,t)=&\sqrt{\frac{-35}{\Delta_4}}\Bigg(-6B\left(-6C+B^2\right) \\ &+72\left(-\frac{B}{2}+\frac{\sqrt{B^2-4C}\left(C_1\sinh\left(\frac{\sqrt{B^2-4C}}{2}(x-vt)\right)+C_2\cosh\left(\frac{\sqrt{B^2-4C}}{2}(x-vt)\right)\right)}{2\left(C_1\cosh\left(\frac{\sqrt{B^2-4C}}{2}(x-vt)\right)+C_2\sinh\left(\frac{\sqrt{B^2-4C}}{2}(x-vt)\right)\right)}\right)C \\ &+36\left(-\frac{B}{2}+\frac{\sqrt{B^2-4C}\left(C_1\sinh\left(\frac{\sqrt{B^2-4C}}{2}(x-vt)\right)+C_2\cosh\left(\frac{\sqrt{B^2-4C}}{2}(x-vt)\right)\right)}{2\left(C_1\cosh\left(\frac{\sqrt{B^2-4C}}{2}(x-vt)\right)+C_2\sinh\left(\frac{\sqrt{B^2-4C}}{2}(x-vt)\right)\right)}\right)^2 B \\ &+24\left(-\frac{B}{2}+\frac{\sqrt{B^2-4C}\left(C_1\sinh\left(\frac{\sqrt{B^2-4C}}{2}(x-vt)\right)+C_2\cosh\left(\frac{\sqrt{B^2-4C}}{2}(x-vt)\right)\right)}{2\left(C_1\cosh\left(\frac{\sqrt{B^2-4C}}{2}(x-vt)\right)+C_2\sinh\left(\frac{\sqrt{B^2-4C}}{2}(x-vt)\right)\right)}\right)^3\Bigg) \\ &\times e^{i\left(-\kappa x+\omega t+\sigma W(t)-\sigma^2 t\right)}, \end{aligned} \tag{55}$$

and

$$r_{11}(x,t)=\chi q_{11}(x,t). \tag{56}$$

Putting $C_1=0, C_2\neq 0$, we have the singular-soliton solutions

$$q_{11}(x,t)=\left(-6\sqrt{-35\Delta_4^{-1}}B\left(-6C+B^2\right)\right.$$
$$+72\left(-\frac{B}{2}+\frac{\sqrt{B^2-4C}}{2}\coth\left(\frac{\sqrt{B^2-4C}}{2}(x-vt)\right)\right)\sqrt{-35\Delta_4^{-1}}C$$
$$+36\left(-\frac{B}{2}+\frac{\sqrt{B^2-4C}}{2}\coth\left(\frac{\sqrt{B^2-4C}}{2}(x-vt)\right)\right)^2\sqrt{-35\Delta_4^{-1}}B \tag{57}$$
$$\left.+24\left(-\frac{B}{2}+\frac{\sqrt{B^2-4C}}{2}\coth\left(\frac{\sqrt{B^2-4C}}{2}(x-vt)\right)\right)^3\sqrt{-35\Delta_4^{-1}}\right)\times e^{i\left(-\kappa x+\omega t+\sigma W(t)-\sigma^2 t\right)}.$$

When putting $C_2=0, C_1\neq 0$, one gets the dark-soliton solutions

$$q_{11}(x,t)=\left(-6\sqrt{-35\Delta_4^{-1}}B\left(-6C+B^2\right)\right.$$
$$+72\left(-\frac{B}{2}+\frac{\sqrt{B^2-4C}}{2}\tanh\left(\frac{\sqrt{B^2-4C}}{2}(x-vt)\right)\right)\sqrt{-35\Delta_4^{-1}}C$$
$$+36\left(-\frac{B}{2}+\frac{\sqrt{B^2-4C}}{2}\tanh\left(\frac{\sqrt{B^2-4C}}{2}(x-vt)\right)\right)^2\sqrt{-35\Delta_4^{-1}}B \tag{58}$$
$$\left.+24\left(-\frac{B}{2}+\frac{\sqrt{B^2-4C}}{2}\tanh\left(\frac{\sqrt{B^2-4C}}{2}(x-vt)\right)\right)^3\sqrt{-35\Delta_4^{-1}}\right)\times e^{i\left(-\kappa x+\omega t+\sigma W(t)-\sigma^2 t\right)},$$

and $r_{11}(x,t)=\chi q_{11}(x,t)$.

In the case of $B^2-4C<0$ one gets

$$q_{12}(x,t)=\sqrt{\frac{-35}{\Delta_4}}\left(-6B\left(-6C+B^2\right)\right.$$
$$+72\left(-\frac{B}{2}+\frac{\sqrt{-B^2+4C}\left(-C_1\sin\left(\frac{\sqrt{-B^2+4C}}{2}(x-vt)\right)+C_2\cos\left(\frac{\sqrt{-B^2+4C}}{2}(x-vt)\right)\right)}{2\left(C_1\cos\left(\frac{\sqrt{-B^2+4C}}{2}(x-vt)\right)+C_2\sin\left(\frac{\sqrt{-B^2+4C}}{2}(x-\mathcal{V}t)\right)\right)}\right)C$$
$$+36\left(-\frac{B}{2}+\frac{\sqrt{-B^2+4C}\left(-C_1\sin\left(\frac{\sqrt{-B^2+4C}}{2}(x-vt)\right)+C_2\cos\left(\frac{\sqrt{-B^2+4C}}{2}(x-vt)\right)\right)}{2\left(C_1\cos\left(\frac{\sqrt{-B^2+4C}}{2}(x-vt)\right)+C_2\sin\left(\frac{\sqrt{-B^2+4C}}{2}(x-vt)\right)\right)}\right)^2B \tag{59}$$
$$\left.+24\left(-\frac{B}{2}+\frac{\sqrt{-B^2+4C}\left(-C_1\sin\left(\frac{\sqrt{-B^2+4C}}{2}(x-vt)\right)+C_2\cos\left(\frac{\sqrt{-B^2+4C}}{2}(x-vt)\right)\right)}{2\left(C_1\cos\left(\frac{\sqrt{-B^2+4C}}{2}(x-vt)\right)+C_2\sin\left(\frac{\sqrt{-B^2+4C}}{2}(x-vt)\right)\right)}\right)^3\right)$$
$$\times e^{i\left(-\kappa x+\omega t+\sigma W(t)-\sigma^2 t\right)},$$

and

$$r_{12}(x,t)=\chi q_{12}(x,t). \tag{60}$$

When $C_1=0, C_2\neq 0$, one obtains the periodic solutions

$$q_{12}(x,t)=\left(-6\sqrt{-35\Delta_4^{-1}}B\left(-6C+B^2\right)\right.$$
$$+72\left(-\frac{B}{2}+\frac{\sqrt{-B^2+4C}}{2}\cot\left(\frac{\sqrt{-B^2+4C}}{2}(x-vt)\right)\right)\sqrt{-35\Delta_4^{-1}}C$$
$$+36\left(-\frac{B}{2}+\frac{\sqrt{-B^2+4C}}{2}\cot\left(\frac{\sqrt{-B^2+4C}}{2}(x-vt)\right)\right)^2\sqrt{-35\Delta_4^{-1}}B \tag{61}$$
$$\left.+24\left(-\frac{B}{2}+\frac{\sqrt{-B^2+4C}}{2}\cot\left(\frac{\sqrt{-B^2+4C}}{2}(x-vt)\right)\right)^3\sqrt{-35\Delta_4^{-1}}\right)e^{i\left(-\kappa x+\omega t+\sigma W(t)-\sigma^2 t\right)}.$$

When $C_2=0, C_1\neq 0$, one arrives at the periodic solutions

$$q_{12}(x,t)=\left(-6\sqrt{-35\Delta_4^{-1}}B\left(-6C+B^2\right)\right.$$
$$+72\left(-\frac{B}{2}-\frac{\sqrt{-B^2+4C}}{2}\tan\left(\frac{\sqrt{-B^2+4C}}{2}(x-vt)\right)\right)\sqrt{-35\Delta_4^{-1}}C$$
$$+36\left(-\frac{B}{2}-\frac{\sqrt{-B^2+4C}}{2}\tan\left(\frac{\sqrt{-B^2+4C}}{2}(x-vt)\right)\right)^2\sqrt{-35\Delta_4^{-1}}B \tag{62}$$
$$\left.+24\left(-\frac{B}{2}-\frac{\sqrt{-B^2+4C}}{2}\tan\left(\frac{\sqrt{-B^2+4C}}{2}(x-vt)\right)\right)^3\sqrt{-35\Delta_4^{-1}}\right)e^{i\left(-\kappa x+\omega t+\sigma W(t)-\sigma^2 t\right)},$$

and $r_{12}(x,t)=\chi q_{12}(x,t)$.

When $B^2-4C=0$, the rational solutions are obtained as

$$q_{13}(x,t)=\left(3\sqrt{\frac{-35}{\Delta_4}}B^3+18\left(\frac{C_2}{C_1+C_2(x-vt)}-\frac{B}{2}\right)\sqrt{\frac{-35}{\Delta_4}}B^2\right.$$
$$\left.+36\left(\frac{C_2}{C_1+C_2(x-vt)}-\frac{B}{2}\right)^2\sqrt{\frac{-35}{\Delta_4}}B+24\left(\frac{C_2}{C_1+C_2(x-vt)}-\frac{B}{2}\right)^3\sqrt{\frac{-35}{\Delta_4}}\right)e^{i\left(-\kappa x+\omega t+\sigma W(t)-\sigma^2 t\right)}, \tag{63}$$

and

$$r_{13}(x,t)=\chi q_{13}(x,t). \tag{64}$$

$$\textbf{Set 2:}\left\{\begin{array}{c}
A_0=\frac{3}{1343}i\left(-4200iC-293iB^2+3B^2\sqrt{2399}-12C\sqrt{2399}\right)B\sqrt{\frac{-35}{\Delta_4}},\\
A_1=\frac{18}{1343}i\left(-993iB^2-1400iC+B^2\sqrt{2399}-4C\sqrt{2399}\right)\sqrt{\frac{-35}{\Delta_4}},\\
A_2=36\sqrt{\frac{-35}{\Delta_4}}B, A_3=24\sqrt{\frac{-35}{\Delta_4}}, \Delta_1=\frac{83}{2686}\left(B^2-4C\right)\left(293+3i\sqrt{2399}\right),\\
\Delta_2=\frac{1}{3607298}\left(44697365+1880673i\sqrt{2399}\right)\left(B^2-4C\right)^2,\\
\Delta_3=\frac{630}{1803649}\left(32129+879i\sqrt{2399}\right)\left(B^2-4C\right)^3.
\end{array}\right\} \tag{65}$$

Inserting Eq. (65) into Eq. (53), one can obtain the following solutions.
If $B^2-4C>0$, we have

$$q_{14}(x,t)=\sqrt{\frac{-35}{\Delta_4}}\left(\frac{3}{1343}i\left(-4200iC-293iB^2+3B^2\sqrt{2399}-12C\sqrt{2399}\right)B\right.$$
$$+\frac{18}{1343}i\left(-\frac{B}{2}+\frac{\sqrt{B^2-4C}\left(C_1\sinh(\Psi)+C_2\cosh(\Psi)\right)}{2\left(C_1\cosh(\Psi)+C_2\sinh(\Psi)\right)}\right)$$
$$\times\left(-C\left(4\sqrt{2399}+1400i\right)+B^2\left(\sqrt{2399}-993i\right)\right) \quad (66)$$
$$+36\left(-\frac{B}{2}+\frac{\sqrt{B^2-4C}\left(C_1\sinh(\Psi)+C_2\cosh(\Psi)\right)}{2\left(C_1\cosh(\Psi)+C_2\sinh(\Psi)\right)}\right)^2B$$
$$\left.+24\left(-\frac{B}{2}+\frac{\sqrt{B^2-4C}\left(C_1\sinh(\Psi)+C_2\cosh(\Psi)\right)}{2\left(C_1\cosh(\Psi)+C_2\sinh(\Psi)\right)}\right)^3\right)e^{i\left(-\kappa x+\omega t+\sigma W(t)-\sigma^2 t\right)},$$

and

$$r_{14}(x,t)=\chi q_{14}(x,t), \quad (67)$$

where $\Psi=\frac{\sqrt{B^2-4C}}{2}\left(x-\mathcal{V}t\right)$.

Putting $B^2-4C>0$ and $\Delta_4<0$ into Eqs. (66) and (67) leads to complexions.
If $B^2-4C<0$, we obtain

$$q_{15}(x,t)=\sqrt{\frac{-35}{\Delta_4}}\left(\frac{3}{1343}i\left(-4200iC-293iB^2+3B^2\sqrt{2399}-12C\sqrt{2399}\right)B\right.$$
$$+\frac{18}{1343}i\left(-\frac{B}{2}+\frac{\sqrt{-B^2+4C}\left(-C_1\sin(\Upsilon)+C_2\cos(\Upsilon)\right)}{2\left(C_1\cos(\Upsilon)+C_2\sin(\Upsilon)\right)}\right)$$
$$\times\left(-C\left(4\sqrt{2399}+1400i\right)+B^2\left(\sqrt{2399}-993i\right)\right) \quad (68)$$
$$+36\left(-\frac{B}{2}+\frac{\sqrt{-B^2+4C}\left(-C_1\sin(\Upsilon)+C_2\cos(\Upsilon)\right)}{2\left(C_1\cos(\Upsilon)+C_2\sin(\Upsilon)\right)}\right)^2B$$
$$\left.+24\left(-\frac{B}{2}+\frac{\sqrt{-B^2+4C}\left(-C_1\sin(\Upsilon)+C_2\cos(\Upsilon)\right)}{2\left(C_1\cos(\Upsilon)+C_2\sin(\Upsilon)\right)}\right)^3\right)e^{i\left(-\kappa x+\omega t+\sigma W(t)-\sigma^2 t\right)},$$

and

$$r_{15}(x,t)=\chi q_{15}(x,t), \quad (69)$$

with $\Upsilon=\frac{\sqrt{-B^2+4C}}{2}\left(x-vt\right)$. The above solutions correspond to complexions solutions if the conditions $B^2-4C<0$ and $\Delta_4<0$ are held.

When $B^2-4C=0$, one reveals the rational solutions

$$q_{16}(x,t)=\sqrt{\frac{-35}{\Delta_4}}\left(3B^3+18\left(\frac{C_2}{C_1+C_2(x-vt)}-\frac{B}{2}\right)B^2\right.$$
$$\left.+36\left(\frac{C_2}{C_1+C_2(x-vt)}-\frac{B}{2}\right)^2B+24\left(\frac{C_2}{C_1+C_2(x-vt)}-\frac{B}{2}\right)^3\right)e^{i\left(-\kappa x+\omega t+\sigma W(t)-\sigma^2 t\right)}, \quad (70)$$

and

$$r_{16}(x,t)=\chi q_{16}(x,t). \quad (71)$$

4.3. Implementation of the simplest-equation method

Using the homogeneous balance principle in Eq. (9), we find $N = 3$. Then Eq. (24) degenerates to the following form:

$$q(\xi)=k_0+k_1w(\xi)+k_2w(\xi)^2+k_3w(\xi)^3. \tag{72}$$

The Bernoulli equation

Substituting Eq. (72) and Eq. (25) into Eq. (9), collecting the coefficients of $w(\xi)^i$ and setting them to zero, we arrive at an algebraic system. Solving this system by Maple, we derive the following sets of parameters.

Set 1:
$$\left\{\begin{array}{c} k_0=-6a^3\sqrt{-\frac{35}{\Delta_4}},k_1=0,k_2=36\sqrt{-\frac{35}{\Delta_4}}b^2a,k_3=24\sqrt{-\frac{35}{\Delta_4}}b^3,\\ \Delta_1=-83a^2,\Delta_2=946a^4,\Delta_3=1260a^6. \end{array}\right\} \tag{73}$$

Consequently, the solutions of Eqs. (1) and (2) are given by

$$\begin{aligned} q_{17}(x,t)=&\left(-6a^3\sqrt{-\frac{35}{\Delta_4}}+36\sqrt{-\frac{35}{\Delta_4}}b^2a^3\frac{\left(\cosh(a(x-vt+C))+\sinh(a(x-vt+C))\right)^2}{\left(1-b\cosh(a(x-vt+C))-b\sinh(a(x-vt+C))\right)^2}\right.\\ &\left.+24\sqrt{-\frac{35}{\Delta_4}}b^3a^3\frac{\left(\cosh(a(x-vt+C))+\sinh(a(x-vt+C))\right)^3}{\left(1-b\cosh(a(x-vt+C))-b\sinh(a(x-vt+C))\right)^3}\right)e^{i(-\kappa x+\omega t+\sigma W(t)-\sigma^2 t)}, \end{aligned} \tag{74}$$

and

$$r_{17}(x,t)=\chi q_{17}(x,t). \tag{75}$$

Set 2:
$$\left\{\begin{array}{l} k_0=-6a^3\left(\sqrt{-\frac{35}{\Delta_4}}-\frac{3}{2686}\left(993i\sqrt{35}+\sqrt{83965}\right)\frac{1}{\sqrt{\Delta_4}}\right),k_2=36\sqrt{-\frac{35}{\Delta_4}}b^2a,\\ k_1=\frac{18}{1343}\left(993i\sqrt{35}+\sqrt{83965}\right)ba^2\frac{1}{\sqrt{\Delta_4}},k_3=24\sqrt{-\frac{35}{\Delta_4}}b^3,\\ \Delta_1=-83a^2\left(\sqrt{-\frac{35}{\Delta_4}}-\frac{3}{2686}\left(993i\sqrt{35}+\sqrt{83965}\right)\frac{1}{\sqrt{\Delta_4}}\right)\sqrt{-\frac{\Delta_4}{35}},\\ \Delta_2=-\frac{1}{47005}a^4\left(23752190+\frac{1880673}{2686}\sqrt{-35}\left(993\sqrt{-35}+\sqrt{83965}\right)\right),\\ \Delta_3=-\frac{36}{1343}a^6\left(10955+\frac{879}{2686}\sqrt{-35}\left(993\sqrt{-35}+\sqrt{83965}\right)\right). \end{array}\right\} \tag{76}$$

The solutions of Eqs. (1) and (2) are as follows:

$$\begin{aligned} q_{18}(x,t)=&\left(-6a^3\left(\sqrt{-\frac{35}{\Delta_4}}-\frac{3}{2686}\left(993i\sqrt{35}+\sqrt{83965}\right)\frac{1}{\sqrt{\Delta_4}}\right)\right.\\ &+\frac{18}{1343}\left(993i\sqrt{35}+\sqrt{83965}\right)ba^3\frac{\left(\cosh(a(x-vt+C))+\sinh(a(x-vt+C))\right)}{\left(1-b\cosh(a(x-vt+C))-b\sinh(a(x-vt+C))\right)}\frac{1}{\sqrt{\Delta_4}}\\ &+36\sqrt{-\frac{35}{\Delta_4}}b^2a^3\frac{\left(\cosh(a(x-vt+C))+\sinh(a(x-vt+C))\right)^2}{\left(1-b\cosh(a(x-vt+C))-b\sinh(a(x-vt+C))\right)^2}\\ &\left.+24\sqrt{-\frac{35}{\Delta_4}}b^3a^3\frac{\left(\cosh(a(x-vt+C))+\sinh(a(x-vt+C))\right)^3}{\left(1-b\cosh(a(x-vt+C))-b\sinh(a(x-vt+C))\right)^3}\right)e^{i(-\kappa x+\omega t+\sigma W(t)-\sigma^2 t)}, \end{aligned} \tag{77}$$

and

$$r_{18}(x,t) = \chi q_{18}(x,t). \tag{78}$$

If the condition $\Delta_4 > 0$ is satisfied, the solutions of Eq. (77) and (78) correspond to complexions.

The Riccati equation

Substituting Eq. (72) and (27) into Eq. (9), collecting the coefficients of $w(\xi)^i$ and putting them to zero, we obtain an algebraic system. Solving this system with Maple, we derive the following sets of parameters.

$$\textbf{Set 1:}\quad \left\{\begin{aligned}
&k_0 = \frac{3}{1343}B\sqrt{35}\left(\sqrt{-\frac{1}{\Delta_4}}\left(293B^2+4200DA\right)+3\sqrt{2399}\sqrt{\frac{\left(-B^2+4DA\right)^2}{\Delta_4}}\right),\\
&k_1 = \frac{18}{1343}\sqrt{35}\left(\sqrt{-\frac{1}{\Delta_4}}\left(1400DA+993B^2\right)+\sqrt{2399}\sqrt{\frac{\left(-B^2+4DA\right)^2}{\Delta_4}}\right)A,\\
&k_2 = 36\sqrt{-\frac{35}{\Delta_4}}A^2B,\quad k_3 = 24\sqrt{-\frac{35}{\Delta_4}}A^3,\\
&\Delta_1 = -\frac{83\sqrt{-\Delta_4}}{2686}\left(293\sqrt{-\frac{1}{\Delta_4}}\left(-B^2+4DA\right)-3\sqrt{2399}\sqrt{\frac{\left(-B^2+4DA\right)^2}{\Delta_4}}\right),\\
&\Delta_2 = \frac{\sqrt{-\Delta_4}}{3607298}\left(-B^2+4DA\right)\\
&\times\left(44697365\sqrt{-\frac{1}{\Delta_4}}\left(-B^2+4DA\right)-1880673\sqrt{2399}\sqrt{\frac{\left(-B^2+4DA\right)^2}{\Delta_4}}\right),\\
&\Delta_3 = -\frac{630\sqrt{-\Delta_4}}{1803649}\left(-B^2+4DA\right)^2\\
&\times\left(32129\sqrt{-\frac{1}{\Delta_4}}\left(-B^2+4DA\right)-879\sqrt{2399}\sqrt{\frac{\left(-B^2+4DA\right)^2}{\Delta_4}}\right).
\end{aligned}\right\} \tag{79}$$

Then the dark-soliton solutions of Eq. (1) and (2) are given by

$$\begin{aligned}
q_{19}(x,t) = {}&\frac{3\sqrt{35}}{1343}\sqrt{B^2-4DA}\tanh\left(\frac{\sqrt{B^2-4DA}}{2}(x-vt+C)\right)\\
&\times\left(-1050\sqrt{-\frac{1}{\Delta_4}}\left(4AD-B^2\right)-3\sqrt{2399}\sqrt{\frac{\left(-B^2+4DA\right)^2}{\Delta_4}}\right.\\
&\left.+1343\left(\tanh\left(\frac{\sqrt{B^2-4DA}}{2}(x-vt+C)\right)\right)^2\sqrt{-\frac{1}{\Delta_4}}\left(4AD-B^2\right)\right)\\
&\times e^{i\left(-\kappa x+\omega t+\sigma W(t)-\sigma^2 t\right)},
\end{aligned} \tag{80}$$

and

$$r_{19}(x,t) = \chi q_{19}(x,t). \tag{81}$$

Using the Riccati equation, the other hyperbolic-function solutions can be written out:

$$q_{20}(x,t)=\left(\frac{3}{1343}B\sqrt{35}\left(\sqrt{-\frac{1}{\Delta_4}\left(293B^2+4200DA\right)}+3\sqrt{2399}\sqrt{\frac{\left(-B^2+4DA\right)^2}{\Delta_4}}\right)\right.$$
$$+\frac{18}{1343}\sqrt{35}\left(\sqrt{-\frac{1}{\Delta_4}\left(1400DA+993B^2\right)}+\sqrt{2399}\sqrt{\frac{\left(-B^2+4DA\right)^2}{\Delta_4}}\right)AH \tag{82}$$
$$\left.+36\sqrt{-\frac{35}{\Delta_4}}A^2BH^2+24\sqrt{-\frac{35}{\Delta_4}}A^3H^3\right)\times e^{i\left(-\kappa x+\omega t+\sigma W(t)-\sigma^2 t\right)},$$

and

$$r_{20}(x,t)=\chi q_{20}(x,t). \tag{83}$$

Here we have

$$H=-\frac{B+\sqrt{B^2-4DA}\tanh\left(\frac{\sqrt{B^2-4DA}}{2}(x-vt)\right)}{2A}$$
$$+\frac{\operatorname{sech}\left(\frac{\sqrt{B^2-4DA}}{2}(x-vt)\right)}{C\cosh\left(\frac{\sqrt{B^2-4DA}}{2}(x-vt)\right)-2\frac{A\sinh\left(\frac{\sqrt{B^2-4DA}}{2}(x-vt)\right)}{\sqrt{B^2-4DA}}}. \tag{84}$$

5. Discussion

Here we present the numerical simulations to illustrate the physical properties of the solutions obtained by us. 3D plots of some of our results have been obtained using Maple. With the three methods employed, we have obtained a total of seven different parameter sets and the corresponding solutions. These solutions involve a large number of parameters. Because these parameters affect the shapes of the solutions, one can generate a wide variety of plots by taking random values for the parameters. Using these plots, one can ascertain the nature of the solitons.

While drawing the plots, we have selected the parameter values given by

$$\left\{\begin{array}{c} A=1, B=8, C=1, C_1=1, C_2=0, a_{11}=1, a_{12}=1, a_{13}=\frac{46}{9}, a_{14}=\frac{14}{9}, a_{15}=-1, \\ a_{16}=-1, a_{23}=\frac{46}{9}, a_{24}=1, a_{25}=1, a_{26}=1, b_1=15, c_1=\frac{9}{4}, c_2=\frac{126}{25}, \chi=\frac{1}{2}, d_1=2, \\ d_2=\frac{42}{25}, e_1=3, f_1=\frac{20}{11}, f_2=\frac{25}{28}, \mu_1=1/2, \mu_2=1/4, v_1=1, v_2=\frac{1}{3}, \omega=1, \\ \alpha_1=0.5, \delta_1=2, \eta_1=1, \eta_2=3, \gamma_1=2, \lambda_1=1, \lambda_2=1, \sigma=1, \theta_1=1, \theta_2=2, W(t)=t \end{array}\right\} \tag{85}$$

$$\left\{\begin{array}{c} a=-2, b=-2, C=0, a_{11}=1, a_{12}=1, a_{13}=\frac{46}{9}, a_{14}=\frac{14}{9}, a_{15}=-1, \\ a_{16}=-1, a_{23}=\frac{46}{9}, a_{24}=1, a_{25}=1, a_{26}=1, b_1=1, c_1=12, c_2=4, \chi=3, d_1=2, \\ d_2=\frac{42}{25}, e_1=3, f_1=\frac{20}{11}, f_2=\frac{25}{28}, \mu_1=1/2, \mu_2=1/4, v_1=1, v_2=\frac{1}{3}, \omega=1, \\ \alpha_1=3, \delta_1=2, \eta_1=1, \eta_2=3, \gamma_1=2, \lambda_1=\frac{193}{33}, \lambda_2=1, \sigma=1, \theta_1=1, \theta_2=\frac{13}{7}, W(t)=t \end{array}\right\} \tag{86}$$

for the cases illustrated by Fig. 1 and Fig. 3, respectively.

Fig. 1 and Fig. 3 display the dark-soliton solutions for the squares of modules of the solutions of Eqs. (55) and (56) and Eqs. (74) and (75), respectively. Fig. 2a shows the plot of $|q_{11}(x,t)|^2$ for $t=1,2,3,4$. This wave propagates to the left along the *x*–axis. Fig. 2b and Fig. 2c depict the plots of $\mathrm{Re}(q_{11}(x,t))$ and $\mathrm{Im}(q_{11}(x,t))$ for the cases $\sigma=1.3, 2.1, 2.41, 3$. After examining carefully all the plots, one can conclude that Fig. 2b and Fig. 2c imply that the wavelength of the soliton decreases with increasing noise effect. Fig. 4a shows a graph of $|q_{17}(x,t)|^2$ for $t=1,2,3,4$. One can see that the wave moves to the right along the *x*–axis. Fig. 4b and Fig. 4c depict the plots of $\mathrm{Re}(q_{17}(x,t))$ and $\mathrm{Im}(q_{17}(x,t))$ for the cases $\sigma=0.1, 0.5, 0.95, 1.3$. Here the noise effect is observed as a fluctuation on the soliton. This effect becomes the greatest at $\sigma=1.3$.

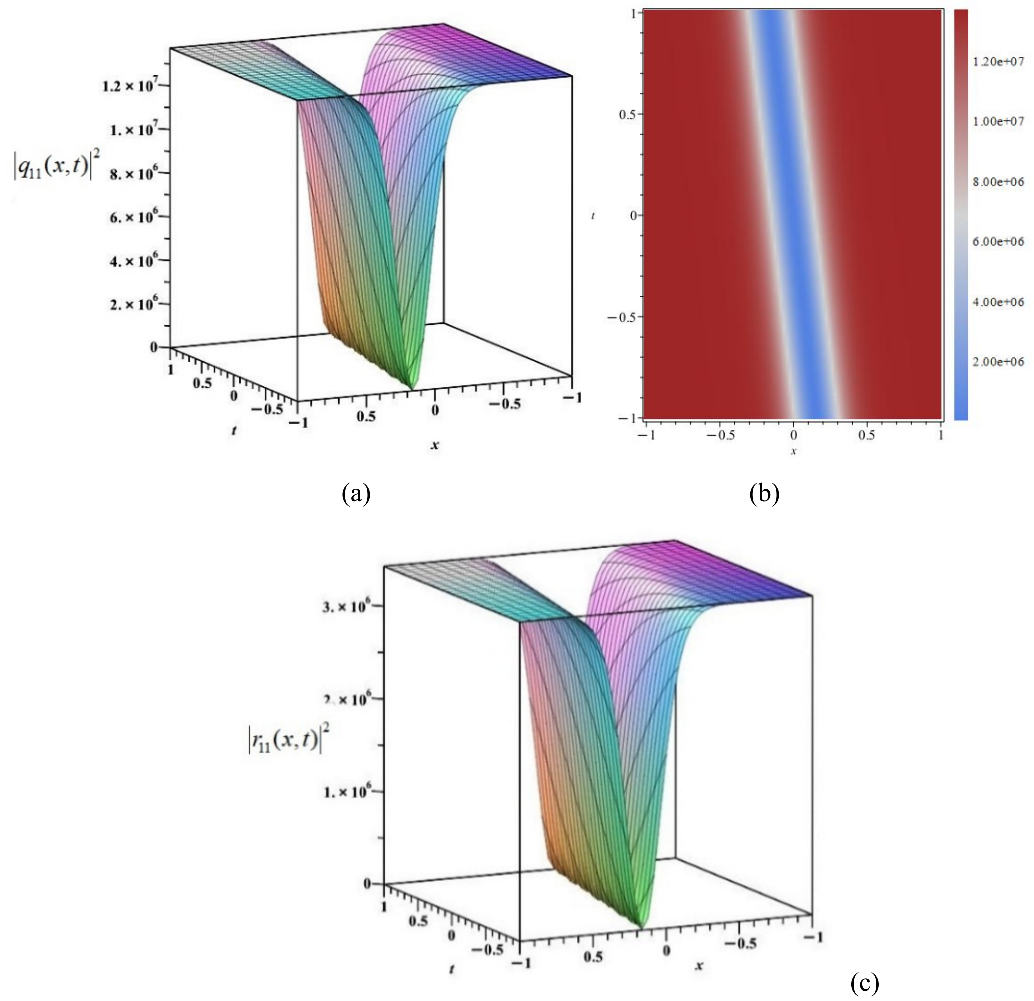
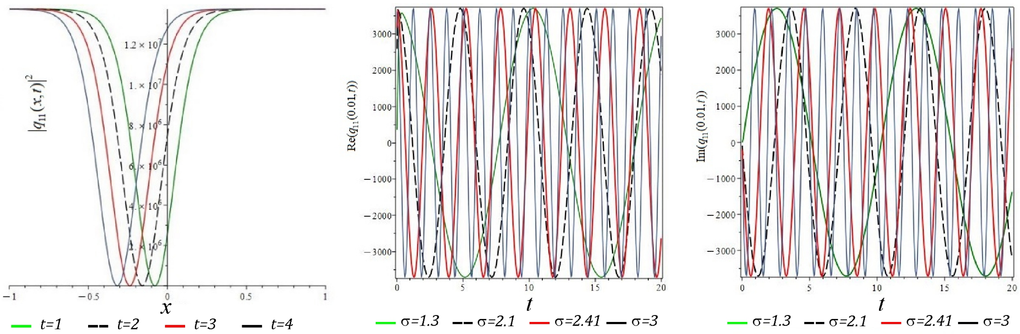
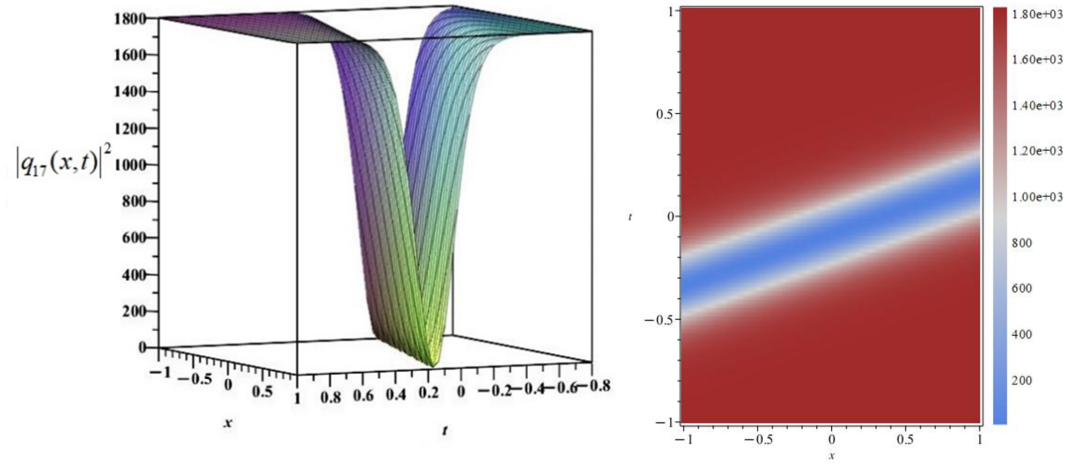


Fig. 1. Profiles of dark-soliton solutions given by Eqs. (55) and (56): (a) surface plot of square of the modulus of the solution $q_{11}(x,t)$, (b) density plot of square of the modulus of the solution $q_{11}(x,t)$, and (c) surface plot of square of the modulus of the solution $r_{11}(x,t)$ ..

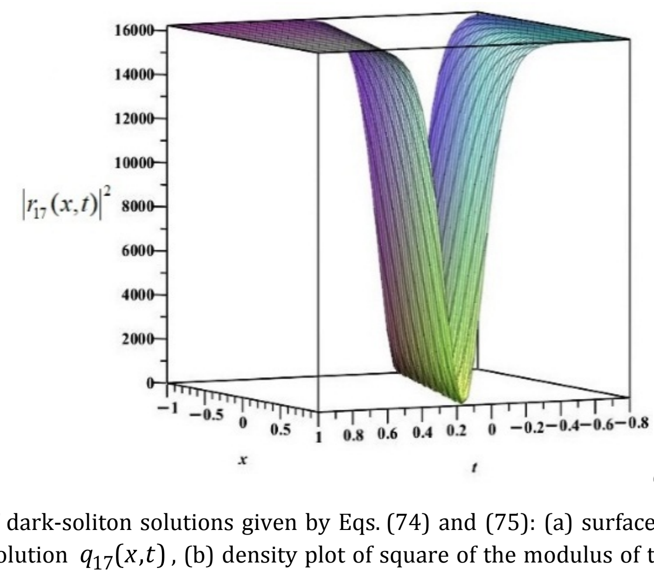


(a) (b) (c)

Fig. 2. Plots of $q_{11}(x,t)$: (a) square of the modulus of the solution $q_{11}(x,t)$ for different t , (b) real part of the solution $q_{11}(0.01,t)$ for different noise values σ , and (c) imaginary part of the solution $q_{11}(0.01,t)$ for different noise values σ .

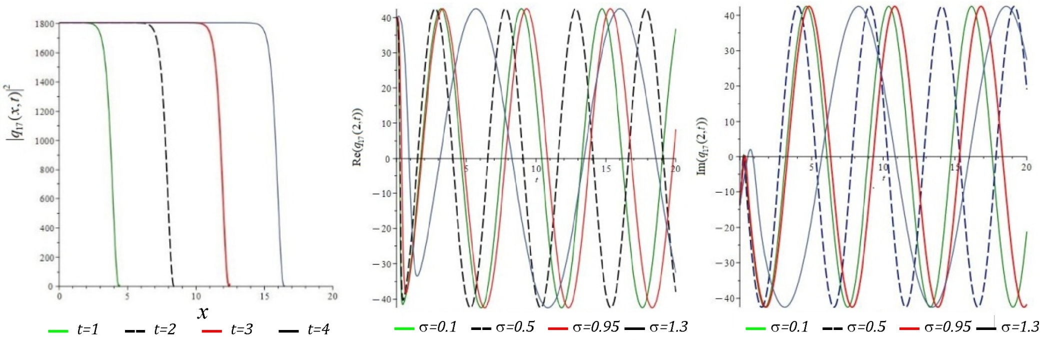


(a) (b)



(c)

Fig. 3. Profiles of dark-soliton solutions given by Eqs. (74) and (75): (a) surface plot of square of the modulus of the solution $q_{17}(x,t)$, (b) density plot of square of the modulus of the solution $q_{17}(x,t)$, and (c) surface plot of square of the modulus of the solution $r_{17}(x,t)$.



(a) (b) (c)

Fig. 4. Plots of $q_{17}(x,t)$: (a) square of the modulus of the solution $q_{17}(x,t)$ for different t , (b) real part of the solution $q_{17}(2,t)$ for different noise values σ , and (c) imaginary part of the solution $q_{17}(2,t)$ for different noise values σ .

6. Conclusions

Using the $\exp(-f(\xi))$-expansion method, the (G'/G)-expansion technique and the simplest-equation method, we have obtained a number of general soliton solutions to the underlying model. The results obtained by all of our methods are concerned with some unknown parameters. They can reveal the structures associated with either trigonometric (tan, sec) or hyperbolic (tanh, sech) functions, as well as rational or exponential functions. If the parameters are selected according to the Sets 1 and 2, the solutions of Eqs. (63) and (64) and Eqs. (70) and (71) in Section 4.2 can be reduced to those of Eqs. (38) and (39), Eqs. (40) and (41), Eqs. (49) and (50) and Eqs. (51) and (52), which are calculated in Section 4.1. A total of four families of the analytical solutions have been obtained using the first and the second methods. On the other hand, the three families of the analytical solutions have been obtained using the third method. Although these methods are not too complicated in terms of their implementation, they represent suitable and reliable techniques for finding the exact solutions.

The results obtained in the present work can be compared to those obtained by Zayed et al. [26] with Eqs. (1) and (2). Using the addendum Kudryashov's method and the unified Riccati-equation expansion method, the authors [26] have obtained the solutions of different types. A comparison of our results with those reported in Ref. [26] shows that the soliton solutions obtained with the appropriate parameter values are similar. To be clearer, the results obtained for the Sets 1 and 2, using the (G'/G)-expansion technique, and the hyperbolic solutions of Eqs. (55) and (56) and Eqs. (66) and (67) in this work are the same as the solutions of Eqs. (49) and (50) obtained in the work [26]. This emphasizes that the method applied by us is a powerful mathematical tool, from which some new exact solutions can be derived, since the both methods mentioned above are able to obtain the solutions already known from the literature.

Moreover, the trigonometric solutions of Eqs. (59) and (60) and Eqs. (68) and (69) in this work are the same as the soliton solutions of Eqs. (55) and (56) obtained in the study [26]. The mathematical structure of the solutions of Eqs. (80) and (81) obtained by the simplest-equation method in the current study is similar to that of the solutions of Eqs. (51) and (52) obtained in the work [26].

We have also obtained the solutions which are similar to those reported in the other studies mentioned above. Besides of those similar solutions, some solutions obtained in the current study are structurally different from those reported earlier. They are found for the first time.

Hence, the methods applied in this work to extract the exact solutions of nonlinear Schrödinger equations are reliable, efficient and well-suited. By selecting the appropriate parameter values, we have displayed both 3D and 2D plots of some representative solutions, including the dark solitons. Considering the results analyzed above, one can see that the efficiency of our methods is evident. Moreover, we have also discussed the effect of the multiplicative white noise on the solutions. It is evident that the increase in the noise-strength parameter can cause fluctuations in the soliton. Finally, further researches can be done in the future to obtain different kinds of the soliton solutions, using different fractal-fractional derivative operators.

Contributions. Yeşim Sağlam Özkan: conceptualization, methodology, writing (original draft); Emrullah Yaşar: conceptualization, software, methodology, writing (original draft), validation.

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***Анотація**. Розглянуто високодисперсійну стохастичну збурену модель Фокаса–Ленелля для волоконних бреггівських ґраток із просторово-часовою дисперсією та генерованим білим шумом у розумінні Іто. Для одержання солітонних рішень використано метод* $\exp(-f(\xi))$*-розкладу, метод (G/G')-розкладу та метод найпростішого рівняння. Для ліпшого розуміння характеру розповсюдження хвиль побудовано тривимірні та двовимірні графіки. Виконано порівняння одержаних солітонів різних типів, таких як темні, сингулярні, періодичні, раціональні та комбіновані солітони. Оцінено ефективність наших методів для базової моделі.*

***Ключові слова:** високодисперсійна стохастична збурена модель Фокаса–Ленеллса, метод* $\exp(-f(\xi))$*-розкладу, метод* (G/G')*-розкладу, метод найпростіших рівнянь, солітони.*