Ukrainian Journal of Physical Optics
SPECIAL ISSUE 5, 2024

Pure-Quartic Stationary Optical Bullets for (3+1)-Dimensional Nonlinear Schrödinger's Equation with Fourth-Order Dispersive Effects and parabolic law of nonlinearity

Abdul-Majid Wazwaz

Department of Mathematics, Saint Xavier University, Chicago, IL 60655 USA, wazwaz@sxu.edu

Received: 14.03.2024

Abstract. This work addresses a (3+1)-dimensional nonlinear Schrödinger's equation with three fourth-order dispersive terms that usually give pure-quartic bullets. It is known that pure-quartic bullets, balanced by fourth-order dispersion and nonlinearity, differ from traditional solitons. We derive various solutions in the form of bright and dark optically modulated bullets. The solutions obtained are useful for exploring the transmission of bullets through optical nanofibers.

Keywords: optical solitons, higher-order nonlinear Schrödinger's equation, fourth-order dispersion, optical fibers
UDC: 535.32
DOI: 10.3116/16091833/Ukr.J.Phys.Opt.2024.S1131

1. Introduction

In [1–6], pure-quartic solitons were examined and defined by shape-maintaining pulses that appear in optical materials with dominant fourth-order dispersion. This class of bright and dark soliton formation results purely from the exact balance between positive self-phase modulation and negative quartic dispersion [1–6]. A nonlinear Schrödinger (NLS) equation arises while studying the nonlinear scientific fields in inhomogeneous and dispersive media. It is frequently used in nonlinear optics, soft-condensed matter physics, plasma physics, and so on [1–14]. Also, this equation was used to investigate the modulational instability of modulated envelope waves such as bright modulated envelope solitons, dark modulated envelope solitons, etc. Significant results have been reported in nonlinear optics by studying various NLS systems with different forms of nonlinearities and orders.

The cubic NLS reads:

$$iu_t + u_{xx} + u|u|^2 = 0, \tag{1}$$

which is integrable [9–12]. Here, the function $u(x, y, t)$ is a complex-valued function of the spatial variables x, y and the temporal variable t, and denotes the varying envelope wave.

Blanco-Redondo et al. [1] examined the quartic dispersion and showed that it plays an important role in constructing the pure-quartic solitons. The pure-quartic solitons arise from a balance between the negative quartic dispersion and the Kerr nonlinear effect. This is unlike the conventional solitons that appear due to a balance between weak nonlinear steepening and dispersion effect [15–22]. Moreover, pure-quartic solitons have the properties of decaying with oscillatory tails at edges. In other words, pure-quartic solitons result from the interaction between fourth-order dispersion and self-phase modulation, resulting in oscillating tails [22–29].

Recently, the transmission of light beams through a pure-quartic diffraction material with Kerr nonlinearity [1–4] was given by the following NLS:

$$iu_t + \frac{\beta_4}{24} u_{xxx} + \sigma |u|^2 u = 0, \tag{2}$$

where $u(x,t)$ is the complex envelope of the electric field, t is the temporal variable, x is the coordinate, β_4 represents the fourth-order diffraction/dispersion coefficient, and σ denotes the cubic nonlinearity coefficient of $|u|^2u$ [1–3]. This equation has been used to study the propagation dynamics of pure-quartic solitons in a silicon photonic crystal waveguide [1–3]. Recent works show that pure quartic solitons have merit in energy-width scaling over traditional solitons. The pure-quartic dispersive phenomena of weak nonlocality can support quartic bright solitons as well as quartic dark solitons, where both take different structural forms. Pure quartic soliton has recently attracted much research because of its applications in microresonators and amplifiers [2–8].

We aim in this work to construct an extended form of the NLS (2), where we propose an extended model, that also incorporates higher-order even terms in (3+1)-dimensions and one more nonlinear term, that takes the form:

$$iu_t + a_1 u_{xxxx} + a_2 u_{yyyy} + a_3 u_{zzzz} + b_1 |u|^2 u + b_2 |u|^4 u = 0, \tag{3}$$

where, $u \equiv u(x,y,z,t)$ and the coefficients a_1, a_2, a_3, b_1, and b_2 are real coefficients depending on the nature of the physical model under consideration. It is obvious that the extended form (3) admits three dispersion terms, namely u_{xxxx}, u_{yyyy}, and u_{zzzz}, and two nonlinear terms $|u|^2 u$ and $|u|^4 u$.

In this work, we plan to use various schemes to study the extended NLSE (3) to furnish new optical modulated envelope soliton solutions. The employed techniques will enable us to get distinct modulated wave structures including bright solitons, dark solitons, periodic solutions, and others.

2. Stationary optical bullet solution

To obtain stationary bright optical bullet solutions for Eq. (3), we introduce the ansatz [28-31]:

$$\mathrm{u} = R\,\mathrm{sech}(\alpha x + \beta y + \gamma z) e^{i(kx+ry+sz-nt)}, \tag{4}$$

where $(R,\alpha,\beta,\gamma,k,r,s)$ and n are arbitrary constants that will be determined. Here α, β and γ are the direction ratios of the bullets while k, r, and s are the wave numbers along the x-, y- and z-direction respectively. Lastly, n is the frequency of the soliton and R is the width of the soliton. By combining Eqs. (3) and (4) and equating the coefficients of the resulting hyperbolic functions, we obtain:

$$\begin{gathered}
\left(\frac{3}{2}\alpha^2 k^2 - \frac{1}{4}\alpha^4 - \frac{1}{4}k^4\right) a_1 + \left(\frac{3}{2}\beta^2 r^2 - \frac{1}{4}\beta^4 - \frac{1}{4}r^4\right) a_2 + \left(\frac{3}{2}\gamma^2 s^2 - \frac{1}{4}\gamma^4 - \frac{1}{4}s^4\right) a_3 - \frac{1}{3}n = 0, \\
(\alpha^3 k - \alpha k^3) a_1 + (\beta^3 r - \beta r^3) a_2 + (\gamma^3 s - \gamma s^3) a_3 = 0, \\
(5\alpha^4 - 3\alpha^2 k^2) a_1 + (5\beta^4 - 3\beta^2 r^2) a_2 + (5\gamma^4 - 3\gamma^2 s^2) a_3 - \frac{1}{4} R^2 b_1 = 0, \\
\alpha^3 a_1 k + \beta^3 a_2 r + \gamma^3 a_3 s = 0, \qquad 6\alpha^4 a_1 + 6\beta^4 a_2 + 6\gamma^4 a_3 + \frac{1}{4} b_2 R^4 = 0.
\end{gathered} \tag{5}$$

Solving system (5) yields:

$$\begin{aligned}
&\alpha = -\frac{\gamma k}{s},\\
&\beta = -\frac{\gamma r}{s},\\
&a_1 = -\frac{a_3 s^4 - a_2 r^4}{k^4},\\
&b_1 = \frac{8 a_3 \gamma \left(5\gamma^2 - 3s^2\right)}{R^2},\\
&b_2 = -\frac{48 a_3 \gamma^4}{R^4},\\
&n = -2a_3\left(\gamma^4 - 6\gamma^2 s^2 + s^4\right),
\end{aligned} \tag{6}$$

where other parameters are left free. Substituting Eqs. (6) into Eq. (4) gives the bright bullet solution.

3. Singular optical bullet solutions

In order to obtain stationary singular solutions to Eq. (3), we introduce the ansatz:

$$\begin{aligned}
&\mathrm{u} = R\,\mathrm{csch}\left(\alpha x + \beta y + \gamma z\right)\\
&\times e^{i\left(kx + ry + sz - nt\right)},
\end{aligned} \tag{7}$$

where $(R,\alpha,\beta,\gamma,k,r,s)$ and n are arbitrary constants and do not equal zero.

By combining Eqs. (7) and (3) and equating the coefficients of the resulting hyperbolic functions, we obtain the same results obtained earlier in Eq. (6), except for b_1, where we obtained:

$$b_1 = -\frac{8 a_3 \gamma \left(5\gamma^2 - 3s^2\right)}{R^2}. \tag{8}$$

Substituting Eq. (8) into Eq. (7) gives the singular bullet solution.

4. Dark optical bullets solutions

To get some stationary dark bullet solutions to Eq. (3), we use the following ansatz:

$$\begin{aligned}
&\mathrm{u} = R\tanh\left(\alpha x + \beta y + \gamma z\right)\\
&\times e^{i\left(kx + ry + sz - nt\right)},
\end{aligned} \tag{11}$$

where the constants $(R,\alpha,\beta,\gamma,k,r,s)$ do not equal zero.

Inserting Eq. (11) into Eq. (3) and equating the coefficients of the hyperbolic functions in the obtained result, we get:

$$\begin{aligned}
a_1 k^4 + a_2 r^4 + a_3 s^4 + b_1 R^2 + b_2 R^4 + N = 0,\\
(\alpha^3 k - \frac{1}{4}\alpha k^3)a_1 + (\beta^3 r - \frac{1}{4}\beta r^3)a_2 + a_3\gamma(\gamma^2 - \frac{1}{4}s^2) = 0,\\
(\alpha^4 - \frac{3}{2}\alpha^2 k^2)a_1 + (\beta^4 - \frac{3}{2}\beta^2 r^2)a_2 + (\gamma^4 - \frac{3}{2}\gamma^2 s^2)a_3 + \frac{1}{4}R^2(b_2 R^2 + \frac{1}{2b_2}) = 0,\\
\alpha^3 a_1 k + \beta^3 a_2 r + \gamma^3 a_3 s = 0,\\
\alpha^4 a_1 + \beta^4 a_2 + \gamma^4 a_3 + \frac{1}{24} b_2 R^4 = 0.
\end{aligned} \tag{12}$$

Solving system (12) yields:

$$
\begin{aligned}
&\alpha = -\frac{\gamma k}{s}, s \neq 0,\\
&\beta = -\frac{\gamma r}{s},\\
&a_1 = \frac{a_3 s^4 - a_2 r^4}{k^4}, k \neq 0,\\
&b_1 = \frac{8 a_3 \gamma^2 \left(10\gamma^2 + 3s^2\right)}{R^2}, R \neq 0,\\
&b_2 = -\frac{48 a_3 \gamma^4}{R^4},\\
&n = a_3\left(16\gamma^4 + 12\gamma^2 s^2 + s^4\right),
\end{aligned} \tag{13}
$$

where other parameters are left free. Substituting Eqs. (13) into Eq. (11) gives the dark bullet solution. Moreover, in a like manner, we show that Eq. (3) gives a stationary singular optical bullet solution given by:

$$
\begin{aligned}
&u = R\coth\left(\alpha x + \beta y + \gamma z\right)\\
&\quad \times e^{i\left(kx + ry + sz - nt\right)},
\end{aligned} \tag{14}
$$

obtained under the same parameters obtained in (13).

5. Conclusion

A (3+1)-dimensional nonlinear Schrödinger equation with fourth-order dispersive terms has been studied. Based on well-known schemes, a variety of distinct solutions in the form of bright and dark optical modulated solutions, are determined. Moreover, other solutions, including periodic, singular modulated solutions, and complex solutions, are obtained.

Data Availability: Data sharing does not apply to this article as no data sets were generated or analyzed during the current study.

Conflicts of Interest: The author declares that he has no conflicts of interest.

Ethical approval: The author declares that he has adhered to the ethical standards of research execution.

References

1. Blanco-Redondo, A., De Sterke, C. M., Sipe, J. E., Krauss, T. F., Eggleton, B. J., & Husko, C. (2016). Pure-quartic solitons. *Nature Communications*, *7*(1), 10427.
2. Triki, H., Pan, A., & Zhou, Q. (2023). Pure-quartic solitons in presence of weak nonlocality. *Physics Letters A*, *459*, 128608.
3. Soltani, M., Triki, H., Azzouzi, F., Sun, Y., Biswas, A., Yıldırım, Y., Alshehri, H. M. & Zhou, Q. (2023). Pure–quartic optical solitons and modulational instability analysis with cubic–quintic nonlinearity. *Chaos, Solitons & Fractals*, *169*, 113212.
4. Zhu, Z., Yang, S., He, C., & Lin, X. (2023). Vector pure-quartic soliton molecule fiber laser. *Chaos, Solitons & Fractals*, *175*, 113978.
5. Wazwaz, A. M., & Xu, G. Q. (2023). Variety of optical solitons for perturbed Fokas–Lenells equation through modified exponential rational function method and other distinct schemes. *Optik*, *287*, 171011.
6. Osman, M. S., & Ghanbari, B. (2018). New optical solitary wave solutions of Fokas-Lenells equation in presence of perturbation terms by a novel approach. *Optik*, *175*, 328-333.
7. Triki, H., & Wazwaz, A. M. (2017). New types of chirped soliton solutions for the Fokas–Lenells equation. *International Journal of Numerical Methods for Heat & Fluid Flow*, *27*(7), 1596-1601.
8. Ding, Y., Osman, M. S., & Wazwaz, A. M. (2019). Abundant complex wave solutions for the nonautonomous Fokas–Lenells equation in presence of perturbation terms. *Optik*, *181*, 503-513.
9. Hosseini, K., Hincal, E., Mirekhtiary, F., Sadri, K., Obi, O. A., Denker, A., & Mirzazadeh, M. (2023). A fourth-order nonlinear Schrödinger equation involving power law and weak nonlocality: its solitary waves and modulational instability analysis. *Optik*, *284*, 170927.

10. Wazwaz, A. M., & Xu, G. Q. (2020). Bright, dark and Gaussons optical solutions for fourth-order Schrödinger equations with cubic–quintic and logarithmic nonlinearities. *Optik*, *202*, 163564.
11. Xu, G. Q. (2011). New types of exact solutions for the fourth-order dispersive cubic–quintic nonlinear Schrödinger equation. *Applied Mathematics and Computation*, *217*(12), 5967-5971.
12. Xu, G. Q. (2019). Painlevé analysis, lump-kink solutions and localized excitation solutions for the (3+ 1)-dimensional Boiti–Leon–Manna–Pempinelli equation. *Applied Mathematics Letters*, *97*, 81-87.
13. Xu, G. Q., Liu, Y. P., & Cui, W. Y. (2022). Painlevé analysis, integrability property and multiwave interaction solutions for a new (4+ 1)-dimensional KdV–Calogero–Bogoyavlenkskii–Schiff equation. *Applied Mathematics Letters*, *132*, 108184.
14. Guo, C., & Guo, B. (2019). The existence of global solutions for the fourth-order nonlinear Schrödinger equations. *J. Appl. Anal. Comput.*, *9*(3), 1183-1192.
15. Wazwaz, A. M., Alhejaili, W., & El-Tantawy, S. A. (2022). Bright and dark envelope optical solitons for a (2+ 1)-dimensional cubic nonlinear Schrödinger equation. *Optik*, *265*, 169525.
16. Wazwaz, A. M., Alhejaili, W., AL-Ghamdi, A. O., & El-Tantawy, S. A. (2023). Bright and dark modulated optical solitons for a (2+ 1)-dimensional optical Schrödinger system with third-order dispersion and nonlinearity. *Optik*, *274*, 170582.
17. El-Tantawy, S. A., Alharbey, R. A., & Salas, A. H. (2022). Novel approximate analytical and numerical cylindrical rogue wave and breathers solutions: An application to electronegative plasma. *Chaos, Solitons & Fractals*, *155*, 111776.
18. Biswas, A., & Khalique, C. M. (2011). Stationary solution of the nonlinear Schrödinger's equation with log law nonlinearity by Lie symmetry analysis. *Waves in Random and Complex Media*, *21*(4), 554-558.
19. Khalique, C. M. (2012). Exact solutions and conservation laws of a coupled integrable dispersionless system. *Filomat*, *26*(5), 957-964.
20. Özkan, Y. S., Yaşar, E., & Khalique, C. M. (2023). Multi-wave, breather and interaction solutions to (3+ 1) dimensional sine-Gordon equation arising in nonlinear physical sciences. *Journal of Computational Science*, *73*, 102144.
21. Adeyemo, O. D., & Khalique, C. M. (2023). Shock waves, periodic, topological kink and singular soliton solutions of a new generalized two dimensional nonlinear wave equation of engineering physics with applications in signal processing, electromagnetism and complex media. *Alexandria Engineering Journal*, *73*, 751-769.
22. Triki, H., Biswas, A., Zhou, Q., Moshokoa, S. P., & Belic, M. (2019). Chirped envelope optical solitons for Kaup–Newell equation. *Optik*, *177*, 1-7.
23. Wazwaz, A. M., Alhejaili, W., & El-Tantawy, S. A. (2022). Bright and dark envelope optical solitons for a (2+ 1)-dimensional cubic nonlinear Schrödinger equation. *Optik*, *265*, 169525.
24. Hosseini, K., Hincal, E., Mirzazadeh, M., Salahshour, S., Obi, O. A., & Rabiei, F. (2023). A nonlinear Schrödinger equation including the parabolic law and its dark solitons. *Optik*, *273*, 170363.
25. Wazwaz, A. M. (2009). Multiple soliton solutions and multiple singular soliton solutions for (2+1)-dimensional shallow water wave equations. *Physics Letters A*, *373*(33), 2927-2930.
26. Wazwaz, A. M. (2010). *Partial differential equations and solitary waves theory*. Springer Science & Business Media.
27. AWazwaz, A. M., Albalawi, W., & El-Tantawy, S. A. (2022). Optical envelope soliton solutions for coupled nonlinear Schrödinger equations applicable to high birefringence fibers. *Optik*, *255*, 168673.
28. Elsherbeny, A. M., Mirzazadeh, M., Arnous, A. H., Biswas, A., Yıldırım, Y., & Asiri, A. (2023). Optical bullets and domain walls with cross-spatio dispersion having parabolic law of nonlinear refractive index. *Journal of Optics*, 1-12.
29. Elsherbeny, A. M., Mirzazadeh, M., Arnous, A. H., Biswas, A., Yildirim, Y., Dakova, A., & Asiri, A. (2023). Optical bullets and domain walls with cross spatio-dispersion and having Kudryashov's form of self-phase modulation. *Contemporary Mathematics*, 505-517.
30. Biswas, A. (2002). Theory of optical Bullets, *Progress in Electromagnetic Research*, *36*, 21-59.
31. Al-Ghafri, K. S., Krishnan, E. V., Khan, S., & Biswas, A. (2022). Optical bullets and their modulational instability analysis. *Applied Sciences*, *12*(18), 9221.

Abdul-Majid Wazwaz. (2024). Pure-Quartic Stationary Optical Bullets for (3+1)-Dimensional Nonlinear Schrödinger's Equation with Fourth-Order Dispersive Effects and parabolic law of nonlinearity. *Ukrainian Journal of Physical Optics*, *25*(5), S1131 – S1136.
doi: 10.3116/16091833/Ukr.J.Phys.Opt.2024.S1131

Анотація. *У цій роботі розглядається (3+1)-вимірне нелінійне рівняння Шредінгера з трьома дисперсійними членами четвертого порядку, яке зазвичай дає чисті квартичні кулі. Відомо, що чисті квартичні кулі, збалансовані дисперсією четвертого порядку і нелінійністю, відрізняються від традиційних солітонів. Ми отримали різні рішення у* *вигляді яскравих і темних оптично модульованих куль. Отримані рішення є корисними для дослідження поширення куль через оптичні нановолокна.*

***Ключові слова**: оптичні солітони, нелінійне рівняння Шредінгера вищого порядку, дисперсія четвертого порядку, оптичні волокна*