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Cnoidal Waves and Solitons to Three-Coupled Nonlinear Schrodinger's Equation with Spatially-Dependent Coefficients

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Abstract. This paper uses the extended auxiliary equation method to obtain the exact solutions of the nonlinear Schrödinger equations with variable-coefficient. As a result, solitary wave solutions, trigonometric function solutions, rational function solutions, and Jacobi elliptic functions solutions are obtained. The solitons are guaranteed to exist, provided the chromatic dispersion coefficients are Riemann integrable. Further, some of the obtained solutions are presented by 3D and 2D graphs to demonstrate the behavior of solutions. The results show that the extended auxiliary equation method, with the help of a computer symbolic computation system, is reliable and effective in finding various exact solutions of nonlinear evolution equations with variable coefficients in mathematical physics.

Keywords: Nonlinear Schrödinger equations with variable coefficient, extended auxiliary equation method, optical soliton, elliptic functions solutions
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1. Introduction

Optical solitons are solitary light waves indeed played a significant role in the revolution of optical communications. In the context of optical fibers widely used for long-distance communications, solitons can counteract the dispersion that causes signal degradation over long distances. One of the major advantages of solitons is their ability to propagate over long distances without significant distortion. They maintain their shape and quality, allowing for high-speed and long-haul data transmission. This characteristic makes it ideal for applications in telecommunications, where the demand for higher bandwidth and faster data rates continues to grow. The breakthrough in using solitons for optical communications Akira Hasegawa and coworkers demonstrated the transmission of soliton pulses over long distances in optical fibers [1-5]. This discovery paved the way for a new era in high-capacity optical communication systems [6]. In [7], the history of solitons used for optical communications and the technical development of soliton transmission is reviewed, and the cause of bit errors in long-distance soliton transmission is presented. The interaction of solitons and amplifier noise prevents their use in high-capacity transoceanic submarine systems unless frequency and timing controls are employed, for example, through the simple insertion of in-line bandpass filters. In addition to their use in long-haul communications, solitons have also found applications in other areas of optical communications. For example, they are utilized in ultrafast laser systems for generating short and intense pulses used in scientific research, medical imaging, and industrial applications. The revolution in soliton-based optical communications has led to significant advancements in the field, including increased data transmission rates, improved signal quality, and extended reach of optical networks. These developments have played a crucial role in shaping the modern telecommunications infrastructure and facilitating the growth of digital technologies. It's worth noting that while solitons have brought about a revolution in optical communications, they are just one aspect of the broader advancements and innovations in the field. Many other technologies in fiber optics, wavelength division multiplexing, and advanced modulation formats have contributed to the progress in optical communications alongside solitons. Solitons have been instrumental in enabling high-capacity optical communication systems such as dense wavelength division multiplexing (DWDM), where multiple information channels can be transmitted simultaneously over a single optical fiber using different wavelengths [8-12]. By leveraging solitons, DWDM systems can transmit terabits of data over long distances [13, 14].

In the framework of nonlinear spin optics self–confined light beams in reorientation nematic liquid crystals have been investigated using modulation theory and numerical experiments [15]. Experimental results of nonreciprocity from the relativistic Sugnac – Fizeau Optical drag effect by proposing a spinning nonlinear resonator to achieve non-reciprocal control of optical solitons [16]. The vortex solitons are described in self–defocusing Kerr – media whose phase line is not parallel to the propagation direction but is perpendicular (or) titled almost arbitrary angles [17]. Gyrating solitons in a circular array of 2N coupled nonlinear optical waveguides, their stability, and tuning the gain-loss coefficient have been analyzed [18]. A scheme is proposed to realize an optical Bessel potential with parity-time (PT) symmetry and investigate the existence, propagation, and manipulation of multidimensional optical solitons through the interplay among diffraction, Kerr nonlinearity, and potential confinement in a cold atomic gas under the condition of electromagnetically induced transparency (EIT) [19]. The intracavity Brillouin laser pumping scheme enabled access to the soliton states with a blue–detuned input pump through an easy operation of the Brillouin Kerr soliton microcomb with excellent performance, making the scheme promising for practical applications [20]. The giant enhancement of the cross–phase modulation for the two polarization components of the probe pulse can be obtained under the condition of double EIT and shows the system supports the stable temporal optical thirring solitons, which have ultralow generation power and ultrashort propagation velocity [21]. The generation of soliton crystals is investigated in the presence of nonlinear mode coupling, which can induce a modulation on the background wave and modify the cavity dynamics under the condition of suitable wave vector mismatch, nonlinear coupling coefficient, and highly deterministic perfect soliton crystals [22]. It is reported that exciton–polariton superfluids can also sustain dark–soliton molecules, although the interactions are connected to the driven dissipative nature of the polariton fluid [23]. The existence and stability of nonlocal vector solitons with Pseudo–Spin–Orbit coupling are investigated, and the results show symmetric and non–central symmetric vector solitons [24]. Also, Bright solitons can compress optical signals by converting long-duration pulses into shorter ones. This compression technique allows for higher data transmission rates and increased bandwidth efficiency. These can counteract the dispersion effects, which is the broadening of optical pulses as they propagate through the fiber. The nonlinear property of bright solitons helps compensate for dispersion, enabling long-haul transmission of data without significant signal degradation. Bright solitons are used in wavelength-division multiplexing (WDM) systems to transmit multiple information channels simultaneously over a single fiber. The robust nature of solitons allows for efficient isolation and transmission of individual channels without interference. Dark solitons can separate individual channels in WDM systems [25, 26]. By utilizing the dark regions of the soliton, different channels can be effectively isolated and transmitted without interference [27-30]. Dark solitons can be manipulated to switch on or off certain channels within a WDM system [31-41]. The dark solitons can be adjusted by applying appropriate control signals to block or transmit specific channels, allowing for dynamic and reconfigurable optical networks [42-49]. It's important to note that bright and dark solitons have unique characteristics and applications within fiber optic communication systems [50-66]. Their properties, such as intensity distribution and interaction behaviors, enable different functionalities that contribute to the efficient transmission and manipulation of optical signals in high-speed communication networks.

Putting forth the above novelty of applications of optical solitons, we derive novel solutions of bright and dark solitons of three coupled nonlinear Schrödinger equation (3-CNLS) with variable coefficient in this paper. Consider the 3-CNLS with the variable coefficient of the form [67]

$$\begin{aligned} &i\phi_{1z}+\frac{1}{2}\alpha(z)\phi_{1tt}+\beta(z)\left(|\phi_1|^2+|\phi_2|^2+|\phi_3|^2\right)\phi_1-i\gamma(z)\phi_1=0,\\ &i\phi_{2z}+\frac{1}{2}\alpha(z)\phi_{2tt}+\beta(z)\left(|\phi_1|^2+|\phi_2|^2+|\phi_3|^2\right)\phi_2-i\gamma(z)\phi_2=0,\\ &i\phi_{3z}+\frac{1}{2}\alpha(z)\phi_{3tt}+\beta(z)\left(|\phi_1|^2+|\phi_2|^2+|\phi_3|^2\right)\phi_3-i\gamma(z)\phi_3=0, \end{aligned} \tag{1}$$

where ϕ_j is the complex amplitude of the j-th-field component $(j=1,2,3)$ of the variables z,t and $\alpha(z),\beta(z),\gamma(z)$ are the group velocity dispersion (GVD), nonlinearity, and fiber gain/loss coefficients, respectively [67]. Eq. (1) has been considered as a model to describe the amplification or attenuation of the picosecond pulse propagation in the inhomogeneous multicomponent optical fiber with different frequencies or polarizations. In [68], based on the Lax pair, infinitely many conservation laws of Eq. (1) are obtained. The authors of reference [68] have discovered an uncountable or infinite number of these conservation laws that often involve quantities like energy, momentum, charge, or other physical properties. This is a valuable insight for understanding the behavior of the system and can have important implications for physics or mathematics. Further, two mixed-type vector soliton solutions are derived via the Hirota method and symbolic computation.

This paper mainly aims to derive various types of new exact solutions to the considered model using the extended auxiliary equation method. The proposed method has gained considerable attention in recent years, and many researchers have utilized various nonlinear models [69-74].

2. Description of the extended auxiliary equation method

This section provides a quick overview of the extended auxiliary equation method [70]. First, consider the nonlinear variable coefficient evolution equation with independent variables $Z=(z,t)$ and dependent variable u:

$$F\left(u,u_t,u_z,u_{tz},u_{tt},u_{zz},........\right)=0, \tag{2}$$

where $u=u(z,t)$ is an unknown function, F is a polynomial in u, and $u_t,u_z,u_{tz},u_{tt},u_{zz}$ is its various partial derivatives.

Step 1. We assume that Eq. (2) has solutions of the form:

$$u(Z)=\sum_{i=1}^{n}a_i(Z)F^i(\xi)+a_0(Z), \qquad a_n(Z)\neq 0, \tag{3}$$

where n is an integer to be determined by balancing the highest-order derivative terms in Eq. (2), $a_i(Z)(i=0,1,...,n)$ and $\xi=\xi(Z)$ are all functions of Z to be determined later. $F=F(\xi)$ is a solution of the following auxiliary ordinary differential equation:

$$F'^2(\xi)=c_0+c_1F(\xi)+c_2F^2(\xi)+c_3F^3(\xi)+c_4F^4(\xi), \tag{4}$$

where $F'=\dfrac{\mathrm{d}F}{\mathrm{d}\xi}$, $c_i(i=0,1,...,4)$ are constants, hence we have

$$F'F''=\frac{c_1}{2}F'+c_2FF'+\frac{3}{2}c_3F^2F'+2c_4F^3F', \tag{5}$$

$$F'''=c_2F'+3c_3FF'+6c_4F^2F', \tag{6}$$

where F' is the derivative of polynomial with respect to ξ.

Step 2. Substituting Eq. (3) into Eq. (2) and using Eqs. (4-6), and then setting the coefficients of $F^iF'^j(i=0,1,2,...,j=0,1,2,...)$ to zero, we obtain a set of over-determined partial differential equations (PDEs) for $a_i(Z),(i=0,1,...,n)$ and $\xi=\xi(Z)$.

Step 3. Solve the over-determined PDEs obtained in Step 2 for $a_i(Z)$ $(i=0,1,...,n)$ and $\xi=\xi(Z)$ using Mathematical software.

Step 4. Substituting $a_i(Z)(i=0,1,...,n)$ and $\xi=\xi(Z)$ obtained in the above steps and well-known solutions $F(\xi)$ of Eq. (4) (see in [70]) into Eq. (3), we obtain the exact solutions of Eq. (2).

3. Implementation of the proposed method

In order to obtain the exact solution of Eq. (1), firstly, we assume that the wave transformation of the form

$$\phi_j=v_j(z,t)e^{iu(z,t)}, \quad j=1,2,3, \tag{7}$$

where $v(z,t)$ and $u(z,t)$ are amplitude and phase functions, respectively. Substituting Eq. (7) into Eq. (1) and separating the real and imaginary parts, we get

$$\begin{aligned}&2\beta(z)\left(v_j(z,t)v_1(z,t)^2+v_j(z,t)v_2(z,t)^2+v_j(z,t)v_3(z,t)^2\right)\\&-\alpha(z)\left(v_j(z,t)u_{jt}(z,t)^2-v_{jtt}(z,t)\right)-2v_j(z,t)u_{jz}(z,t)=0,\end{aligned} \tag{8}$$

$$\begin{aligned}&-2\gamma(z)v_j(z,t)+2\alpha(z)u_{jt}(z,t)v_{jt}(z,t)\\&+\alpha(z)v_j(z,t)u_{jtt}(z,t)+2v_{jz}(z,t)=0,\end{aligned} \tag{9}$$

where $j=1,2,3$. Balancing the highest order derivative terms with the nonlinear terms in Eq. (8), we obtained $n=1$ in Eq. (3). Thus, we have

$$\xi = p(z)t + q(z),\ u(z,t) = \Gamma(z)t + \Omega(z), \tag{10}$$

and

$$v_j(z,t) = f_j(z) + g_j(z)F(\xi),\ \ j = 1,2,3, \tag{11}$$

where $\Gamma(z), \Omega(z), f_1(z), f_2(z)$ are functions of z to be determined, $p(z)$ and $q(z)$ are related to pulse width and group velocity, respectively.

Substituting Eqs. (10) and (11) into Eqs. (8) and (9) and setting each coefficient of polynomial in $F^iF'^j$ (i,j=0, 1, 2, 3) to zero, we obtain a set of over-determined PDEs for $\Gamma(z), \Omega(z)$, $f_1(z), f_2(z), p(z)$ and $q(z)$. By solving the above system of PDEs with the help of Mathematica, we obtain the following two types of results:

Result 1:

$$\begin{aligned}
&c_1 = c_3 = 0,\quad \Gamma'(z) = 0,\quad \Gamma(z) = A_1,\\
&p(z) = A_2,\quad f_1(z) = f_2(z) = f_3(z) = 0,\\
&g_1(z) = A_3 e^{\int \gamma(z)dz},\quad g_2(z) = A_4 e^{\int \gamma(z)dz},\\
&g_3(z) = A_5 e^{\int \gamma(z)dz},\quad q(z) = A_6 - A_1A_2\int \alpha(z)dz\ ,\\
&\Omega(z) = A_7 + \frac{1}{2}\left(-A_1^2 + A_2^2c_2\right)\int \alpha(z)dz,\\
&\beta(z) = -\frac{e^{-2\int \gamma(z)dz}A_2^2c_4\alpha(z)}{A_3^2 + A_4^2 + A_5^2},
\end{aligned} \tag{12}$$

where A_i with $(i = 0, 1, 2, 3, 4)$ are arbitrary constants. We substitute this result into Eq. (7) along with Eqs. (10) and (11) to obtain the fundamental solutions of Eq. (1), which depend on the solution $F(\xi)$ of Eq. (4). Given different values of c_i $(i = 0, 2, 4)$, Eq. (1) has many kinds of solutions, which are listed below:

1. If $c_0 = 0,\ c_2 > 0,\ c_4 < 0$, then the Eq. (1) has the bright soliton solutions:

$$\begin{aligned}
\phi_1(z,t) = {} & A_3 e^{\int \gamma(z)dz}\sqrt{-\frac{c_2}{c_4}}\ \mathrm{sech}\left[\sqrt{c_2}\ \xi\right]\\
&\times e^{i\left(A_1t + A_7 + \frac{1}{2}\left(-A_1^2 + A_2^2c_2\right)\int \alpha(z)dz\right)},
\end{aligned} \tag{13}$$

$$\begin{aligned}
\phi_2(z,t) = {} & A_4 e^{\int \gamma(z)dz}\sqrt{-\frac{c_2}{c_4}}\ \mathrm{sech}\left[\sqrt{c_2}\ \xi\right]\\
&\times e^{i\left(A_1t + A_7 + \frac{1}{2}\left(-A_1^2 + A_2^2c_2\right)\int \alpha(z)dz\right)},
\end{aligned} \tag{14}$$

$$\begin{aligned}
\phi_3(z,t) = {} & A_5 e^{\int \gamma(z)dz}\sqrt{-\frac{c_2}{c_4}}\ \mathrm{sech}\left[\sqrt{c_2}\ \xi\right]\\
&\times e^{i\left(A_1t + A_7 + \frac{1}{2}\left(-A_1^2 + A_2^2c_2\right)\int \alpha(z)dz\right)},
\end{aligned} \tag{15}$$

where $\xi = A_2t + A_6 - A_1A_2\int \alpha(z)dz$.

The bright solitons (13)-(15) will exist provided $\alpha(z)$ is Riemann integrable. For the choice of sinusoidal function for inhomogeneous profiles, a soliton is transmitted as a snake soliton through the multimode fiber as shown in Fig. 1(a). Moreover, we infer that the period of oscillation, width, and intensity of soliton is invariant as depicted in Fig. 1(b).

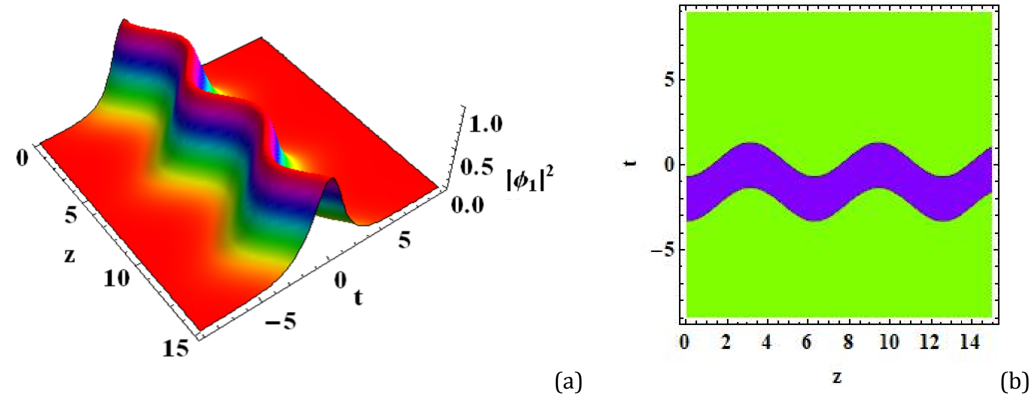

Fig. 1. One bright soliton evolution (a) and density plot (b).

As portrayed in Fig. 2 (a), a bright soliton propagates through an inhomogeneous multimode fiber without gain or loss. It shows the stable propagation of optical soliton where amplitude, width, and phase are constant along the propagation direction. This is a significant property of optical solitons.

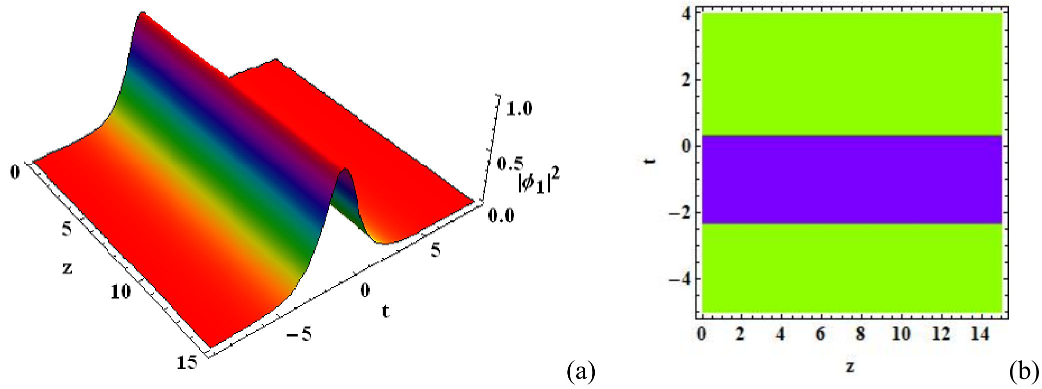

Fig. 2. One bright soliton evolution(a) and density plot (b).

One bright soliton gets compression during the propagation in a three-core or three-mode inhomogeneous optical fiber. Till reaching the $z=0$, no change in the width of the optical soliton as clearly shown in Fig. 3 (a). In practice, pulse compression can be achieved by employing dispersion decreasing fiber where dispersion is gradually decreasing in the fiber core.

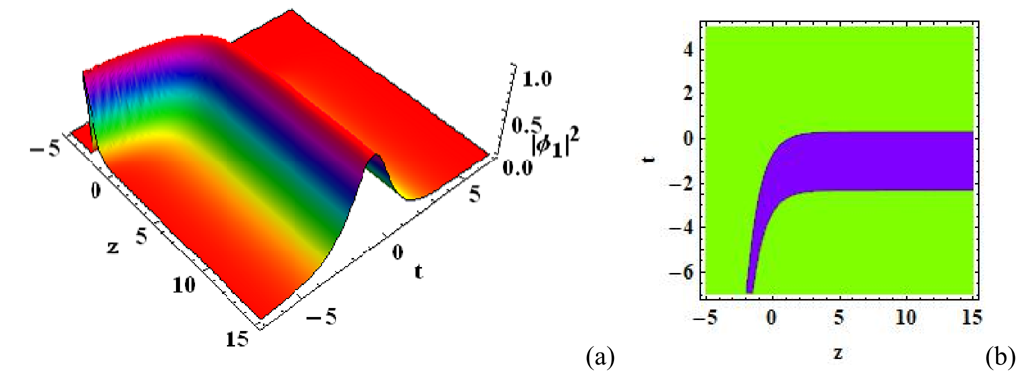

Fig. 3. One bright soliton evolution (a) and density plot (b).

2. If $c_0 = \frac{c_2^2}{4c_4}, c_2 < 0, c_4 < 0$, then the Eq. (1) has the dark soliton solutions:

$$\phi_1(z,t) = A_3 e^{\int \gamma(z)dz} \sqrt{-\frac{c_2}{2c_4}} \tanh\left[\sqrt{-\frac{c_2}{2}}\,\xi\right] e^{i\left(A_1 t + A_7 + \frac{1}{2}\left(-A_1^2 + A_2^2 c_2\right)\int \alpha(z)dz\right)}, \tag{16}$$

$$\phi_2(z,t) = A_4 e^{\int \gamma(z)dz} \sqrt{-\frac{c_2}{2c_4}} \tanh\left[\sqrt{-\frac{c_2}{2}}\,\xi\right] e^{i\left(A_1 t + A_7 + \frac{1}{2}\left(-A_1^2 + A_2^2 c_2\right)\int \alpha(z)dz\right)}, \tag{17}$$

$$\phi_3(z,t) = A_5 e^{\int \gamma(z)dz} \sqrt{-\frac{c_2}{2c_4}} \tanh\left[\sqrt{-\frac{c_2}{2}}\,\xi\right] e^{i\left(A_1 t + A_7 + \frac{1}{2}\left(-A_1^2 + A_2^2 c_2\right)\int \alpha(z)dz\right)}, \tag{18}$$

where $\xi = A_2 t + A_6 - A_1 A_2 \int \alpha(z)dz$.

The soliton pulse is compressed when propagating through a three-mode optical fiber with a specific dispersion profile. When compared with the compression of bright soliton, width is gradually reduced along the propagation as displayed in Fig. 4 (a). Here phase and width are simultaneously varying in the fiber optic communication system.

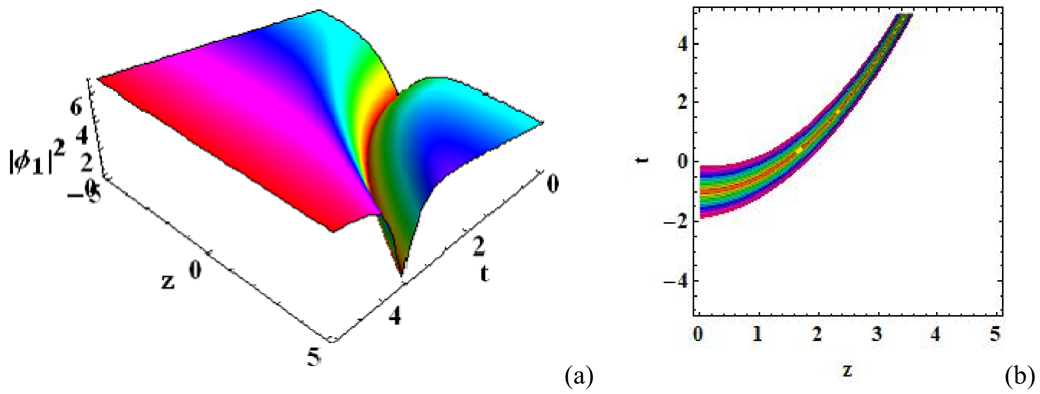

Fig. 4. Dark soliton progation (a) and corresponding density plot (b).

As shown in Fig. 5 (a), dark soliton is periodically oscillated with constant oscillation and width. In the lossless fiber, dark soliton propagated without attenuation as clearly shown in Fig. 5 (a). The oscillating behavior of soliton is called snake soliton.

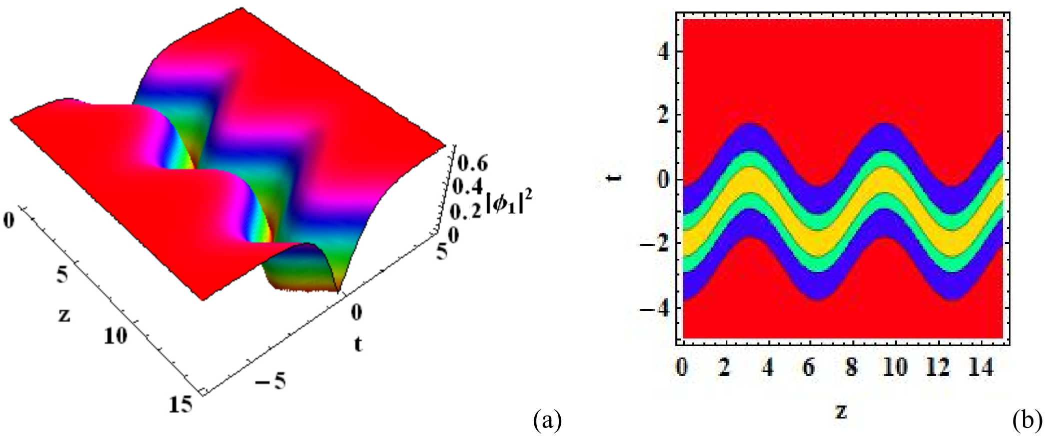

Fig. 5. Periodic evolution of dark soliton (a) and corresponding density plot (b).

3. If $c_0 = \frac{c_2^2}{4c_4}, c_2 < 0, c_4 > 0$, then the Eq. (1) has the trigonometric function solutions:

$$\phi_1(z,t) = A_3 e^{\int \gamma(z)dz} \sqrt{-\frac{c_2}{c_4}} \sec\left[\sqrt{-c_2}\ \xi\right] e^{i\left(A_1 t + A_7 + \frac{1}{2}(-A_1^2 + A_2^2 c_2)\int \alpha(z)dz\right)}, \tag{19}$$

$$\phi_2(z,t) = A_4 e^{\int \gamma(z)dz} \sqrt{-\frac{c_2}{c_4}} \sec\left[\sqrt{-c_2}\ \xi\right] e^{i\left(A_1 t + A_7 + \frac{1}{2}(-A_1^2 + A_2^2 c_2)\int \alpha(z)dz\right)}, \tag{20}$$

$$\phi_3(z,t) = A_5 e^{\int \gamma(z)dz} \sqrt{-\frac{c_2}{c_4}} \sec\left[\sqrt{-c_2}\ \xi\right] e^{i\left(A_1 t + A_7 + \frac{1}{2}(-A_1^2 + A_2^2 c_2)\int \alpha(z)dz\right)}, \tag{21}$$

where $\xi = A_2 t + A_3 - A_1 A_2 \int \alpha(z)dz$ while, if $c_0 = \frac{c_2^2}{4c_4}, c_2 > 0, c_4 < 0$, then the Eq. (1) has the following trigonometric function solutions:

$$\phi_1(z,t) = A_3 e^{\int \gamma(z)dz} \sqrt{\frac{c_2}{2c_4}} \tan\left[\sqrt{\frac{c_2}{2}}\ \xi\right] e^{i\left(A_1 t + A_7 + \frac{1}{2}(-A_1^2 + A_2^2 c_2)\int \alpha(z)dz\right)}, \tag{22}$$

$$\phi_2(z,t) = A_4 e^{\int \gamma(z)dz} \sqrt{\frac{c_2}{2c_4}} \tan\left[\sqrt{\frac{c_2}{2}}\ \xi\right] e^{i\left(A_1 t + A_7 + \frac{1}{2}(-A_1^2 + A_2^2 c_2)\int \alpha(z)dz\right)}, \tag{23}$$

$$\phi_3(z,t) = A_5 e^{\int \gamma(z)dz} \sqrt{\frac{c_2}{2c_4}} \tan\left[\sqrt{\frac{c_2}{2}}\ \xi\right] e^{i\left(A_1 t + A_7 + \frac{1}{2}(-A_1^2 + A_2^2 c_2)\int \alpha(z)dz\right)}, \tag{24}$$

where $\xi = A_2 t + A_6 - A_1 A_2 \int \alpha(z)dz$.

4. If $c_0 = c_2 = 0, c_4 < 0$, then the Eq. (1) has the rational function solution:

$$\phi_1(z,t) = -A_3 e^{\int \gamma(z)dz} \frac{1}{\sqrt{c_4}\xi}\ e^{i\left(A_1 t + A_7 + \frac{1}{2}(-A_1^2 + A_2^2 c_2)\int \alpha(z)dz\right)}, \tag{25}$$

$$\phi_2(z,t) = -A_4 e^{\int \gamma(z)dz} \frac{1}{\sqrt{c_4}\xi}\ e^{i\left(A_1 t + A_7 + \frac{1}{2}(-A_1^2 + A_2^2 c_2)\int \alpha(z)dz\right)}, \tag{26}$$

$$\phi_3(z,t) = -A_5 e^{\int \gamma(z)dz} \frac{1}{\sqrt{c_4}\xi}\ e^{i\left(A_1 t + A_7 + \frac{1}{2}(-A_1^2 + A_2^2 c_2)\int \alpha(z)dz\right)}, \tag{27}$$

where $\xi = A_2 t + A_6 - A_1 A_2 \int \alpha(z)dz$.

5. $c_0 = -\frac{c_2^2 m^2 (1-m^2)}{c_4 (2m^2-1)^2}$, $c_2 > 0$, $c_4 < 0$, (where m denotes the modulus of Jacobi elliptic function, where 0<m<1) then the Eq. (1) has the Jacobi elliptic solutions:

$$\phi_1(z,t) = A_3 e^{\int \gamma(z)dz} \sqrt{-\frac{c_2 m^2}{c_4(2m^2-1)}}\ \mathrm{cn}\left[\sqrt{\frac{c_2}{(2m^2-1)}}\ \xi\right] e^{i\left(A_1 t + A_7 + \frac{1}{2}(-A_1^2 + A_2^2 c_2)\int \alpha(z)dz\right)}, \tag{28}$$

$$\phi_2(z,t) = A_4 e^{\int \gamma(z)dz} \sqrt{-\frac{c_2 m^2}{c_4(2m^2-1)}}\ \mathrm{cn}\left[\sqrt{\frac{c_2}{(2m^2-1)}}\ \xi\right] e^{i\left(A_1 t + A_7 + \frac{1}{2}(-A_1^2 + A_2^2 c_2)\int \alpha(z)dz\right)}, \tag{29}$$

$$\phi_3(z,t) = A_5 e^{\int \gamma(z)dz} \sqrt{-\frac{c_2 m^2}{c_4(2m^2-1)}}\ \mathrm{cn}\left[\sqrt{\frac{c_2}{(2m^2-1)}}\ \xi\right] e^{i\left(A_1 t + A_7 + \frac{1}{2}(-A_1^2 + A_2^2 c_2)\int \alpha(z)dz\right)}, \tag{30}$$

where $\xi = A_2 t + A_6 - A_1 A_2 \int \alpha(z) dz$.

Note that, if $m \to 1$, then $\mathrm{cn}\xi \to \mathrm{sech}\xi$, then we have the same bright soliton solutions (13), (14), and (15).

6. If $c_0 = \dfrac{c_2^2(1-m^2)}{c_4(2-m^2)^2}$, $c_2 > 0$, $c_4 < 0$, then the Eq. (1) has the Jacobi elliptic solutions:

$$\phi_1(z,t) = A_3 e^{\int \gamma(z)dz} \sqrt{-\frac{c_2}{c_4(2-m^2)}}\, \mathrm{dn}\left[\sqrt{\frac{c_2}{(2-m^2)}}\,\xi\right] e^{i\left(A_1 t + A_7 + \frac{1}{2}(-A_1^2 + A_2^2 c_2)\int \alpha(z)dz\right)}, \tag{31}$$

$$\phi_2(z,t) = A_4 e^{\int \gamma(z)dz} \sqrt{-\frac{c_2}{c_4(2-m^2)}}\, \mathrm{dn}\left[\sqrt{\frac{c_2}{(2-m^2)}}\,\xi\right] e^{i\left(A_1 t + A_7 + \frac{1}{2}(-A_1^2 + A_2^2 c_2)\int \alpha(z)dz\right)}, \tag{32}$$

$$\phi_3(z,t) = A_5 e^{\int \gamma(z)dz} \sqrt{-\frac{c_2}{c_4(2-m^2)}}\, \mathrm{dn}\left[\sqrt{\frac{c_2}{(2-m^2)}}\,\xi\right] e^{i\left(A_1 t + A_7 + \frac{1}{2}(-A_1^2 + A_2^2 c_2)\int \alpha(z)dz\right)}, \tag{33}$$

where $\xi = A_2 t + A_6 - A_1 A_2 \int \alpha(z) dz$.

Note that, if $m \to 1$, then $\mathrm{dn}\xi \to \mathrm{sech}\xi$, then we have the same bright soliton solutions (13), (14), and (15).

7. If $c_0 = \dfrac{c_2^2 m^2}{c_4(m^2+1)^2}$, $c_2 > 0$, $c_4 < 0$, then the Eq. (1) has the Jacobi elliptic solutions:

$$\phi_1(z,t) = A_3 e^{\int \gamma(z)dz} \sqrt{-\frac{c_2 m^2}{c_4(m^2+1)}}\, \mathrm{sn}\left[\sqrt{\frac{-c_2}{(m^2+1)}}\,\xi\right] e^{i\left(A_1 t + A_7 + \frac{1}{2}(-A_1^2 + A_2^2 c_2)\int \alpha(z)dz\right)}, \tag{34}$$

$$\phi_2(z,t) = A_4 e^{\int \gamma(z)dz} \sqrt{-\frac{c_2 m^2}{c_4(m^2+1)}}\, \mathrm{sn}\left[\sqrt{\frac{-c_2}{(m^2+1)}}\,\xi\right] e^{i\left(A_1 t + A_7 + \frac{1}{2}(-A_1^2 + A_2^2 c_2)\int \alpha(z)dz\right)}, \tag{35}$$

$$\phi_3(z,t) = A_5 e^{\int \gamma(z)dz} \sqrt{-\frac{c_2 m^2}{c_4(m^2+1)}}\, \mathrm{sn}\left[\sqrt{\frac{-c_2}{(m^2+1)}}\,\xi\right] e^{i\left(A_1 t + A_7 + \frac{1}{2}(-A_1^2 + A_2^2 c_2)\int \alpha(z)dz\right)}, \tag{36}$$

where $\xi = A_2 t + A_6 - A_1 A_2 \int \alpha(z) dz$.

Note that, if $m \to 1$, then $\mathrm{sn}\xi \to \tanh\xi$, then we have the same dark soliton solutions (16), (17), and (18).

Result 2:

$$\begin{gathered}
c_0 = c_1 = 0,\ \Gamma'(z) = 0,\ \Gamma(z) = A_1,\ p'(z) = 0,\ p(z) = A_2, \\
g_1(z) = A_3 e^{\int \gamma(z)dz},\ g_2(z) = A_4 e^{\int \gamma(z)dz}, \\
g_3(z) = A_5 e^{\int \gamma(z)dz},\ q(z) = A_6 - A_1 A_2 \int \alpha(z)dz\ , \\
f_1(z) = A_7 e^{\int \gamma(z)dz},\ f_2(z) = A_8 e^{\int \gamma(z)dz}, \\
f_3(z) = A_9 e^{\int \gamma(z)dz}, \\
\Omega(z) = A_{10} - \frac{1}{2A_5^2}\left(A_1^2 A_5^2 + 2A_2^2 A_9^2 c_4\right)\int \alpha(z)dz, \\
\beta(z) = -\frac{e^{-2\int \gamma(z)dz} A_2^2 c_4 \alpha(z)}{A_3^2 + A_4^2 + A_5^2}.
\end{gathered} \tag{37}$$

We substitute this result into Eq. (7) along with Eqs. (10) and (11) to obtain the fundamental solutions of Eq. (1) which depend on the solution $F(\xi)$ of Eq. (4). Given different values of c_i $(i = 2, 3, 4)$, Eq. (1) has the following soliton solutions:

$$\phi_1(z,t) = \left(A_7 e^{\int \gamma(z)dz} + A_3 e^{\int \gamma(z)dz} \frac{c_2 \operatorname{sech}\left[\frac{1}{2}\sqrt{c_2}\,\xi\right]^2}{2\sqrt{c_2 c_4}\tanh\left[\frac{1}{2}\sqrt{c_2}\,\xi\right] - c_3} \right) \tag{38}$$
$$\times e^{i\left(A_1 t + A_{10} - \frac{1}{2A_5^2}\left(A_1^2 A_5^2 + 2A_2^2 A_9^2 c_4\right)\int \alpha(z)dz \right)},$$

$$\phi_2(z,t) = \left(A_8 e^{\int \gamma(z)dz} + A_4 e^{\int \gamma(z)dz} \frac{c_2 \operatorname{sech}\left[\frac{1}{2}\sqrt{c_2}\,\xi\right]^2}{2\sqrt{c_2 c_4}\tanh\left[\frac{1}{2}\sqrt{c_2}\,\xi\right] - c_3} \right) \tag{39}$$
$$\times e^{i\left(A_1 t + A_{10} - \frac{1}{2A_5^2}\left(A_1^2 A_5^2 + 2A_2^2 A_9^2 c_4\right)\int \alpha(z)dz \right)},$$

$$\phi_3(z,t) = \left(A_9 e^{\int \gamma(z)dz} + A_5 e^{\int \gamma(z)dz} \frac{c_2 \operatorname{sech}\left[\frac{1}{2}\sqrt{c_2}\,\xi\right]^2}{2\sqrt{c_2 c_4}\tanh\left[\frac{1}{2}\sqrt{c_2}\,\xi\right] - c_3} \right) \tag{40}$$
$$\times e^{i\left(A_1 t + A_{10} - \frac{1}{2A_5^2}\left(A_1^2 A_5^2 + 2A_2^2 A_9^2 c_4\right)\int \alpha(z)dz \right)},$$

where $\xi = A_2 t + A_6 - A_1 A_2 \int \alpha(z)dz$, $c_2 = \frac{4A_9^2 c_4}{A_5^2}$, $c_3 = \frac{4A_9 c_4}{A_5}$, $A_7 = \frac{A_3 A_9}{A_5}$, and $A_8 = \frac{A_4 A_9}{A_5}$. Provided that $c_2 > 0$ and $A_5 \neq 0$.

We observe kink and anti-kink solitons in a three-mode optical fiber system when control parameters are appropriately adjusted. The three-dimensional schematic propagation of kink and anti-kink solitons is propagating in opposite directions, as illustrated in Fig. 6(a) and 6(b). In the constant values, the positive and negative signs determine the kink and anti-kink solitons, respectively. On the other hand, the velocity and amplitude of kink solitons are determined by constant values present in the obtained solutions. As seen in Fig. 6, the shape of both kink and anti-kink solitons are unchanged during the propagation, which is the same as that of optical solitons.

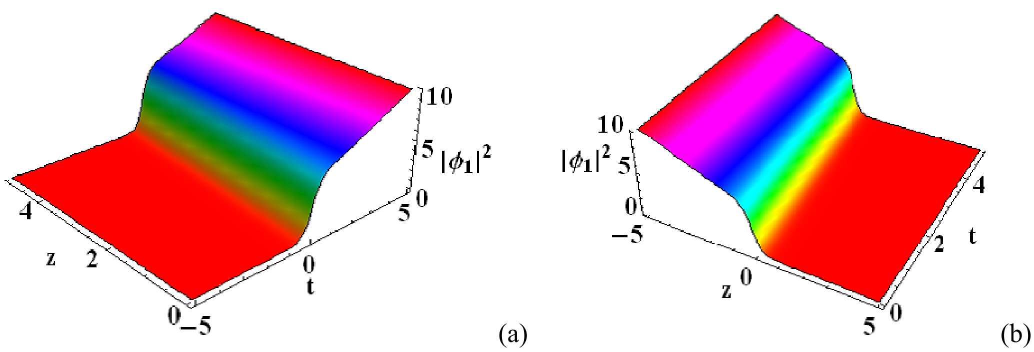

Fig. 6. Plot for kink soliton(a) and plot for anti-kink soliton (b).

Since $c_3 = 2\sqrt{c_2 c_4}$, $A_7 = \frac{A_3 A_9}{A_5}$, and $A_8 = \frac{A_4 A_9}{A_5}$, solution (38), (39) and (40) degenerate to the following soliton solutions:

$$\phi_1(z,t) = -\frac{A_3 A_9}{A_5} e^{\int \gamma(z)dz} \tanh\left[\frac{A_9}{A_5}\sqrt{c_4}\ \xi\right] e^{i\left(A_1 t + A_{10} - \frac{1}{2A_5^2}\left(A_1^2 A_5^2 + 2A_2^2 A_9^2 c_4\right)\int \alpha(z)dz\right)}, \tag{41}$$

$$\phi_2(z,t) = -\frac{A_4 A_9}{A_5} e^{\int \gamma(z)dz} \tanh\left[\frac{A_9}{A_5}\sqrt{c_4}\ \xi\right] e^{i\left(A_1 t + A_{10} - \frac{1}{2A_5^2}\left(A_1^2 A_5^2 + 2A_2^2 A_9^2 c_4\right)\int \alpha(z)dz\right)}, \tag{42}$$

$$\phi_3(z,t) = -A_9 e^{\int \gamma(z)dz} \tanh\left[\frac{A_9}{A_5}\sqrt{c_4}\ \xi\right] e^{i\left(A_1 t + A_{10} - \frac{1}{2A_5^2}\left(A_1^2 A_5^2 + 2A_2^2 A_9^2 c_4\right)\int \alpha(z)dz\right)}, \tag{43}$$

where $\xi = A_2 t + A_6 - A_1 A_2 \int \alpha(z)dz$.

4. Conclusions

In this study, the extended auxiliary equation method is described to obtain more exact solutions of the nonlinear Schrödinger equation with variable coefficients. Using the proposed method we successfully obtained some new exact solutions to the equation under different conditions. These exact solutions include solitary wave solutions, trigonometric function solutions, rational function solutions, and Jacobi elliptic function solutions. The characteristics of some obtained solutions are analyzed via 3D and density plots. The results show that the extended auxiliary equation method is direct, powerful, and can be used for many other high-dimensional nonlinear differential equations in mathematical physics.

Declarations

The authors have no relevant financial or non-financial interests to disclose.

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Анотація. *У цій статті використовується метод розширеного допоміжного рівняння для отримання точних розв'язків нелінійних рівнянь Шредінгера зі змінним коефіцієнтом. У результаті отримані розв'язки ізольованих хвиль, розв'язки тригонометричних функцій, розв'язки раціональних функцій та розв'язки еліптичних функцій Якобі. Солітони однозначно існують за умови, що коефіцієнти хроматичної дисперсії інтегровні за Ріманом. Крім того, деякі з отриманих рішень представлені 3D і 2D графіками, щоб продемонструвати поведінку рішень. Результати показують, що метод розширеного допоміжного рівняння за допомогою комп'ютерної системи символьних обчислень є надійним та ефективним у знаходженні різних точних розв'язків нелінійних еволюційних рівнянь зі змінними коефіцієнтами в математичній фізиці.*

Ключові слова: *нелінійні рівняння Шредінгера зі змінним коефіцієнтом, метод розширених допоміжних рівнянь, оптичний солітон, розв'язки еліптичних функцій.*