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BRIGHT, DARK, AND W-SHAPED SOLITONS OF BISWAS–ARSHED EQUATION VIA VARIATIONAL ITERATION METHOD

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Abstract. In this work, we adopt the Biswas-Arshed equation, as innovative framework for modeling soliton transmission through optical fibers in a media with Kerr-type nonlinearity. Bright, dark, and W-shaped solitons were successfully acquired for this innovative model by using the variational iteration method. This effective technique continues to rise in popularity for numerically addressing model equations from various physical phenomena, including photonics. The study is novel in that it employs an iterative variational approach to recover soliton-type solutions for the model numerically. This approach eliminates the necessity to assume linearizations or discretizations, which could potentially affect the physical characteristics of the model. The algorithm presents the results with a very low error rate.

Keywords: Biswas–Arshed equation, Kerr law nonlinearity, variational iteration method, solitons

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1. Introduction

Optical solitons are of critical importance in the fields of mathematical physics and nonlinear optics. The stability and regulation of solitons' evolution within optical fiber render them advantageous in optics and resilient in numerous scientific disciplines. Solitons are crucial in the propagation of optical continuum and can be utilized to transmit data over extremely long distances. In addition to being crucial in nonlinear optics, optical solitons inspire research in various photonics subfields. Investigating nonlinear Schrodinger equations has emerged as a prominent area of academic research in recent decades. These equations are extremely useful in the fields of physics, including plasma physics, superconductive properties, nonlinear optics, biological physics, star formation, quantum mechanics, and others [1- 7]. The telecommunication industry can benefit from the renowned Biswas–Arshed model [15], first proposed by Biswas and Arshed in 2018. This model finds application in various domains, including light routing, photo-controlled switching devices, and soliton transmission via optical fiber media. Two nonlinear iterations of this model were introduced, namely the power law and the Kerr law. This model is distinguished by the absence of self-phase modulation and its low count of group velocity dispersion. An additional noteworthy characteristic of this model is its incorporation of second and third-order spatiotemporal dispersions, which replenishes for the depleted group velocity dispersion. Many dependable and efficient integration techniques have been investigated to represent the Biswas–Arshed model's soliton solution [8-14].

Although there is an extensive variety of analytical methods available for solving the Biswas-Arshed equation, only a limited number of numerical schemes are visible and applicable. The aforementioned techniques include the variational iteration method (VIM), finite element method, Adomian decomposition approach, and Laplace-Adomian decomposition. In this paper, VIM is implemented to resolve Kerr-type nonlinearity in the dynamics of soliton models centered on the Biswas-Arshed equation. For the obtained solitons, the surface plot, contour plot, and error plot are all displayed. The remainder of the paper is devoted to describing the specifics of VIM and its implementation in the model.

2. Governing model

The Biswas-Arshed model [15] is described by:

$$\begin{aligned} &iq_t + a_1 q_{xx} + a_2 q_{xt} + i\left(b_1 q_{xxx} + b_2 q_{xxt}\right) \\ &= i\left[\sigma\left(|q|^2 q\right)_x + \mu\left(|q|^2\right)_x q + \theta |q|^2 q_x\right]. \end{aligned} \tag{1}$$

In Eq. (1), $q(x,t)$ represents a complex field envelope, and x and t are spatial and temporal variables, respectively. The first component is denoted as the temporal evolution of pulses, and the coefficients a_1 and a_2 in this equation provide the existence of spatiotemporal dispersion and group velocity dispersion, respectively. The nonlinear terms denoting self-steepening and nonlinear dispersions are guaranteed by the sequential coefficients of σ, μ, and θ. The existence of third-order dispersion and third-order spatiotemporal dispersion is definitively established by the sequential coefficients b_1 and b_2. This study aims to examine the Biswas-Arshed equation using the VIM. Consequently, solutions for bright, dark, and W-shaped soliton waves, along with their corresponding existence constraints, are derived.

3. A brief overview of the VIM

The VIM is well-known and frequently employed to solve problems involving nonlinear differential equations with initial conditions. This section will describe in minimal detail the algorithm that will be implemented to conduct numerical simulations. The algorithm is a direct result of the aforementioned methodology.

The VIM is utilized to analyze the following nonlinear partial differential equation:

$$\begin{cases} Lq(x,t) + Rq(x,t) + Nq(x,t) = g(x,t), \\ q(x,0) = q_0(x). \end{cases} \tag{2}$$

In the given equation, $L = \partial/\partial t$, R and N denote linear and nonlinear operators, respectively, and an inhomogeneous term (or source) is represented by $g(x,t)$.

The VIM admits the use of the correction functional for Eq. (2) which can be written as:

$$q_{n+1}(x,t) = q_n(x,t) + \int_0^t \lambda(\xi) \begin{pmatrix} Lq_n(x,\xi) + R\tilde{q}_n(x,\xi) \\ +N\tilde{q}_n(x,\xi) - g(x,\xi) \end{pmatrix} d\xi, \quad n \geq 0. \tag{3}$$

In the given equation, $\lambda(\xi)$ represents a general Lagrange multiplier that can be most adequately determined through variational theory [19-22]. The correction term, denoted as $\tilde{q}_n$, is regarded as a restricted variation, with $\delta\tilde{q}_n = 0$.

The stationary conditions for Eq. (3) can be determined in a subsequent manner:

$$\begin{cases} \lambda'(\xi)|_{\xi=t} = 0, \\ 1+\lambda(\xi)|_{\xi=t} = 0. \end{cases} \tag{4}$$

Consequently, the Lagrange multiplier can be identified as $\lambda(\xi) = -1$.

When the obtained multiplier is substituted into Eq. (3), the resulting iteration formula is as follows:

$$\begin{aligned} q_{n+1}(x,t) &= q_n(x,t) - \int_0^t \left(Lq_n(x,\xi) + Rq_n(x,\xi) Nq_n(x,\xi) - g(x,\xi) \right) d\xi, \\ &= q_n(x,t) - \int_0^t Lq_n(x,\xi) d\xi - \int_0^t \left(Rq_n(x,\xi) + Nq_n(x,\xi) - g(x,\xi) \right) d\xi, \\ &= q_0(x,t) - \int_0^t \left(Rq_n(x,\xi) + Nq_n(x,\xi) - g(x,\xi) \right) d\xi,\ n \geq 0. \end{aligned} \tag{5}$$

We have employed:

$$q_n(x,0) = q_0(x,t) = q(x,0). \tag{6}$$

Because of the generic iterative approach to solving Eq. (2), the following can be written:

$$\begin{cases} q_0(x,t) = q(x,0), \\ q_{n+1}(x,t) = q_0(x,t) - \int_0^t \left(Rq_n(x,\xi) + Nq_n(x,\xi) - g(x,\xi) \right) d\xi, \quad n \geq 0. \end{cases} \tag{7}$$

The elements of sequence $\{q_n\}$ are constructed in accordance with the variational iteration method so that the sequence converges to the exact solution [23]. The iterative approximations of the solution, denoted as q_{n+1}, $n \geq 0$, can be easily obtained by selecting an appropriate trial function q_0. As a result, the solution can be expressed as:

$$\lim_{n \to \infty} q_n(x,t) = q(x,t). \tag{8}$$

Essentially, the correction functional (3) will provide several approximations, and the exact solution is eventually reached as the limit of these consecutive approximations.

Numerous users have demonstrated that the VIM is highly effective and can be implemented directly, eliminating the necessity for linearizing nonlinear terms that could potentially alter the problem's physical characteristics.

4. Solution of the Biswas-Arshed equation by VIM

This section provides an overview of how VIM is utilized to derive an explicit solution to Eq. (1) given the initial condition $q(x,0) = q_0(x,t)$. Expressing the Biswas-Arshed model as a homogeneous equation, we have:

$$iq_t + a_1 q_{xx} + a_2 q_{xt} + i\left(b_1 q_{xxx} + b_2 q_{xxt}\right) - i\left[\sigma(|q|^2 q)_x + \mu\left(|q|^2\right)_x q + \theta |q|^2 q_x\right] = 0. \tag{9}$$

According to the scheme of Eq. (2), the linear component is:

$$R = a_1 q_{xx} + a_2 q_{xt} + i\left(b_1 q_{xxx} + b_2 q_{xxt}\right), \tag{10}$$

$g = 0$ and the nonlinear part is given by:

$$N = -i\left[\sigma\left(|q|^2 q\right)_x + \mu\left(|q|^2\right)_x q + \theta|q|^2 q_x\right]. \tag{11}$$

Applying VIM to Eq. (9), we obtain:

$$q_0(x,t) = q(x,0), \tag{12}$$

and for $n \geq 0$:

$$\begin{aligned} q_{n+1}(x,t) &= q_0(x,t) \\ &- \int_0^t \left[\left(a_1\frac{\partial^2}{\partial x^2} + a_2\frac{\partial^2}{\partial x\partial\xi} + ib_1\frac{\partial^3}{\partial x^3} + ib_2\frac{\partial^3}{\partial x^2\partial\xi}\right)q(x,\xi)\right]d\xi \\ &- i\int_0^t \left[\begin{array}{l}\sigma\left\{|q(x,\xi)|^2 q(x,\xi)\right\}_x \\ +\mu\left\{|q(x,\xi)|^2\right\}_x q(x,\xi) + \theta|q(x,\xi)|^2 q_x(x,\xi)\end{array}\right]dx. \end{aligned} \tag{13}$$

It is important to note that the initial condition $q_0(x,t) = q(x,0)$, is precisely the first term of the series solution obtained from the VIM.

5. The numerical computations and graphic results

In this part, we will use the suggested approach to solve the nonlinear Biswas-Arshed equation with Kerr law nonlinearity, as described by Eq. (1), utilizing different initial conditions. We conduct numerical simulations to analyze optical solitons with different characteristics, including bright, dark, and W-shaped solitons. Every stage of the computation process is executed using the MATHEMATICA software program.

5.1. Simulation of bright solitons

Consider Eq. (9), with initial conditions [16-18]:

$$q_0(x,t) = A_1\text{sech}\left[B_1(x)\right]e^{i[-\kappa x+\Omega_0]}, \tag{14}$$

where κ represents the wavevector of the soliton and Ω_0 represents the phase center. The soliton's amplitude A_1 is determined using the following equation:

$$A_1 = \pm\sqrt{\frac{-2\left(\omega+\kappa\left(a_1\kappa - a_2\omega\right)+\kappa^2\left(b_1\kappa - b_2\omega\right)\right)}{\kappa\left(\sigma+\theta\right)}}, \tag{15}$$

where ω is the frequency, the soliton's width, denoted as B_1, is determined by:

$$B_1 = \pm\sqrt{\frac{\omega+\kappa\left(a_1\kappa - a_2\omega\right)+\kappa^2\left(b_1\kappa - b_2\omega\right)}{a_1 + 3\kappa b_1 - \omega b_2 - \nu\left(a_2 + 2\kappa b_2\right)}}. \tag{16}$$

Table 1 compares the absolute error of the exact and approximate solutions for different coefficients. Figs. 1, 2, and 3 display the graphical representations of the approximate solutions obtained by the use of VIM, the contour plot of the wave amplitude $|q|^2$, and the absolute error, respectively.

Table 1. Bright optical solitons.

Cases	a_1	a_2	b_1	b_2	σ	μ	θ	n	$\|Max\,Error\|$
B_I	0.5	6.2	8.8	6.5	3.5	3.1	2.2	16	4.0×10^{-8}
B_{II}	0.6	1.9	6.3	6.7	4.3	3.3	2.6	16	6.0×10^{-9}
B_{III}	0.1	−1.5	0.9	0.9	2.9	1.1	0.8	16	1.0×10^{-8}

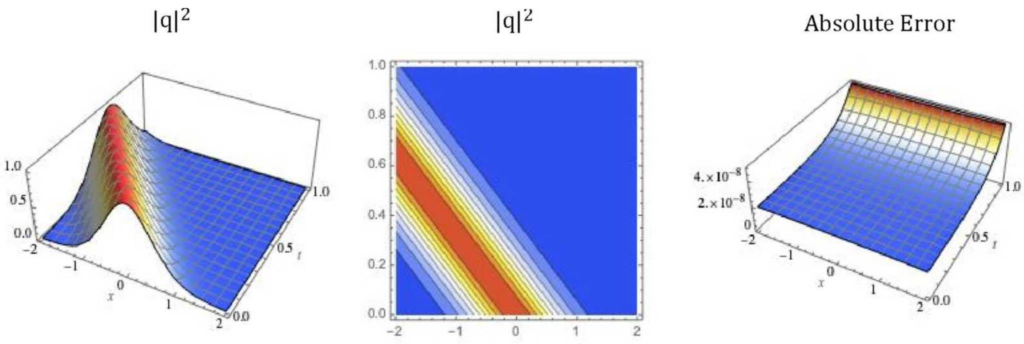

Fig. 1. Graphical representation of the example given in case B_I : (left) 3D bright soliton, (center) 2D waveform evolution, and (right) absolute error with $n=16$ iteration steps.

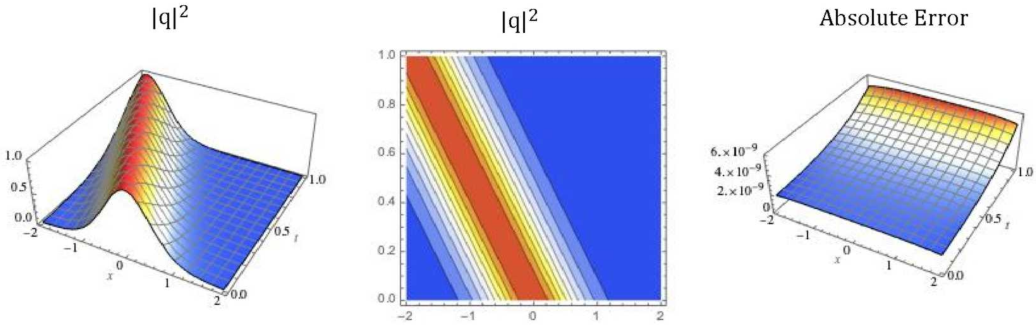

Fig. 2. Graphical representation of the example given in case B_{II} : (left) 3D bright soliton, (center) 2D waveform evolution, and (right) absolute error with $n=16$ iteration steps.

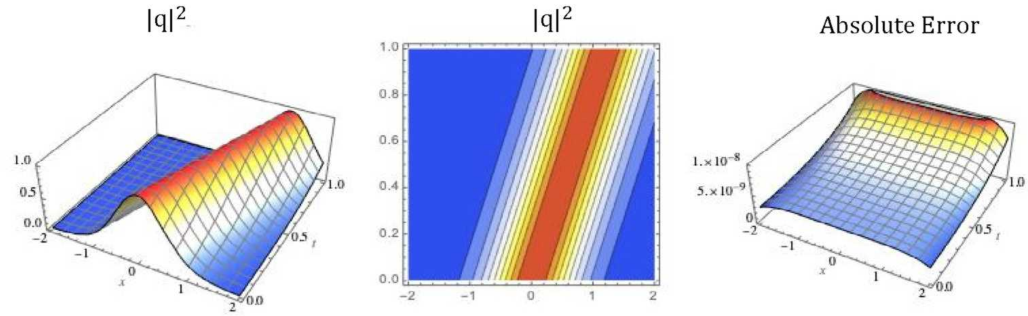

Fig. 3. Graphical representation of the example given in case B_{III} : (left) 3D bright soliton, (center) 2D waveform evolution, and (right) absolute error with $n=16$ iteration steps.

5.2. Simulation of dark solitons

Consider Eq. (9), with initial conditions [16-18]:

$$q_0(x,t)=A_2\tanh\left[B_2(x)\right]e^{i[-\kappa x+\Omega_0]}. \tag{17}$$

The soliton's amplitude A_2 may be determined using the following equation:

$$A_2=\pm\sqrt{\frac{-(\omega+\kappa(a_1\kappa-a_2\omega)+\kappa^2(b_1\kappa-b_2\omega))}{\kappa(\sigma+\theta)}}, \tag{18}$$

and the soliton's width, denoted as B_2, is determined by:

$$B_2=\pm\sqrt{\frac{-(\omega+\kappa(a_1\kappa-a_2\omega)+\kappa^2(b_1\kappa-b_2\omega))}{2(a_1+3\kappa b_1-\omega b_2-\nu(a_2+2\kappa b_2))}}. \tag{19}$$

Table 2 compares the absolute error of the exact and approximate solutions for different coefficients. Figs. 4, 5, and 6 display the graphical representations of the approximate solutions obtained by using VIM, the contour plot of the wave amplitude $|q|^2$, and the absolute error, respectively.

Table 2. Dark optical solitons.

Cases	a_1	a_2	b_1	b_2	σ	μ	θ	n	$\lvert Max\,Error \rvert$
D_I	1.6	0.2	4.3	0.7	3.0	0.5	1.7	16	1.0×10^{-8}
D_{II}	5.6	−2.7	7.1	0.4	3.1	0.4	6.6	16	2.0×10^{-8}
D_{III}	1.8	0.7	2.2	3.7	0.9	4.2	4.8	16	1.5×10^{-8}

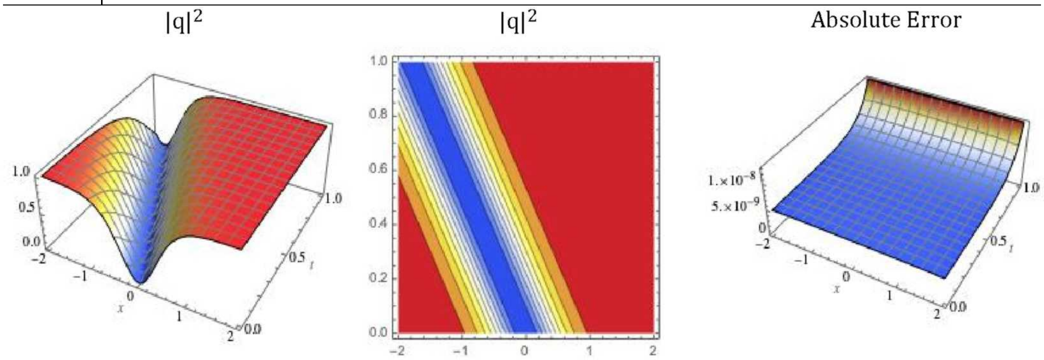

Fig. 4. Graphical representation of the example given in case D_I : (left) 3D dark soliton, (center) 2D waveform evolution, and (right) absolute error with n=16 iteration steps.

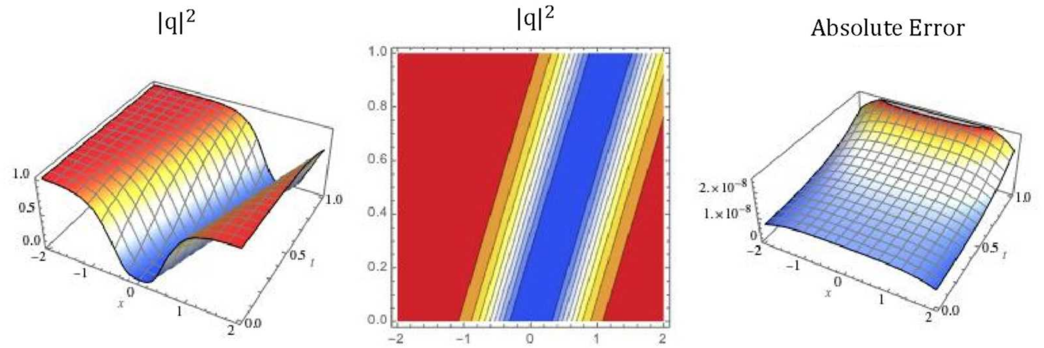

Fig. 5. Graphical representation of the example given in case D_{II} : (left) 3D dark soliton, (center) 2D waveform evolution, and (right) absolute error with n=16 iteration steps.

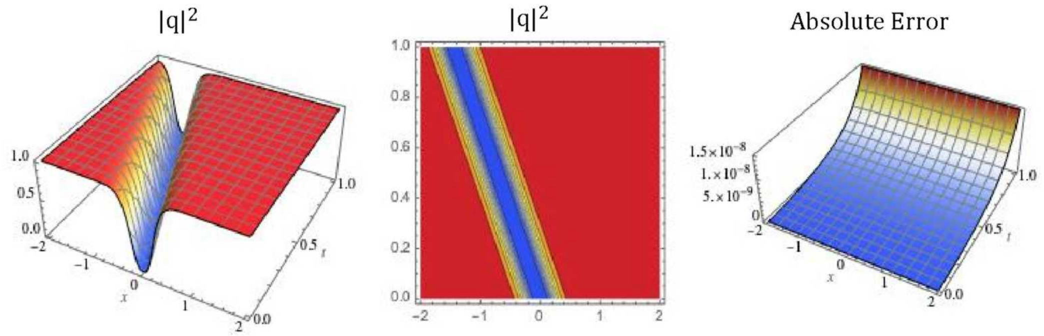

Fig. 6. Graphical representation of the example given in case D_{III} : (left) 3D dark soliton, (center) 2D waveform evolution, and (right) absolute error with n=16 iteration steps.

5.3. Simulation of W-shaped solitons

Consider Eq. (9), with initial conditions [16-18]:

$$q_0(x,t) = \left(\Gamma + A_3 \operatorname{sech}\left[B_3(x)\right]\right) e^{i[-\kappa x + \Omega_0]}. \tag{20}$$

The soliton's amplitude A_3 satisfies the following relation:

$$A_3 = \sqrt{\sigma^2 + 2\Gamma^2}. \tag{21}$$

The soliton's width, denoted as B_3, is determined by:

$$B_3 = -\frac{\sigma^2 \Gamma}{a_1}. \tag{22}$$

In order to guarantee the existence of a W-shaped soliton, Γ must be a real number such that:

$$|\Gamma| < |A_3| \quad and \quad \Gamma A_3 < 0. \tag{23}$$

Table 3 compares the absolute error of the exact and approximate solutions for different coefficients. Figs. 7, 8, and 9 display the graphical representations of the approximate solutions obtained by using VIM, the contour plot of the wave amplitude of $|q|^2$, and the absolute error, respectively.

Table 3. W-shaped optical solitons.

Cases	a_1	a_2	b_1	b_2	σ	μ	θ	n	$\lvert Max\,Error \rvert$
W_I	2.4	7.2	1.1	3.5	6.0	0.1	5.9	16	4.0×10^{-8}
W_{II}	9.1	5.4	0.6	2.2	5.1	2.4	0.5	16	4.0×10^{-8}
W_{III}	3.8	5.8	7.3	6.9	2.2	0.5	2.6	16	1.5×10^{-8}

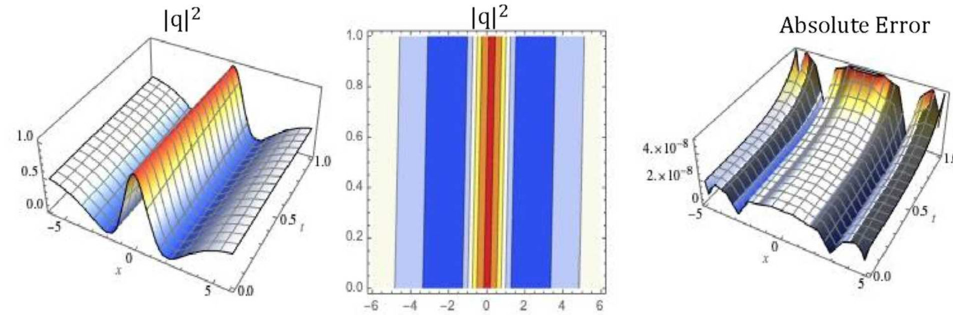

Fig. 7. Graphical representation of the example given in case W_I : (left) 3D W-shaped soliton, (center) 2D waveform evolution, and (right) absolute error with n=16 iteration steps.

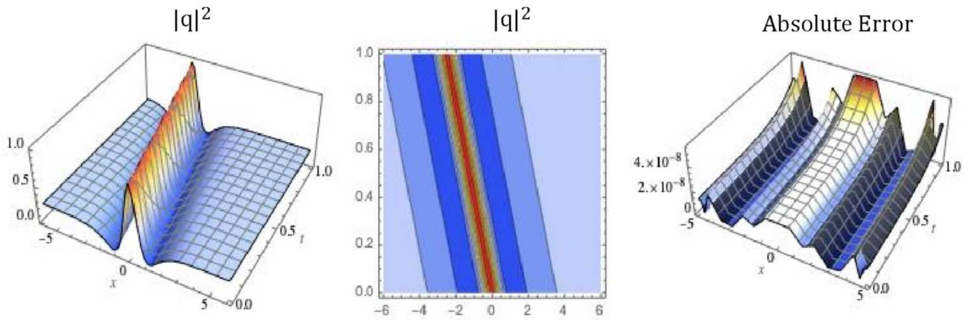

Fig. 8. Graphical representation of the example given in case W_{II} : (left) 3D W-shaped soliton, (center) 2D waveform evolution, and (right) absolute error with n=16 iteration steps.

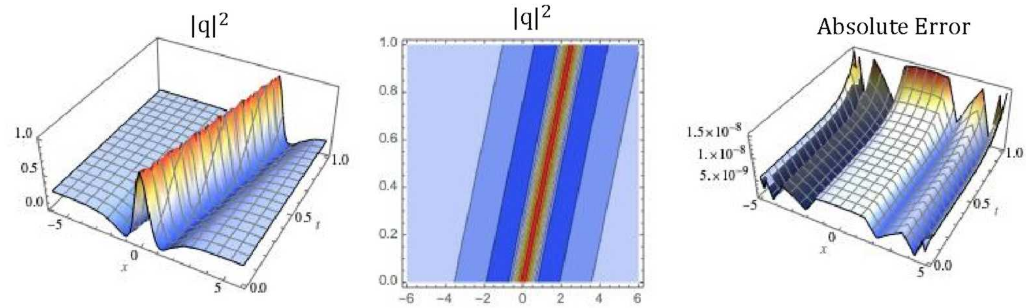

Fig. 9. Graphical representation of the example given in case W_{III} : (left) 3D W-shaped soliton, (center) 2D waveform evolution, and (right) absolute error with *n*=16 iteration steps.

6. Conclusions

Optical solitons of the Biswas-Arshed equation with Kerr-law nonlinearity and higher order dispersions have been successfully addressed in this article employing the implementation of the variational iteration method. We have obtained W-shaped, dark, and bright optical solitons using the aforementioned scheme considering certain constraints. Consequently, the integration technique demonstrated tremendous efficacy when applied to the model selected for the present investigation. Later on, the equation will be studied with full nonlinearity using effective integration methodologies. The valuable findings will be promptly presented.

References

1. Yildirim, Y., Biswas, A., Moraru, L., Alghamdi, A. A. (2023). Quiescent optical solitons for the concatenation model with nonlinear chromatic dispersion. *Mathematics*, *1*(7), 1709.
2. Malomed, B. A. (2022). Multidimensional dissipative solitons and solitary vortices. *Chaos Solitons Fractals*, *163*, 112526.
3. González-Gaxiola, O., Biswas, A., Ruiz de Chavez, J., Asiri, A. (2023). Bright and dark optical solitons for the concatenation model by the Laplace-Adomian decomposition scheme. *Ukrainian Journal of Physical Optics*, *24*(3), 222-234.
4. Yıldırım, Y., Biswas, A., Ekici, M., González-Gaxiola, O., Khan, S., Triki, H., Moraru, L., Alzahrani, A. K., Belic, M. R. (2020). Optical solitons with Kudryashov's model by a range of integration norms. *Chinese Journal of Physics*, *66*, 660-672.
5. González-Gaxiola, O., Biswas, A., Moraru, L., Alghamdi, A. A. (2023). Solitons in neurosciences by the Laplace–Adomian decomposition scheme. *Mathematics*, *11*(5), 1080.
6. Yue, C., Seadawy, A., Lu, D. (2016). Stability analysis of the soliton solutions for the generalized quintic derivative nonlinear Schrödinger equation. *Results in Physics*, *6*, 911-916.
7. Yildrim, Y., Biswas, A., Dakova, A., Guggilla, P., Khan, S., Alshehri, H. M., Belic, M. R. (2021). Cubic-quartic optical solitons having quadratic-cubic nonlinearity by sine-Gordon equation approach. *Ukrainian Journal of Physical Optics*, *22*(4), 255–269.
8. Yildrim, Y. (2019). Optical solitons of Biswas–Arshed equation by trial equation technique. *Optik*, *182*, 876-883.
9. Ullah, M. S., Abdeljabbar, A., Roshid, H-O., Ali, M. Z. (2022). Application of the unified method to solve the Biswas–Arshed model. *Results in Physics*, *42*, 105946.
10. Kudryashov, N. A. (2020). Periodic and solitary waves of the Biswas–Arshed equation. *Optik*, *200*, 163442.
11. Akram, G., Sadaf, M., Zainab, I. (2022). The dynamical study of Biswas–Arshed equation via modified auxiliary equation method. *Optik*, *255*, 168614.
12. Korpinar, Z., Inc, M., Bayram, M., Hashemi, M. S. (2020). New optical solitons for Biswas–Arshed equation with higher order dispersions and full nonlinearity. *Optik*, *206*, 163332.
13. Kumar, S., Niwas, M. (2022). New optical soliton solutions of Biswas–Arshed equation using the generalised exponential rational function approach and Kudryashov's simplest equation approach. *Pramana – J. Phys.*, *96*, 204.
14. Das, P. K. (2019). The rapidly convergent approximation method to solve system ofequations and its application to the Biswas-Arshed equation. *Optik*, *195*, 163134.

15. Biswas, A., Arshed, S. (2018). Optical solitons in presence of higher order dispersions and absence of self-phase modulation. *Optik*, *174*, 452–459.
16. Rehmana, H. U., Jafar, S., Javed, A., Hussain, S., Tahir, M. (2020). New optical solitons of Biswas-Arshed equation using different techniques. *Optik*, *206*, 163670.
17. Aouadi, S., Bouzida, A., Daoui, A. K., Triki, H., Zhou, Q., Liu, S. (2019). W-shaped, bright and dark solitons of Biswas-Arshed equation. *Optik*, 182, 227-232.
18. Li, Z., Li, L., Tian, H., Zhou, G. (2000). New types of solitary wave solutions for the higher order nonlinear Schrödinger equation. *Physical Review Letters*, *84*(18), 4096-4099.
19. He, J. H., Wu, X. H. (2006). Construction of solitary solution and compacton like solution by variational iteration method. *Chaos, Solitons & Fractals* , *29*(1), 108-113.
20. Momani, S., Abuasad, S. (2006). Application of He's variational iteration method to Helmholtz equation. *Chaos, Solitons & Fractals* , *27*(5), 1119-1123.
21. Wazwaz, A. M. (2007). The variational iteration method for rational solutions for KdV, K(2,2) Burgers, and cubic Boussinesq equations. *Journal of Computational and Applied Mathematics*, *207*(1), 18–23.
22. Wazwaz, A. M. (2019). Bright and dark optical solitons for (2+1) – dimensional Schrödinger (NLS) equations in the anomalous dispersion regimesand the normal dispersive regimes. *Optik*, *192*, 162948.
23. Odibat, Z. M. (2010). A study on the convergence of variational iteration method. *Mathematical and Computer Modelling*, *51*, 1181-1192.

O. González-Gaxiola, Yakup Yildirim. (2024). Bright, Dark, and W-Shaped Solitons of Biswas–Arshed Equation via Variational Iteration Method. *Ukrainian Journal of Physical Optics*, *25*(5), S1151 – S1159. doi: 10.3116/16091833/Ukr.J.Phys.Opt.2024.S1151

Анотація*. У цій роботі ми використали рівняння Бісваса-Аршеда, як інноваційний підхід до моделювання передачі солітонів через оптичні волокна в середовищі з керрівською нелінійністю. Були отримані яскраві, темні та W-подібні солітони в рамках цієїцієї інноваційної моделі за допомогою методу варіаційних ітерацій. Це ефективний метод, популярність якого продовжує зростати для чисельної рішення модельних рівнянь різних фізичних явищ, у тому числі фотоніки. Дослідження є новим у тому, що воно використовує ітераційний варіаційний підхід для чисельного відновлення рішень солітонного типу. Цей підхід усуває необхідність припускати лінеаризацію або дискретизацію, які потенційно можуть вплинути на фізичні характеристики моделі. Алгоритм надає результати з дуже низьким рівнем помилок.*

Ключові слова*: рівняння Бісваса–Аршеда, нелінійність закону Керра, метод варіаційних ітерацій, солітони.*