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Optical Soliton Parameters by Variational Principle: Polynomial and Triple-Power Laws (Super-Gaussons and Super-Sech Pulses)

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Abstract. In this article, the parameter dynamics of super–Gaussian and super–sech pulses for the perturbed nonlinear Schrödinger's equation with polynomial and triple–power nonlinearity laws are recovered. The variational principle successfully recovers this dynamical system.

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1. Introduction

The parameter dynamics of optical solitons is an imperative feature in optoelectronics phenomena [1–7]. Therefore, it is imperative to address this feature to the fullest. These dynamics lead to many further studies ranging from the computation of collision-induced timing jitter, four-wave mixing, bifurcation analysis, and several other features. The current paper is a sequel to the previous work on the parabolic and dual–power law of self–phase modulation (SPM) [1]. Additionally, the dynamics of Gaussons for dispersion-managed (DM) solitons have been retrieved in the past. Later, the DM soliton perturbation with super–sech was reported. Lately, the parameter dynamics of such solitons with quadratic–cubic, anti–cubic and generalized anti–cubic solitons were also recovered. The current work therefore addresses the retrieval of the soliton parameter dynamics with polynomial law as well as triple–power law of SPM. The variational principle is applied to make this retrieval possible. The perturbation terms are subsequently included, the extended parameter dynamics with such terms are laid down, and these perturbation terms are taken up with generalized intensity.

2. Polynomial law nonlinearity

The governing model of such an equation is written as:

$$iq_t + aq_{xx} + \left(b_1|q|^2 + b_2|q|^4 + b_3|q|^6\right)q = 0, \tag{1}$$

where $q = q(x,t)$ is a complex-valued function representing the wave profile and $i = \sqrt{-1}$. The first term in Eq. (1) is the linear temporal evolution. The constants $a, b_{1,2,3}$ are the coefficients of chromatic dispersion (CD) and self–phase modulation (SPM), respectively.

2.1. Variational principle

The Lagrangian (L_g) is associated with Eq. (1) and is written as:

$$L_g = \int_{-\infty}^{\infty}\left[\frac{i}{4}\left(q^* q_t - q q_t^*\right) - \frac{a}{2}|q_x|^2 + \frac{b_1}{4}|q|^4 + \frac{b_2}{6}|q|^6 + \frac{b_3}{8}|q|^8\right]dx, \tag{2}$$

where q^* is the complex conjugate of q and the conserved energy and momentum are:

$$E = \int_{-\infty}^{\infty} |q|^2\, dx, \tag{3}$$

$$M = i\int_{-\infty}^{\infty}\left(q^* q_x - q q_x^*\right)dx, \tag{4}$$

respectively, while, the Hamiltonian is given by:

$$H = \int_{-\infty}^{\infty}\left[a|q_x|^2 - \frac{b_1}{4}|q|^4 - \frac{b_2}{6}|q|^6 - \frac{b_3}{8}|q|^8\right]dx. \tag{5}$$

Now, the pulse hypothesis $q = q(x,t)$ of Eq. (1) is given by:

$$q(x,t) = A(t) f\left[B(t)\{x - \bar{x}(t)\}\right]\exp\left[-i\kappa(t)\{x - \bar{x}(t)\} + i\theta_0(t)\right], \tag{6}$$

where f represents the shape of the pulse. It could be a Gaussian type or super-Gaussian type pulse. Here, $A(t)$ is the soliton amplitude, $B(t)$ is the pulse width, $\bar{x}(t)$ is the center position of the soliton, $\kappa(t)$ is the soliton wavevector, and $\theta_0(t)$ is the soliton phase. For convenience, we define the following integral:

$$I_{a,b,c} = \int_{-\infty}^{\infty} s^a f^b(s)\left(\frac{df(s)}{ds}\right)^c ds \tag{7}$$

where b, c are non-negative integers. Now, substituting (6) into (2) and using the formula:

$$ds = B(t)dx. \tag{8}$$

Consequently, the Lagrangian (2) reduces to:

$$L_g = -\frac{A^2(t)}{2B(t)}\left(\frac{d\theta_0(t)}{dt} + \kappa(t)\frac{d\bar{x}(t)}{dt} + a\kappa^2(t)\right)I_{0,2,0} - \frac{1}{2}aA^2(t)B^3(t)I_{0,0,2} + \frac{b_1A^4(t)}{4B(t)}I_{0,4,0} + \frac{b_2A^6(t)}{6B(t)}I_{0,6,0} + \frac{b_3A^8(t)}{8B(t)}I_{0,8,0}. \tag{9}$$

For such a pulse form, given by (6), we have the integrals of motion as:

$$E = \frac{A^2(t)}{B(t)}I_{0,2,0}, \tag{10}$$

$$M = \frac{2A^2(t)\kappa(t)}{B(t)}I_{0,2,0}, \tag{11}$$

while the Hamiltonian is given by:

$$H = \frac{A^2(t)}{B(t)}\left[\begin{array}{l} aB^4(t)I_{0,0,2} + a\kappa^2(t)I_{0,2,0} - \frac{b_1A^2(t)}{4}I_{0,4,0} \\ -\frac{b_2A^4(t)}{6}I_{0,6,0} - \frac{b_3A^6(t)}{8}I_{0,8,0} \end{array}\right]. \tag{12}$$

2.2. Parameter dynamics

In this subsection, we derive the dynamic system by introducing the following Euler–Lagrange (EL) equation [4, 8]:

$$\frac{\partial L_g}{\partial p} - \frac{d}{dt}\left(\frac{\partial L_g}{\partial p_t}\right) = 0, \tag{13}$$

where p is one of the five soliton parameters $A(t)$, $B(t)$, $\bar{x}(t)$, $\kappa(t)$ and $\theta_0(t)$, respectively. Substituting (9) into (13) to get the following dynamic system:

$$\left[\frac{d\theta_0(t)}{dt} + \kappa(t)\frac{d\bar{x}(t)}{dt} + a\kappa^2(t)\right]I_{0,2,0} + aB^4(t)I_{0,0,2} - b_1A^2(t)I_{0,4,0} - b_2A^4(t)I_{0,6,0} - b_3A^6(t)I_{0,8,0} = 0, \tag{14}$$

$$\left[\frac{d\theta_0(t)}{dt} + \kappa(t)\frac{d\bar{x}(t)}{dt} + a\kappa^2(t)\right]I_{0,2,0} - 3aB^4(t)I_{0,0,2} - \frac{1}{2}b_1A^2(t)I_{0,4,0} - \frac{1}{3}b_2A^4(t)I_{0,6,0} - \frac{1}{4}b_3A^6(t)I_{0,8,0} = 0, \tag{15}$$

$$\frac{2}{A(t)}\frac{dA(t)}{dt} - \frac{1}{B(t)}\frac{dB(t)}{dt} + \frac{1}{\kappa(t)}\frac{d\kappa(t)}{dt} = 0, \tag{16}$$

$$\frac{d\bar{x}(t)}{dt} = -2a\kappa(t), \tag{17}$$

$$\frac{dA(t)}{dt} = \frac{A(t)}{2B(t)}\frac{dB(t)}{dt}. \tag{18}$$

The Eqs. (14)–(18) represent the general forms of the soliton parameter dynamics of Eq. (1) for the pulse form given by (6). The dynamic system (14)–(18) can be reduced to become:

$$\frac{d\theta_0(t)}{dt} = \frac{48a\kappa^2(t)I_{0,2,0} + 39b_3A^6(t)I_{0,8,0} + 40b_2A^4(t)I_{0,6,0} + 42b_1A^2(t)I_{0,4,0}}{48I_{0,2,0}}, \tag{19}$$

$$B(t)=\left[\frac{9b_3A^6(t)I_{0,8,0}+8b_2A^4(t)I_{0,6,0}+6b_1A^2(t)I_{0,4,0}}{48aI_{0,0,2}}\right]^{\frac{1}{4}}, \tag{20}$$

$$\frac{d\bar{x}(t)}{dt}=-2a\kappa(t), \tag{21}$$

$$\frac{d\kappa(t)}{dt}=0, \tag{22}$$

$$A(t)=K\sqrt{B(t)}, \tag{23}$$

where the constant K is proportional to the square roots of the energy.

2.2.1. Super-Gaussian pulses. For super-Gaussian pulses, we set $f(s)=e^{-s^{2m}}, \quad m>0$. Then, the integrals of motion are given by:

$$E=\frac{A^2(t)}{mB(t)}2^{-\frac{1}{2m}}\Gamma\left(\frac{1}{2m}\right), \tag{24}$$

$$M=\frac{A^2(t)\kappa(t)}{mB(t)}2^{(2m-1)/2m}\Gamma\left(\frac{1}{2m}\right), \tag{25}$$

and the Hamiltonian is given by:

$$\begin{aligned}H=\frac{A^2(t)}{mB(t)}\Bigg[&amB^4(t)(2m-1)2^{(1-2m)/2m}\Gamma\left(1-\frac{1}{2m}\right)+a\kappa^2(t)2^{-1/2m}\Gamma\left(\frac{1}{2m}\right)\\&-\frac{b_1A^2(t)}{2^{(2m+1)/m}}\Gamma\left(\frac{1}{2m}\right)-\frac{b_2A^4(t)}{6^{(2m+1)/2m}}\Gamma\left(\frac{1}{2m}\right)-\frac{b_3A^6(t)}{8^{(2m+1)/2m}}\Gamma\left(\frac{1}{2m}\right)\Bigg],\end{aligned} \tag{26}$$

where $\Gamma(u)$ is the gamma function that is defined for $u>0$. This compels the parameter m to be bounded belowб as given by:

$$m>\frac{1}{2}. \tag{27}$$

Also, we have the evolution equations for the pulse parameters (19)–(23), which reduce to:

$$\frac{d\theta_0(t)}{dt}=\frac{a\kappa^2(t)2^{4-\frac{1}{2m}}+13b_3A^6(t)8^{-\frac{1}{2m}}+\frac{40}{3}b_2A^4(t)6^{-\frac{1}{2m}}+14b_1A^2(t)2^{-\frac{1}{m}}}{2^{4-\frac{1}{2m}}}, \tag{28}$$

$$B(t)=\left[\frac{A^2(t)\left[3b_3A^4(t)8^{-\frac{1}{2m}}+\frac{8}{3}b_2A^2(t)6^{-\frac{1}{2m}}+2b_12^{-\frac{1}{m}}\right]\Gamma\left(\frac{1}{2m}\right)}{am(2m-1)2^{(6m+1)/2m}\Gamma\left(1-\frac{1}{2m}\right)}\right]^{\frac{1}{4}}, \tag{29}$$

$$\frac{d\bar{x}(t)}{dt}=-2a\kappa(t), \tag{30}$$

$$\frac{d\kappa(t)}{dt}=0, \tag{31}$$

$$A(t)=K\sqrt{B(t)}. \tag{32}$$

2.2.2. Super-sech pulses. For super-sech pulses, we set $f(s)=\text{sech}^{2m}(s), \quad m>0$. Then, the integrals of motion are given by:

$$E=\frac{\sqrt{\pi}A^2(t)}{B(t)}\frac{\Gamma(2m)}{\Gamma\left(2m+\frac{1}{2}\right)}, \tag{33}$$

$$M=\frac{2\sqrt{\pi}A^2(t)\kappa(t)}{B(t)}\frac{\Gamma(2m)}{\Gamma\left(2m+\frac{1}{2}\right)}, \tag{34}$$

and the Hamiltonian is given by:

$$\begin{aligned}H=&2amA^2(t)B^3(t)\left[{}_2F_1(1,-2m-1,2m+1,-1)-1-\frac{4m^2\sqrt{\pi}\Gamma(2m)}{(4m+1)\Gamma\left(2m+\frac{1}{2}\right)}\right]\\&+\frac{a\kappa^2(t)A^2(t)}{B(t)}\frac{\sqrt{\pi}\Gamma(2m)}{\Gamma\left(2m+\frac{1}{2}\right)}\\&-\frac{A^4(t)}{B(t)}\left[\frac{b_1}{4}\frac{\sqrt{\pi}\Gamma(4m)}{\Gamma\left(4m+\frac{1}{2}\right)}+\frac{b_2A^2(t)}{6}\frac{\sqrt{\pi}\Gamma(6m)}{\Gamma\left(6m+\frac{1}{2}\right)}+\frac{b_3A^4(t)}{8}\frac{\sqrt{\pi}\Gamma(8m)}{\Gamma\left(8m+\frac{1}{2}\right)}\right],\end{aligned} \tag{35}$$

where Gauss' hypergeometric function in its generalized form is:

$${}_pF_q\left(a_1,\ldots a_p;b_1,\ldots b_q;z\right)=\sum_{k=0}^{\infty}\frac{(a_1)_k\cdots(a_p)_k}{(b_1)_k\cdots(b_q)_k}\frac{z^k}{k!} \tag{36}$$

with the Pochhammer symbol being:

$$(p)_n=\begin{cases}1 & n=0\,,\\ p(p+1)\cdots(p+n-1) & n>0\,.\end{cases} \tag{37}$$

Also, we have the evolution equations for the pulse parameters (19)–(23), which reduce to

$$\frac{d\theta_0(t)}{dt}=\frac{\frac{48a\kappa^2(t)\Gamma(2m)}{\Gamma\left(2m+\frac{1}{2}\right)}+\frac{39b_3A^6(t)\Gamma(8m)}{\Gamma\left(8m+\frac{1}{2}\right)}+\frac{40b_2A^4(t)\Gamma(6m)}{\Gamma\left(6m+\frac{1}{2}\right)}+\frac{42b_1A^2(t)\Gamma(4m)}{\Gamma\left(4m+\frac{1}{2}\right)}}{\left(\frac{48\Gamma(2m)}{\Gamma\left(2m+\frac{1}{2}\right)}\right)} \tag{38}$$

$$B(t)=\left[\frac{\frac{9b_3A^6(t)\sqrt{\pi}\Gamma(8m)}{\Gamma\left(8m+\frac{1}{2}\right)}+\frac{8b_2A^4(t)\sqrt{\pi}\Gamma(6m)}{\Gamma\left(6m+\frac{1}{2}\right)}+\frac{6b_1A^2(t)\sqrt{\pi}\Gamma(4m)}{\Gamma\left(4m+\frac{1}{2}\right)}}{96am\left[{}_2F_1(1,-2m-1,2m+1,-1)-1-\frac{4m^2\sqrt{\pi}\Gamma(2m)}{(4m+1)\Gamma\left(2m+\frac{1}{2}\right)}\right]}\right]^{\frac{1}{4}}, \tag{39}$$

$$\frac{d\bar{x}(t)}{dt}=-2a\kappa(t), \tag{40}$$

$$\frac{d\kappa(t)}{dt}=0, \tag{41}$$

$$A(t)=K\sqrt{B(t)}. \tag{42}$$

2.3. Perturbed nonlinear Schrödinger's equation (NLSE)

The governing model of such an equation is written as follows:

$$iq_t + aq_{xx} + \left(b_1|q|^2 + b_2|q|^4 + b_3|q|^6\right)q = i\epsilon R\left[q,q^*\right], \tag{43}$$

where $R\left[q,q^*\right]$ is given by:

$$\begin{aligned}R = {} & \delta|q|^{2m}q + \alpha q_x + \beta q_{xx} + \lambda\left(|q|^{2m}q\right)_x + \theta\left(|q|^{2m}\right)_x q + \sigma|q|^{2m}q_x \\ & -i\xi\left(q^2q_x^*\right)_x - i\eta q_x^2q^* - i\zeta q^*\left(q^2\right)_{xx} - i\mu\left(|q|^{2m}\right)_x q + \left(\sigma_1 q + \sigma_2 q_x\right)\int_{-\infty}^{x}|q|^{2m}ds,\end{aligned} \tag{44}$$

and $\epsilon,\delta,\alpha,\beta,\lambda,\theta,\sigma,\xi,\eta,\zeta,\mu$, σ_1 and σ_2 are constants, where ϵ is the spectrum relative width i.e. so-called quasimonochromaticity. From (6) and (44), we have

$$\begin{aligned}R = {} & A(t)\begin{cases}\delta A^{2m}(t)f^{2m+1}(s) + \alpha B^2(t)\dfrac{df(s)}{ds} \\ +\left[(2m+1)\lambda + 2m\theta + \sigma\right]A^{2m}(t)B^2(t)f^{2m}(s)\dfrac{df(s)}{ds} \\ -\beta\kappa^2(t)f(s) + 2(\xi-\eta-4\zeta)A^2(t)B^2(t)\kappa(t)f^2(s)\dfrac{df(s)}{ds} \\ +A^{2m}(t)\left[\sigma_1 f(s) + \sigma_2 B^2(t)\dfrac{df(s)}{ds}\right]\displaystyle\int_{-\infty}^{x}f^{2m}(s)ds\end{cases} \\ & -i\left[\begin{aligned}&\alpha\kappa(t)f(s) + 2\beta\kappa(t)B^2(t)\frac{df(s)}{ds} + (\lambda+\sigma)A^{2m}(t)\kappa(t)f^{2m+1}(s) \\ &+(2\xi+\eta+2\zeta)A^2(t)B^4(t)f(s)\left(\frac{df(s)}{ds}\right)^2 + (\xi-4\zeta-\eta)A^2(t)\kappa^2(t)f^3(s) \\ &+2m\mu A^{2m}(t)B^2(t)f^{2m}(s)\frac{df(s)}{ds} + \sigma_2 A^{2m}(t)\kappa(t)f(s)\int_{-\infty}^{x}f^{2m}(s)ds\end{aligned}\right]\Bigg\} \\ & \times\exp\left[-i\frac{\kappa(t)}{B(t)}s + i\theta_0(t)\right].\end{aligned} \tag{45}$$

2.4. Parameter dynamics

In this subsection, we derive the dynamic system of Eq. (43) by introducing the following EL equation:

$$\frac{\partial L_g}{\partial p} - \frac{d}{dt}\left(\frac{\partial L_g}{\partial p_t}\right) = i\epsilon\int_{-\infty}^{\infty}\left(R\frac{\partial q^*}{\partial p} - R^*\frac{\partial q}{\partial p}\right)dx \tag{46}$$

where L_g is given by (9) and p is one of these same five parameters $A(t)$, $B(t)$, $\overline{x}(t)$, $\kappa(t)$ and $\theta_0(t)$, respectively, while R^* is the complex-conjugate of R. Now, we have the following dynamic system:

$$\begin{aligned}&\left[\frac{d\theta_0(t)}{dt} + \kappa(t)\left(2\epsilon\alpha + \frac{d\overline{x}(t)}{dt}\right) + a\kappa^2(t) + 2\epsilon\sigma_2\kappa(t)A^{2m}(t)\int_{-\infty}^{x}f^{2m}(s)ds\right]I_{0,2,0} \\ &-A^2(t)\left[b_1 - 2\epsilon(\xi-\eta-4\zeta)\kappa^2(t)\right]I_{0,4,0} - b_2A^4(t)I_{0,6,0} - b_3A^6(t)I_{0,8,0} + aB^4(t)I_{0,0,2} \\ &= -2\epsilon(\lambda+\sigma)\kappa(t)A^{2m}(t)I_{0,2m+2,0} - 2\epsilon(2\xi+\eta+2\zeta)A^2(t)B^4(t)I_{0,2,2},\end{aligned} \tag{47}$$

$$\left[\frac{d\theta_0(t)}{dt}+\kappa(t)\frac{d\bar{x}(t)}{dt}+a\kappa^2(t)\right]I_{0,2,0}$$
$$-\frac{b_1A^2(t)}{2}I_{0,4,0}-\frac{b_2A^4(t)}{3}I_{0,6,0}-\frac{b_3A^6(t)}{4}I_{0,8,0}-3aB^4(t)I_{0,0,2} \tag{48}$$
$$=8\epsilon\beta B(t)\kappa(t)I_{2,0,2}+8\epsilon m\mu B(t)A^{2m}(t)I_{2,2m,2},$$

$$-\frac{A^2(t)}{2B(t)}\left[\frac{d\bar{x}(t)}{dt}+2a\kappa(t)\right]I_{0,2,0}=0, \tag{49}$$

$$\left[\begin{array}{l}2\kappa(t)B(t)\frac{dA(t)}{dt}+A(t)B(t)\frac{d\kappa(t)}{dt}-A(t)\kappa(t)\frac{dB(t)}{dt}\\ -4\epsilon\kappa(t)A(t)B(t)\left(\sigma_1A^{2m}(t)\int_{-\infty}^{x}f^{2m}(s)ds-\beta\kappa^2(t)\right)\end{array}\right]I_{0,2,0}$$
$$=4\epsilon\delta\kappa(t)B(t)A^{2m+1}(t)I_{0,2m+2,0}+8\epsilon\beta\kappa(t)B^5(t)A(t)I_{0,0,2} \tag{50}$$
$$+8\epsilon m\mu B^5(t)A^{2m+1}(t)I_{0,2m,2},$$

$$\left[\begin{array}{l}\frac{dA(t)}{dt}-\frac{A(t)}{2B(t)}\frac{dB(t)}{dt}\\ -2\epsilon A(t)\left(\sigma_1A^{2m}(t)\int_{-\infty}^{x}f^{2m}(s)ds-\beta\kappa^2(t)\right)\end{array}\right]I_{0,2,0}=2\epsilon\delta A^{2m+1}(t)I_{0,2m+2,0}. \tag{51}$$

The Eqs. (47)–(51) represent the general forms of the soliton parameters dynamics of Eq. (43) for the pulse form given by (6). The dynamic system (47)–(51) can be reduced to become:

$$\frac{d\theta_0(t)}{dt}=a\kappa^2(t)+\frac{2\epsilon\kappa(t)}{3}\left[\alpha+\sigma_2A^{2m}(t)\int_{-\infty}^{x}f^{2m}(s)ds\right]$$
$$+\frac{A^2(t)\left[2\epsilon(\xi-\eta-4\zeta)\kappa^2(t)\right]I_{0,4,0}+13aB^4(t)I_{0,0,2}}{3I_{0,2,0}}$$
$$+\frac{2\epsilon\left[\begin{array}{l}16\beta\kappa(t)B(t)I_{2,0,2}+(2\xi+\eta+2\zeta)A^2(t)B^4(t)I_{0,2,2}\\ +(\lambda+\sigma)\kappa(t)A^{2m}(t)I_{0,2m+2,0}+16m\mu A^{2m}(t)B(t)I_{2,2m,2}\end{array}\right]}{3I_{0,2,0}} \tag{52}$$
$$-\frac{b_2A^4(t)I_{0,6,0}}{9I_{0,2,0}},$$

$$\frac{d\bar{x}(t)}{dt}=-2a\kappa(t), \tag{53}$$

$$\frac{d\kappa(t)}{dt}=\frac{8\epsilon B^4(t)\left[\beta\kappa(t)I_{0,0,2}+m\mu A^{2m}(t)I_{0,2m,2}\right]}{I_{0,2,0}}, \tag{54}$$

$$\frac{dB(t)}{dt}=\frac{2B(t)}{A(t)}\left[\begin{array}{l}\frac{dA(t)}{dt}-2\epsilon A(t)\left(\sigma_1A^{2m}(t)\int_{-\infty}^{x}f^{2m}(s)ds-\beta\kappa^2(t)\right)\\ -\frac{2\epsilon\delta A^{2m+1}(t)I_{0,2m+2,0}}{I_{0,2,0}}\end{array}\right], \tag{55}$$

$$B^4(t)=\frac{\begin{bmatrix}3A^2(t)\left[b_1-4\epsilon(\xi-\eta-4\zeta)\kappa^2(t)\right]I_{0,4,0}\\ -12\epsilon\kappa(t)\left[\alpha+\sigma_2A^{2m}(t)\int_{-\infty}^{x}f^{2m}(s)ds\right]I_{0,2,0}\end{bmatrix}}{12\left[2aI_{0,0,2}+\epsilon(2\xi+\eta+2\zeta)A^2(t)I_{0,2,2}\right]}$$
$$+\frac{3b_3A^6(t)I_{0,8,0}}{8\left[2aI_{0,0,2}+\epsilon(2\xi+\eta+2\zeta)A^2(t)I_{0,2,2}\right]}$$
$$-\frac{4\epsilon\left[\beta\kappa(t)B(t)I_{2,0,2}+m\mu A^{2m}(t)B(t)I_{2,2m,2}\right]}{\left[2aI_{0,0,2}+\epsilon(2\xi+\eta+2\zeta)A^2(t)I_{0,2,2}\right]}$$
$$+\frac{b_2A^4(t)I_{0,6,0}-3\epsilon(\lambda+\sigma)\kappa(t)A^{2m}(t)I_{0,2m+2,0}}{3\left[2aI_{0,0,2}+\epsilon(2\xi+\eta+2\zeta)A^2(t)I_{0,2,2}\right]}. \tag{56}$$

2.4.1. Super-Gaussian pulses. For the super-Gaussian pulses, the dynamic system (52)-(56) reduce to:

$$\frac{d\theta_0(t)}{dt}=a\kappa^2(t)+\frac{2\epsilon\kappa(t)}{3}\left[\alpha+\sigma_2A^{2m}(t)(2m)^{-1-\frac{1}{2m}}\Gamma\left(\frac{1}{2m},2mx^{2m}\right)\right]$$
$$-\frac{b_2A^4(t)2^{1+\frac{1}{2m}}6^{-\frac{1}{2m}-1}}{3}+13am(2m-1)B^4(t)2^{\frac{1}{m}-1}\frac{\Gamma\left(1-\frac{1}{2m}\right)}{3\Gamma\left(\frac{1}{2m}\right)}$$
$$+\frac{2\epsilon\left[m(2m-1)(2\xi+\eta+2\zeta)A^2(t)B^4(t)2^{\frac{3}{2m}-3}\right]\Gamma\left(1-\frac{1}{2m}\right)}{3\Gamma\left(\frac{1}{2m}\right)}$$
$$+\frac{\epsilon\begin{bmatrix}8(2m+1)\beta\kappa(t)B(t)+2(\lambda+\sigma)\kappa(t)A^{2m}(t)(m+1)^{-\frac{1}{2m}}\\ +8m(2m+1)\mu A^{2m}(t)B(t)(m+1)^{-2-\frac{1}{2m}}\\ +(\xi-\eta-4\zeta)A^2(t)\kappa^2(t)2^{1-\frac{1}{2m}}\end{bmatrix}}{3}, \tag{57}$$

$$\frac{d\bar{x}(t)}{dt}=-2a\kappa(t), \tag{58}$$

$$\frac{d\kappa(t)}{dt}=\frac{\epsilon m(2m-1)B^4(t)2^{\frac{1}{m}+2}\left[\beta\kappa(t)+m\mu A^{2m}(t)(m+1)^{\frac{1}{2m}-1}\right]\Gamma\left(1-\frac{1}{2m}\right)}{\Gamma\left(\frac{1}{2m}\right)}, \tag{59}$$

$$\frac{dB(t)}{dt}=\frac{2B(t)}{A(t)}\begin{bmatrix}\frac{dA(t)}{dt}-2\epsilon A(t)\left[\sigma_1A^{2m}(t)(2m)^{-1-\frac{1}{2m}}\Gamma\left(\frac{1}{2m},2mx^{2m}\right)-\beta\kappa^2(t)\right]\\ -2\epsilon\delta A^{2m+1}(t)(m+1)^{-\frac{1}{2m}}\end{bmatrix}, \tag{60}$$

$$B^4(t)=\frac{\begin{bmatrix}3A^2(t)\left(b_1-4\epsilon(\xi-\eta-4\zeta)\kappa^2(t)\right)2^{-\frac{3}{2m}-2}\\ -3\epsilon\kappa(t)2^{-\frac{1}{m}}\left(\alpha+\sigma_2A^{2m}(t)(2m)^{-1-\frac{1}{2m}}\Gamma\left(\frac{1}{2m},2mx^{2m}\right)\right)\end{bmatrix}\Gamma\left(\frac{1}{2m}\right)}{3m\left[a(2m-1)+\epsilon(2\xi+\eta+2\zeta)A^2(t)2^{\frac{1}{2m}-3}(2m-1)\right]\Gamma\left(1-\frac{1}{2m}\right)}$$
$$+\frac{3b_3A^6(t)\left[8^{-\frac{1}{2m}}\Gamma\left(\frac{1}{2m}\right)\right]}{m(2m-1)\Gamma\left(1-\frac{1}{2m}\right)\left[a2^{3+\frac{1}{2m}}+2^{\frac{1}{m}}\epsilon(2\xi+\eta+2\zeta)A^2(t)\right]}\tag{61}$$
$$-\frac{\epsilon(2m+1)2^{-\frac{1}{2m}}\Gamma\left(\frac{1}{2m}\right)\left[\beta\kappa(t)B(t)+m\mu A^{2m}(t)B(t)(m+1)^{-2-\frac{1}{2m}}\right]}{m(2m-1)\Gamma\left(1-\frac{1}{2m}\right)\left[a2^{\frac{1}{2m}}+\epsilon(2\xi+\eta+2\zeta)A^2(t)2^{\frac{1}{m}-3}\right]}$$
$$+\frac{\Gamma\left(\frac{1}{2m}\right)\left[b_2A^4(t)6^{-\frac{1}{2m}}-3\epsilon(\lambda+\sigma)\kappa(t)A^{2m}(t)2^{-\frac{1}{2m}}(m+1)^{-\frac{1}{2m}}\right]}{3m(2m-1)\Gamma\left(1-\frac{1}{2m}\right)\left[a2^{\frac{1}{2m}}+\epsilon(2\xi+\eta+2\zeta)A^2(t)2^{\frac{1}{m}-3}\right]},$$

where $\Gamma(a,x)$ is the incomplete gamma function.

2.4.2. Super-sech pulses. For the super-sech pulses, the dynamic system (52)-(56) reduce to:

$$\frac{d\theta_0(t)}{dt}=a\kappa^2(t)+\frac{2\epsilon\kappa(t)}{3}\left[\alpha-\frac{\sigma_2A^{2m}(t)\mathrm{sech}^{4m^2}(x)}{4m^2}\left[{}_2F_1\left(\frac{1}{2},2m^2,1+2m^2,\mathrm{sech}^2(x)\right)\right]\right]$$
$$+\frac{\begin{bmatrix}2\epsilon\sqrt{\pi}A^2(t)(\xi-\eta-4\zeta)\kappa^2(t)\Gamma(4m)\Gamma\left(2m+\frac{1}{2}\right)\\ +26amB^4(t)\Gamma\left(4m+\frac{1}{2}\right)\Gamma\left(2m+\frac{1}{2}\right)\\ \times\left[{}_2F_1(1,-2m-1,2m+1,-1)-1-\frac{4m^2\sqrt{\pi}\Gamma(2m)}{(4m+1)\Gamma\left(2m+\frac{1}{2}\right)}\right]\end{bmatrix}}{3\sqrt{\pi}\Gamma(2m)\Gamma\left(4m+\frac{1}{2}\right)}\tag{62}$$
$$+\frac{2\epsilon\begin{bmatrix}m(2\xi+\eta+2\zeta)A^2(t)B^4(t)\left[{}_2F_1(1,-4m-1,4m+1,-1)-1\right]\\ \times\Gamma\left(2m+\frac{1}{2}\right)\Gamma\left(2m(m+1)+\frac{1}{2}\right)\\ +\sqrt{\pi}(1+4m)(\lambda+\sigma)\kappa(t)A^{2m}(t)\Gamma(2m(m+1))\Gamma\left(2m+\frac{1}{2}\right)\end{bmatrix}}{3(1+4m)\sqrt{\pi}\Gamma(2m)\Gamma\left(2m(m+1)+\frac{1}{2}\right)}$$
$$+\frac{2\epsilon m^2\beta\kappa(t)B(t)4^{2m+1}\Gamma\left(2m+\frac{1}{2}\right)}{3\sqrt{\pi}\Gamma(2m)}$$

$$\times\left\{\begin{array}{l}\dfrac{{}_4F_3\left(2m,2m,2m,2+4m;1+2m,1+2m,1+2m;-1\right)}{m^3}\\-\dfrac{16\left[{}_4F_3\left(1+2m,1+2m,1+2m,2+4m;2+2m,2+2m,2+2m;-1\right)\right]}{\left[1+2m\right]^3}\\+\dfrac{{}_4F_3\left(2+2m,2+2m,2+2m,2+4m;3+2m,3+2m,3+2m;-1\right)}{\left[1+m\right]^3}\end{array}\right\}$$

$$\times\left\{\begin{array}{l}\dfrac{{}_4F_3\left(\begin{array}{l}2(m+1)m,2(m+1)m,2(m+1)m,2+4(m+1)m;1\\+2(m+1)m,1+2(m+1)m,1+2(m+1)m;-1\end{array}\right)}{m^3(m+1)^3}\\-\dfrac{16\left[{}_4F_3\left(\begin{array}{l}1+2(m+1)m,1+2(m+1)m,1+2(m+1)m,2\\+4(m+1)m;2+2(m+1)m,2+2(m+1)m,2+2(m+1)m;-1\end{array}\right)\right]}{\left[1+2(m+1)m\right]^3}\\+\dfrac{{}_4F_3\left(\begin{array}{l}2+2(m+1)m,2+2(m+1)m,2+2(m+1)m,2\\+4(m+1)m;3+2(m+1)m,3+2(m+1)m,3+2(m+1)m;-1\end{array}\right)}{\left[1+(m+1)m\right]^3}\end{array}\right\}$$
$$+\frac{2\epsilon m^3\mu A^{2m}(t)B(t)\left[4^{2(m+1)m+1}\right]\Gamma\left(2m+\frac{1}{2}\right)}{3\sqrt{\pi}\,\Gamma(2m)}-\frac{b_2A^4(t)\Gamma(6m)\Gamma\left(2m+\frac{1}{2}\right)}{9\Gamma(2m)\Gamma\left(6m+\frac{1}{2}\right)}, \tag{62}$$

$$\frac{d\bar{x}(t)}{dt}=-2a\kappa(t), \tag{63}$$

$$\frac{d\kappa(t)}{dt}=\frac{8\epsilon mB^4(t)\left[\begin{array}{l}2\beta\kappa(t)\left[{}_2F_1(1,-2m-1,2m+1,-1)-1-\dfrac{4m^2\sqrt{\pi}\,\Gamma(2m)}{(4m+1)\Gamma\left(2m+\frac{1}{2}\right)}\right]\\\times\Gamma\left(2m+\frac{1}{2}\right)\Gamma\left(2m(m+1)+\frac{1}{2}\right)\\+\mu A^{2m}(t)\sqrt{\pi}\,\Gamma\left(2m(m+1)\right)\Gamma\left(2m+\frac{1}{2}\right)\end{array}\right]}{\sqrt{\pi}\,\Gamma(2m)\Gamma\left(2m(m+1)+\frac{1}{2}\right)}, \tag{64}$$

$$\frac{dB(t)}{dt}=\frac{2B(t)}{A(t)}\left[\begin{array}{l}\dfrac{dA(t)}{dt}+2\epsilon A(t)\\\times\left(\dfrac{\sigma_1A^{2m}(t)sech^{4m^2}(x)}{4m^2}\left[{}_2F_1\left(\frac{1}{2},2m^2,1+2m^2,sech^2(x)\right)\right]-\beta\kappa^2(t)\right)\end{array}\right]-\frac{4\epsilon\delta A^{2m}(t)B(t)\Gamma\left(2m+\frac{1}{2}\right)\Gamma\left(2m(m+1)\right)}{\Gamma(2m)\Gamma\left(2m(m+1)+\frac{1}{2}\right)}, \tag{65}$$

$$
\begin{aligned}
B^4(t) = {} & \frac{\left\{3A^2(t)\sqrt{\pi}(1+4m)\left[b_1 - 4\epsilon(\xi-\eta-4\zeta)\kappa^2(t)\right]\dfrac{\Gamma(4m)}{\Gamma\left(4m+\frac{1}{2}\right)}\right\}}{\left\{\begin{array}{l} 48am(1+4m)\left[{}_2F_1(1,-2m-1,2m+1,-1)-1-\dfrac{4m^2\sqrt{\pi}\Gamma(2m)}{(4m+1)\Gamma\left(2m+\frac{1}{2}\right)}\right] \\ +\epsilon m(2\xi+\eta+2\zeta)A^2(t)\left[{}_2F_1(1,-4m-1,4m+1,-1)-1\right] \end{array}\right\}} \\
& - \frac{\left\{12\epsilon\sqrt{\pi}\kappa(t)(1+4m)\left[\begin{array}{l}\alpha - \dfrac{\sigma_2 A^{2m}(t)\operatorname{sech}^{4m^2}(x)}{4m^2} \\ \times\left[{}_2F_1\left(\frac{1}{2},2m^2,1+2m^2,\operatorname{sech}^2(x)\right)\right]\end{array}\right]\dfrac{\Gamma(2m)}{\Gamma\left(2m+\frac{1}{2}\right)}\right\}}{\left\{\begin{array}{l} 48am(1+4m)\left[{}_2F_1(1,-2m-1,2m+1,-1)-1-\dfrac{4m^2\sqrt{\pi}\Gamma(2m)}{(4m+1)\Gamma\left(2m+\frac{1}{2}\right)}\right] \\ +\epsilon m(2\xi+\eta+2\zeta)A^2(t)\left[{}_2F_1(1,-4m-1,4m+1,-1)-1\right] \end{array}\right\}} \qquad (66) \\
& + \frac{3b_3A^6(t)\dfrac{\sqrt{\pi}\Gamma(8m)}{\Gamma\left(8m+\frac{1}{2}\right)}}{\left\{\begin{array}{l} 32am\left[{}_2F_1(1,-2m-1,2m+1,-1)-1-\dfrac{4m^2\sqrt{\pi}\Gamma(2m)}{(4m+1)\Gamma\left(2m+\frac{1}{2}\right)}\right] \\ +\dfrac{\epsilon m(2\xi+\eta+2\zeta)A^2(t)\left[{}_2F_1(1,-4m-1,4m+1,-1)-1\right]}{(1+4m)} \end{array}\right\}} \\
& - \frac{4\epsilon m^2\mu A^{2m}(t)B(t)4^{2(m+1)m-1}}{\left\{\begin{array}{l} 4a\left[{}_2F_1(1,-2m-1,2m+1,-1)-1-\dfrac{4m^2\sqrt{\pi}\Gamma(2m)}{(4m+1)\Gamma\left(2m+\frac{1}{2}\right)}\right] \\ +\dfrac{\epsilon(2\xi+\eta+2\zeta)A^2(t)\left[{}_2F_1(1,-4m-1,4m+1,-1)-1\right]}{1+4m} \end{array}\right\}} \\
& \times\left\{\begin{array}{l} \dfrac{{}_4F_3\left(\begin{array}{l}2(m+1)m,2(m+1)m,2(m+1)m,2+4(m+1)m;1 \\ +2(m+1)m,1+2(m+1)m,1+2(m+1)m;-1\end{array}\right)}{m^3(m+1)^3} \\ -\dfrac{16\left[{}_4F_3\left(\begin{array}{l}1+2(m+1)m,1+2(m+1)m,1+2(m+1)m,2 \\ +4(m+1)m;2+2(m+1)m,2+2(m+1)m,2+2(m+1)m;-1\end{array}\right)\right]}{\left[1+2(m+1)m\right]^3} \\ +\dfrac{{}_4F_3\left(\begin{array}{l}2+2(m+1)m,2+2(m+1)m,2+2(m+1)m,2 \\ +4(m+1)m;3+2(m+1)m,3+2(m+1)m,3+2(m+1)m;-1\end{array}\right)}{\left[1+(m+1)m\right]^3} \end{array}\right\}
\end{aligned}
$$

continued on next page

$$
+\frac{b_2 A^4(t)\frac{\sqrt{\pi}\Gamma(6m)}{\Gamma\left(6m+\frac{1}{2}\right)}-3\epsilon(\lambda+\sigma)\kappa(t)A^{2m}(t)\frac{\sqrt{\pi}\Gamma(2m(m+1))}{\Gamma\left(2m(m+1)+\frac{1}{2}\right)}}{\left\{12am\left[{}_2F_1(1,-2m-1,2m+1,-1)-1-\frac{4m^2\sqrt{\pi}\Gamma(2m)}{(4m+1)\Gamma\left(2m+\frac{1}{2}\right)}\right]+\frac{3\epsilon m(2\xi+\eta+2\zeta)A^2(t)\left[{}_2F_1(1,-4m-1,4m+1,-1)-1\right]}{1+4m}\right\}}
$$

$$
-\frac{4\epsilon m\beta\kappa(t)B(t)4^{2m-1}}{\left\{4a\left[{}_2F_1(1,-2m-1,2m+1,-1)-1-\frac{4m^2\sqrt{\pi}\Gamma(2m)}{(4m+1)\Gamma\left(2m+\frac{1}{2}\right)}\right]+\frac{\epsilon(2\xi+\eta+2\zeta)A^2(t)\left[{}_2F_1(1,-4m-1,4m+1,-1)-1\right]}{1+4m}\right\}}
$$

$$
\times\left\{\frac{{}_4F_3(2m,2m,2m,2+4m;1+2m,1+2m,1+2m;-1)}{m^3}-\frac{16\left[{}_4F_3(1+2m,1+2m,1+2m,2+4m;2+2m,2+2m,2+2m;-1)\right]}{[1+2m]^3}+\frac{{}_4F_3(2+2m,2+2m,2+2m,2+4m;3+2m,3+2m,3+2m;-1)}{[1+m]^3}\right\}. \tag{66*}
$$

3. Triple-power law nonlinearity

The governing model of such an equation is written as:

$$
iq_t+aq_{xx}+\left(b_1|q|^{2n}+b_2|q|^{2n+2}+b_3|q|^{2n+4}\right)q=0, \tag{67}
$$

where $q=q(x,t)$ is a complex-valued function representing the wave profile, n is the power-law nonlinearity parameter, and $i=\sqrt{-1}$. The first term in Eq. (67) is the linear temporal evolution. The constants a,b_1,b_2 and b_3 are the coefficients of CD and SPM, respectively.

3.1. Variational principle

The Lagrangian (L_g) is associated with Eq. (67) and is written as:

$$
L_g=\int_{-\infty}^{\infty}\left[\frac{i}{4}(q^*q_t-qq_t^*)-\frac{a}{2}|q_x|^2+\frac{b_1}{2(n+1)}|q|^{2n+2}+\frac{b_2}{2(n+2)}|q|^{2n+4}+\frac{b_3}{2(n+3)}|q|^{2n+6}\right]dx, \tag{68}
$$

where q^* is the complex conjugate of q and the conserved quantities are given by Eqs. (2,3), while, the Hamiltonian is given by:

$$
H=\int_{-\infty}^{\infty}\left[a|q_x|^2-\frac{b_1}{2(n+1)}|q|^{2n+2}-\frac{b_2}{2(n+2)}|q|^{2n+4}-\frac{b_3}{2(n+3)}|q|^{2n+6}\right]dx. \tag{69}
$$

Now, substituting (6) into (68) and using the formula (8); consequently, the Lagrangian (68) reduces to:

* continuation

$$L_g = -\frac{A^2(t)}{2B(t)}\left(\frac{d\theta_0(t)}{dt} + \kappa(t)\frac{d\bar{x}(t)}{dt} + a\kappa^2(t)\right)I_{0,2,0} - \frac{1}{2}aA^2(t)B^3(t)I_{0,0,2}$$
$$+\frac{b_1 A^{2n+2}(t)}{(2n+2)B(t)}I_{0,2n+2,0} + \frac{b_2 A^{2n+4}(t)}{(2n+4)B(t)}I_{0,2n+4,0} + \frac{b_3 A^{2n+6}(t)}{(2n+6)B(t)}I_{0,2n+6,0}. \quad (70)$$

For such a pulse form, given by (6), we have the integrals of motion as:

$$E = \frac{A^2(t)}{B(t)}I_{0,2,0}, \quad (71)$$

$$M = \frac{2A^2(t)\kappa(t)}{B(t)}I_{0,2,0}, \quad (72)$$

while the Hamiltonian is given by:

$$H = \frac{A^2(t)}{B(t)}\left[\begin{matrix} aB^4(t)I_{0,0,2} + a\kappa^2(t)I_{0,2,0} - \frac{b_1 A^{2n}(t)}{(2n+2)}I_{0,2n+2,0} \\ -\frac{b_2 A^{2n+2}(t)}{(2n+4)}I_{0,2n+4,0} - \frac{b_3 A^{2n+4}(t)}{(2n+6)}I_{0,2n+6,0} \end{matrix}\right] \quad (73)$$

3.2. Parameter dynamics

In this subsection, we derive the dynamic system by introducing the following EL equation:

$$\frac{\partial L_g}{\partial p} - \frac{d}{dt}\left(\frac{\partial L_g}{\partial p_t}\right) = 0, \quad (74)$$

where p is one of the five soliton parameters $A(t)$, $B(t)$, $\bar{x}(t)$, $\kappa(t)$ and $\theta_0(t)$, respectively. Substituting (70) into (74) to get the following dynamic system:

$$\left[\frac{d\theta_0(t)}{dt} + \kappa(t)\frac{d\bar{x}(t)}{dt} + a\kappa^2(t)\right]I_{0,2,0} + aB^4(t)I_{0,0,2}$$
$$-b_1 A^{2n}(t)I_{0,2n+2,0} - b_2 A^{2n+2}(t)I_{0,2n+4,0} - b_3 A^{2n+4}(t)I_{0,2n+6,0} = 0, \quad (75)$$

$$\left[\frac{d\theta_0(t)}{dt} + \kappa(t)\frac{d\bar{x}(t)}{dt} + a\kappa^2(t)\right]I_{0,2,0} - \frac{3}{2}aB^4(t)I_{0,0,2}$$
$$-\frac{b_1 A^{2n}(t)I_{0,2n+2,0}}{(2n+2)} - \frac{b_2 A^{2n+2}(t)I_{0,2n+4,0}}{(2n+4)} - \frac{b_3 A^{2n+4}(t)I_{0,2n+6,0}}{(2n+6)} = 0, \quad (76)$$

$$\frac{2}{A(t)}\frac{dA(t)}{dt} - \frac{1}{B(t)}\frac{dB(t)}{dt} + \frac{1}{\kappa(t)}\frac{d\kappa(t)}{dt} = 0, \quad (77)$$

$$d\bar{x}(t)/dt = -2a\kappa(t), \quad (78)$$

$$\frac{dA(t)}{dt} = \frac{A(t)}{2B(t)}\frac{dB(t)}{dt}. \quad (79)$$

The equations (75)–(79) represent the general forms of the soliton parameter dynamics of Eq. (67) for the pulse form given by (6). The dynamic system (75)–(79) can be reduced to become:

$$\frac{d\theta_0(t)}{dt} = \frac{\left(\begin{matrix} 5a\kappa^2(t)(n^3+6n^2+11n+6)I_{0,2,0} + (3n^3+19n^2+38n+24)b_1 A^{2n}(t)I_{0,2n+2,0} \\ +(3n^3+19n^2+37n+21)b_2 A^{2n+2}(t)I_{0,2n+4,0} \end{matrix}\right)}{5(n^3+6n^2+11n+6)I_{0,2,0}}$$
$$+\frac{(3n^3+19n^2+36n+20)b_3 A^{2n+4}(t)I_{0,2n+6,0}}{5(n^3+6n^2+11n+6)I_{0,2,0}}, \quad (80)$$

$$B(t)=\left[\frac{\begin{array}{l}(2n^3+11n^2+17n+6)b_1A^{2n}(t)I_{0,2n+2,0}\\+(2n^3+11n^2+18n+9)b_2A^{2n+2}(t)I_{0,2n+4,0}\\+(2n^3+11n^2+19n+10)b_3A^{2n+4}(t)I_{0,2n+6,0}\end{array}}{5a(n^3+6n^2+11n+6)I_{0,0,2}}\right]^{\frac{1}{4}} \tag{81}$$

$$\frac{d\bar{x}(t)}{dt}=-2a\kappa(t), \tag{82}$$

$$\frac{d\kappa(t)}{dt}=0, \tag{83}$$

$$A(t)=K\sqrt{B(t)}, \tag{84}$$

where the constant K is proportional to the square roots of the energy.

3.2.1. Super-Gaussian pulses. For super-Gaussian pulses, we set $f(s)=e^{-s^{2m}}, \quad m>0$. Then, the integrals of motion are given by:

$$E=\frac{A^2(t)}{mB(t)}2^{-\frac{1}{2m}}\Gamma\left(\frac{1}{2m}\right), \tag{85}$$

$$M=\frac{A^2(t)\kappa(t)}{mB(t)}2^{(2m-1)/2m}\Gamma\left(\frac{1}{2m}\right), \tag{86}$$

and the Hamiltonian is given by:

$$H=\frac{A^2(t)}{mB(t)}\left[\begin{array}{l}\frac{amB^4(t)(2m-1)\Gamma\left(1-\frac{1}{2m}\right)}{2^{(2m-1)/2m}}+\frac{a\kappa^2(t)\Gamma\left(\frac{1}{2m}\right)}{2^{1/2m}}\\-\frac{b_1A^{2n}(t)\Gamma\left(\frac{1}{2m}\right)}{(2n+2)^{(2m+1)/2m}}-\frac{b_2A^{2n+2}(t)\Gamma\left(\frac{1}{2m}\right)}{(2n+4)^{(2m+1)/2m}}-\frac{b_3A^{2n+4}(t)\Gamma\left(\frac{1}{2m}\right)}{(2n+6)^{(2m+1)/2m}}\end{array}\right], \tag{87}$$

Also, we have the evolution equations for the pulse parameters (80)–(84), which reduce to:

$$\frac{d\theta_0(t)}{dt}=\frac{1}{5(n^3+6n^2+11n+6)}\times\left[\begin{array}{l}5a\kappa^2(t)(n^3+6n^2+11n+6)\\+(3n^3+19n^2+38n+24)b_1A^{2n}(t)(n+1)^{-\frac{1}{2m}}\\+(3n^3+19n^2+37n+21)b_2A^{2n+2}(t)(n+2)^{-\frac{1}{2m}}\\+(3n^3+19n^2+36n+20)b_3A^{2n+4}(t)(n+3)^{-\frac{1}{2m}}\end{array}\right], \tag{88}$$

$$B^4(t)=\frac{A^{2n}(t)\Gamma\left(\frac{1}{2m}\right)}{5am(2m-1)(n^3+6n^2+11n+6)2^{\frac{1}{m}-1}\Gamma\left(1-\frac{1}{2m}\right)}\times\left[\begin{array}{l}\frac{(2n^3+11n^2+17n+6)b_1}{(n+1)^{1/2m}}\\+\frac{(2n^3+11n^2+18n+9)b_2A^2(t)}{(n+2)^{1/2m}}\\+\frac{(2n^3+11n^2+19n+10)b_3A^4(t)}{(n+3)^{1/2m}}\end{array}\right], \tag{89}$$

$$\frac{d\bar{x}(t)}{dt}=-2a\kappa(t), \tag{90}$$

$$\frac{d\kappa(t)}{dt}=0, \tag{91}$$

$$A(t)=K\sqrt{B(t)}. \tag{92}$$

3.2.2. Super-sech pulses. For super-sech pulses, we set $f(s)=sech^{2m}(s)$, $m>0$. Then, the integrals of motion are given by:

$$E=\frac{\sqrt{\pi}A^2(t)}{B(t)}\frac{\Gamma(2m)}{\Gamma\left(2m+\frac{1}{2}\right)}, \tag{93}$$

$$M=\frac{2\sqrt{\pi}A^2(t)\kappa(t)}{B(t)}\frac{\Gamma(2m)}{\Gamma\left(2m+\frac{1}{2}\right)}, \tag{94}$$

and the Hamiltonian is given by:

$$\begin{aligned}H=\frac{A^2(t)}{B(t)}\Bigg[&2amB^4(t)\\ &\times\left[{}_2F_1(1,-2m-1,2m+1,-1)-1-\frac{4m^2\sqrt{\pi}\Gamma(2m)}{(4m+1)\Gamma\left(2m+\frac{1}{2}\right)}\right]\\ &+\frac{a\kappa^2(t)\sqrt{\pi}\Gamma(2m)}{\Gamma\left(2m+\frac{1}{2}\right)}\\ &-\frac{b_1A^{2n}(t)}{(2n+2)}\frac{\sqrt{\pi}\Gamma(2m(n+1))}{\Gamma\left(2m(n+1)+\frac{1}{2}\right)}\\ &-\frac{b_2A^{2n+2}(t)}{(2n+4)}\frac{\sqrt{\pi}\Gamma(2m(n+2))}{\Gamma\left(2m(n+2)+\frac{1}{2}\right)}\\ &-\frac{b_3A^{2n+4}(t)}{(2n+6)}\frac{\sqrt{\pi}\Gamma(2m(n+3))}{\Gamma\left(2m(n+3)+\frac{1}{2}\right)}\Bigg],\end{aligned} \tag{95}$$

Also, we have the evolution equations for the pulse parameters (80)–(84), which reduce to:

$$\frac{d\theta_0(t)}{dt}=\frac{\Gamma\left(2m+\frac{1}{2}\right)}{5(n^3+6n^2+11n+6)\Gamma(2m)}\left[\begin{aligned}&\frac{5a\kappa^2(t)(n^3+6n^2+11n+6)\Gamma(2m)}{\Gamma\left(2m+\frac{1}{2}\right)}\\ &+\frac{(3n^3+19n^2+38n+24)b_1A^{2n}(t)\Gamma(2m(n+1))}{\Gamma\left(2m(n+1)+\frac{1}{2}\right)}\\ &+\frac{(3n^3+19n^2+37n+21)b_2A^{2n+2}(t)\Gamma(2m(n+2))}{\Gamma\left(2m(n+2)+\frac{1}{2}\right)}\\ &+\frac{(3n^3+19n^2+36n+20)b_3A^{2n+4}(t)\Gamma(2m(n+3))}{\Gamma\left(2m(n+3)+\frac{1}{2}\right)}\end{aligned}\right], \tag{96}$$

$$B^4(t)=\frac{A^{2n}(t)\sqrt{\pi}}{10am\left(n^3+6n^2+11n+6\right)\left[{}_2F_1\left(1,-2m-1,2m+1,-1\right)-1-\dfrac{4m^2\sqrt{\pi}\Gamma(2m)}{(4m+1)\Gamma\left(2m+\frac{1}{2}\right)}\right]}$$
$$\times\left[\begin{array}{l}\dfrac{\left(2n^3+11n^2+17n+6\right)b_1\Gamma\left(2m(n+1)\right)}{\Gamma\left(2m(n+1)+\frac{1}{2}\right)}+\dfrac{\left(2n^3+11n^2+18n+9\right)b_2A^2(t)\Gamma\left(2m(n+2)\right)}{\Gamma\left(2m(n+2)+\frac{1}{2}\right)}\\+\dfrac{\left(2n^3+11n^2+19n+10\right)b_3A^4(t)\Gamma\left(2m(n+3)\right)}{\Gamma\left(2m(n+3)+\frac{1}{2}\right)}\end{array}\right], \quad (97)$$

$$\frac{d\bar{x}(t)}{dt}=-2a\kappa(t), \quad (98)$$

$$\frac{d\kappa(t)}{dt}=0, \quad (99)$$

$$A(t)=K\sqrt{B(t)}. \quad (100)$$

3.3. Perturbed NLSE

The governing model of such an equation is written as:

$$iq_t+aq_{xx}+\left(b_1|q|^{2n}+b_2|q|^{2n+2}+b_3|q|^{2n+4}\right)q=i\epsilon R[q,q^*], \quad (101)$$

where $R[q,q^*]$ is given by:

$$\begin{aligned}R&=\delta|q|^{2m}q+\alpha q_x+\beta q_{xx}+\lambda\left(|q|^{2m}q\right)_x+\theta\left(|q|^{2m}\right)_x q+\sigma|q|^{2m}q_x\\&-i\xi\left(q^2q_x^*\right)_x-i\eta q_x^2q^*-i\zeta q^*\left(q^2\right)_{xx}-i\mu\left(|q|^{2m}\right)_x q+\left(\sigma_1q+\sigma_2q_x\right)\int_{-\infty}^{x}|q|^{2m}ds,\end{aligned} \quad (102)$$

and $\epsilon,\delta,\alpha,\beta,\lambda,\theta,\sigma,\xi,\eta,\zeta,\mu,$ σ_1 and σ_2 are constants. From (6) and (101), we have:

$$\begin{aligned}R=A(t)\Bigg\{&\delta A^{2m}(t)f^{2m+1}(s)+\alpha B^2(t)\frac{df(s)}{ds}\\&+\left[(2m+1)\lambda+2m\theta+\sigma\right]A^{2m}(t)B^2(t)f^{2m}(s)\frac{df(s)}{ds}\\&-\beta\kappa^2(t)f(s)+2(\xi-\eta-4\zeta)A^2(t)B^2(t)\kappa(t)f^2(s)\frac{df(s)}{ds}\\&+A^{2m}(t)\left[\sigma_1f(s)+\sigma_2B^2(t)\frac{df(s)}{ds}\right]\int_{-\infty}^{x}f^{2m}(s)ds\\&-i\left[\begin{array}{l}\alpha\kappa(t)f(s)+2\beta\kappa(t)B^2(t)\dfrac{df(s)}{ds}\\+(\lambda+\sigma)A^{2m}(t)\kappa(t)f^{2m+1}(s)+(2\xi+\eta+2\zeta)A^2(t)B^4(t)f(s)\left(\dfrac{df(s)}{ds}\right)^2\\+(\xi-4\zeta-\eta)A^2(t)\kappa^2(t)f^3(s)\\+2m\mu A^{2m}(t)B^2(t)f^{2m}(s)\dfrac{df(s)}{ds}+\sigma_2A^{2m}(t)\kappa(t)f(s)\displaystyle\int_{-\infty}^{x}f^{2m}(s)ds\end{array}\right]\Bigg\}\\&\times\exp\left[-i\frac{\kappa(t)}{B(t)}s+i\theta_0(t)\right].\end{aligned} \quad (103)$$

3.4. Parametr dynamics

In this subsection, we derive the dynamic system of Eq. (43) by introducing the following EL equation:

$$\frac{\partial L_g}{\partial p}-\frac{d}{dt}\left(\frac{\partial L_g}{\partial p_t}\right)=i\epsilon\int_{-\infty}^{\infty}\left(R\frac{\partial q^*}{\partial p}-R^*\frac{\partial q}{\partial p}\right)dx \tag{104}$$

where L_g is given by (70) and p is one of these same five parameters $A(t)$, $B(t)$, $\bar{x}(t)$, $\kappa(t)$ and $\theta_0(t)$, respectively, while R^* is the complex conjugate of R. Now, we have the following dynamic system:

$$\begin{aligned}&\left[\frac{d\theta_0(t)}{dt}+\kappa(t)\frac{d\bar{x}(t)}{dt}+a\kappa^2(t)\right]I_{0,2,0}+aB^4(t)I_{0,0,2}-b_1A^{2n}(t)I_{0,2n+2,0}\\&-b_2A^{2n+2}(t)I_{0,2n+4,0}-b_3A^{2n+4}(t)I_{0,2n+6,0}\\&=2\epsilon\alpha\kappa(t)I_{0,2,0}+2\epsilon\sigma_2\kappa(t)A^{2m}(t)I_{0,2,0}\int_{-\infty}^{x}f^{2m}(s)ds\\&+2\epsilon(\lambda+\sigma)A^{2m}(t)\kappa(t)I_{0,2m+2,0}\\&+2\epsilon A^2(t)B^4(t)(2\xi+\eta+2\zeta)I_{0,2,2}+2\epsilon\kappa^2(t)A^2(t)(\xi-\eta-4\zeta)I_{0,4,0},\end{aligned} \tag{105}$$

$$\begin{aligned}&\left[\frac{d\theta_0(t)}{dt}+\kappa(t)\frac{d\bar{x}(t)}{dt}+a\kappa^2(t)\right]I_{0,2,0}-\frac{3}{2}aB^4(t)I_{0,0,2}\\&-\frac{b_1A^{2n}(t)I_{0,2n+2,0}}{(2n+2)}-\frac{b_2A^{2n+2}(t)I_{0,2n+4,0}}{(2n+4)}-\frac{b_3A^{2n+4}(t)I_{0,2n+6,0}}{(2n+6)}\\&=4\epsilon\beta\kappa(t)B(t)I_{2,0,2}+4\epsilon m\mu A^{2m}(t)B(t)I_{2,2m,2},\end{aligned} \tag{106}$$

$$\begin{aligned}&\left[\frac{A(t)\kappa(t)}{B(t)}\frac{dA(t)}{dt}+\frac{A^2(t)}{2B(t)}\frac{d\kappa(t)}{dt}-\frac{A^2(t)\kappa(t)}{2B^2(t)}\frac{dB(t)}{dt}\right]I_{0,2,0}\\&=4\epsilon\beta A^2(t)B^3(t)\kappa(t)I_{0,0,2}+\frac{2\epsilon\delta A^{2m+2}(t)\kappa(t)I_{0,2m+2,0}}{B(t)}\\&-\frac{2\epsilon\beta A^2(t)\kappa^3(t)I_{0,2,0}}{B(t)}+\frac{2\epsilon\sigma_1A^{2m+2}(t)\kappa(t)I_{0,2,0}\int_{-\infty}^{x}f^{2m}(s)ds}{B(t)},\end{aligned} \tag{107}$$

$$-\frac{A^2(t)}{2B(t)}\left[\frac{d\bar{x}(t)}{dt}+2a\kappa(t)\right]I_{0,2,0}=0, \tag{108}$$

$$\begin{aligned}&\left[\frac{A(t)}{B(t)}\frac{dA(t)}{dt}-\frac{A^2(t)}{2B^2(t)}\frac{dB(t)}{dt}\right]I_{0,2,0}\\&=\frac{2\epsilon\delta A^{2m+2}(t)I_{0,2m+2,0}}{B(t)}-\frac{2\epsilon\beta A^2(t)\kappa^2(t)I_{0,2,0}}{B(t)}+\frac{2\epsilon\sigma_1A^{2m+2}(t)I_{0,2,0}\int_{-\infty}^{x}f^{2m}(s)ds}{B(t)}.\end{aligned} \tag{109}$$

Eqs. (105)–(109) represent the general forms of the soliton parameters dynamics of Eq. (101) for the pulse form given by (6). The dynamic system (105)–(109) can be reduced to become:

$$\frac{d\theta_0(t)}{dt}=\frac{\begin{bmatrix}6a\epsilon\left[\alpha\kappa(t)+\sigma_2\kappa(t)A^{2m}(t)\int_{-\infty}^{x}f^{2m}(s)ds\right]I_{0,0,2}\\+a\kappa^2(t)\left[5aI_{0,0,2}-4\epsilon A^2(t)(2\xi+\eta+2\zeta)I_{0,2,2}\right]\end{bmatrix}}{\left[5aI_{0,0,2}-4\epsilon A^2(t)(2\xi+\eta+2\zeta)I_{0,2,2}\right]}$$

$$+\frac{4\epsilon B(t)\left[\beta\kappa(t)I_{2,0,2}+m\mu A^{2m}(t)I_{2,2m,2}\right]}{I_{0,2,0}}$$

$$+\frac{b_1A^{2n}(t)I_{0,2n+2,0}\left[a(3n+4)I_{0,0,2}-2\epsilon A^2(t)(2\xi+\eta+2\zeta)I_{0,2,2}\right]}{(n+1)\left[5aI_{0,0,2}-4\epsilon A^2(t)(2\xi+\eta+2\zeta)I_{0,2,2}\right]I_{0,2,0}} \tag{110}$$

$$+\frac{b_2A^{2n+2}(t)I_{0,2n+4,0}\left[a(3n+7)I_{0,0,2}-2\epsilon A^2(t)(2\xi+\eta+2\zeta)I_{0,2,2}\right]}{(n+2)\left[5aI_{0,0,2}-4\epsilon A^2(t)(2\xi+\eta+2\zeta)I_{0,2,2}\right]I_{0,2,0}}$$

$$+\frac{b_3A^{2n+4}(t)I_{0,2n+6,0}\left[3a(2n+5)I_{0,0,2}+5aI_{0,0,2}-4\epsilon A^2(t)(2\xi+\eta+2\zeta)I_{0,2,2}\right]}{2(n+3)\left[5aI_{0,0,2}-4\epsilon A^2(t)(2\xi+\eta+2\zeta)I_{0,2,2}\right]I_{0,2,0}},$$

$$\frac{d\bar{x}(t)}{dt}=-2a\kappa(t), \tag{111}$$

$$\frac{d\kappa(t)}{dt}=\frac{8\epsilon\left[\beta\kappa(t)I_{0,0,2}+m\mu A^{2m}(t)I_{0,2m,2}\right]B^4(t)}{I_{0,2,0}}, \tag{112}$$

$$\frac{dB(t)}{dt}=-\frac{2B(t)}{A(t)}\frac{dA(t)}{dt}+4\epsilon\beta B(t)\kappa^2(t)$$
$$-\frac{4\epsilon\delta A^{2m}(t)B(t)I_{0,2m+2,0}}{I_{0,2,0}}-4\epsilon\sigma_1A^{2m}(t)B(t)\int_{-\infty}^{x}f^{2m}(s)ds \tag{113}$$

$$B^4(t)=\frac{2\epsilon\left[\alpha\kappa(t)+\sigma_2\kappa(t)A^{2m}(t)\int_{-\infty}^{x}f^{2m}(s)ds\right]I_{0,2,0}}{\left[\frac{5}{2}aI_{0,0,2}-2\epsilon A^2(t)(2\xi+\eta+2\zeta)I_{0,2,2}\right]}$$

$$+\frac{(2n+1)b_1A^{2n}(t)I_{0,2n+2,0}}{(2n+2)\left[\frac{5}{2}aI_{0,0,2}-2\epsilon A^2(t)(2\xi+\eta+2\zeta)I_{0,2,2}\right]}$$

$$+\frac{(2n+3)b_2A^{2n+2}(t)I_{0,2n+4,0}}{(2n+4)\left[\frac{5}{2}aI_{0,0,2}-2\epsilon A^2(t)(2\xi+\eta+2\zeta)I_{0,2,2}\right]} \tag{114}$$

$$+\frac{(2n+5)b_3A^{2n+4}(t)I_{0,2n+6,0}}{(2n+6)\left[\frac{5}{2}aI_{0,0,2}-2\epsilon A^2(t)(2\xi+\eta+2\zeta)I_{0,2,2}\right]}$$

$$+\frac{2\epsilon(\lambda+\sigma)A^{2m}(t)\kappa(t)I_{0,2m+2,0}+2\epsilon\kappa^2(t)A^2(t)(\xi-\eta-4\zeta)I_{0,4,0}}{\left[\frac{5}{2}aI_{0,0,2}-2\epsilon A^2(t)(2\xi+\eta+2\zeta)I_{0,2,2}\right]}$$

$$-\frac{4\epsilon\beta\kappa(t)B(t)I_{2,0,2}-4\epsilon m\mu A^{2m}(t)B(t)I_{2,2m,2}}{\left[\frac{5}{2}aI_{0,0,2}-2\epsilon A^2(t)(2\xi+\eta+2\zeta)I_{0,2,2}\right]}.$$

3.4.1. Super-Gaussian pulses. For the super-Gaussian pulses, the dynamic system (110)-(114) reduce to:

$$\frac{d\theta_0(t)}{dt}=\frac{\left\{\begin{array}{l}6a\epsilon\left[\alpha\kappa(t)+\sigma_2\kappa(t)A^{2m}(t)(2m)^{-(2m+1)/2m}\Gamma\left(\frac{1}{2m},2mx^{2m}\right)\right]2^{(1-2m)/2m}\\+a\kappa^2(t)\left[5a2^{(1-2m)/2m}-\epsilon A^2(t)(2\xi+\eta+2\zeta)2^{(1-m)/m}\right]\end{array}\right\}}{\left[5a2^{(1-2m)/2m}-\epsilon A^2(t)(2\xi+\eta+2\zeta)2^{(1-m)/m}\right]}$$
$$+\epsilon(2m+1)B(t)\left[\beta\kappa(t)+\frac{m\mu A^{2m}(t)}{(m+1)^{(1+4m)/2m}}\right]$$
$$+\frac{b_1A^{2n}(t)\left[a(3n+4)-\epsilon A^2(t)(2\xi+\eta+2\zeta)2^{(1-2m)2m}\right]}{(n+1)^{(1+2m)/2m}\left[5a-\epsilon A^2(t)(2\xi+\eta+2\zeta)2^{(1+4m)/2m}\right]}$$
$$+\frac{b_2A^{2n+2}(t)\left[a(3n+7)-\epsilon A^2(t)(2\xi+\eta+2\zeta)2^{(1-2m)/2m}\right]}{(n+2)^{(2m+1)/2m}\left[5a-\epsilon A^2(t)(2\xi+\eta+2\zeta)2^{1/2m}\right]}$$
$$+\frac{b_3A^{2n+4}(t)\left[(3n+10)a-4\epsilon A^2(t)(2\xi+\eta+2\zeta)2^{(1-2m)/2m}\right]}{(n+3)^{(1-2m)/2m}\left[5a-\epsilon A^2(t)(2\xi+\eta+2\zeta)2^{1/2m}\right]}, \quad (115)$$

$$\frac{d\bar{x}(t)}{dt}=-2a\kappa(t), \quad (116)$$

$$\frac{d\kappa(t)}{dt}=\frac{m(2m-1)\epsilon 2^{(1+2m)/m}\left[\beta\kappa(t)+m\mu A^{2m}(t)(m+1)^{\frac{1}{2m}-1}\right]\Gamma\left(1-\frac{1}{2m}\right)B^4(t)}{\Gamma\left(\frac{1}{2m}\right)}, \quad (117)$$

$$\frac{dB(t)}{dt}=-\frac{2B(t)}{A(t)}\frac{dA(t)}{dt}+4\epsilon\beta B(t)\kappa^2(t)$$
$$-\frac{4\epsilon\delta A^{2m}(t)B(t)}{(m+1)^{1/2m}}-\frac{4\epsilon\sigma_1A^{2m}(t)B(t)\Gamma\left(\frac{1}{2m},2mx^{2m}\right)}{(2m)^{(1+2m)/2m}}, \quad (118)$$

$$B^4(t)=\frac{2\epsilon\left[\alpha\kappa(t)+\sigma_2\kappa(t)A^{2m}(t)(2m)^{-\frac{(2m+1)}{2m}}\Gamma\left(\frac{1}{2m},2mx^{2m}\right)\right]\Gamma\left(\frac{1}{2m}\right)}{m(2m-1)\Gamma\left(1-\frac{1}{2m}\right)\left[5a2^{\frac{(1-2m)}{m}}-\epsilon A^2(t)(2\xi+\eta+2\zeta)2^{\frac{(3-4m)}{2m}}\right]}$$
$$+\frac{(2n+1)b_1A^{2n}(t)\Gamma\left(\frac{1}{2m}\right)}{m(2m-1)(n+1)^{\frac{(1+2m)}{2m}}\left[5a2^{\frac{(1-m)}{m}}-\epsilon A^2(t)(2\xi+\eta+2\zeta)2^{\frac{(3-2m)}{2m}}\right]\Gamma\left(1-\frac{1}{2m}\right)}$$
$$+\frac{(2n+3)b_2A^{2n+2}(t)\Gamma\left(\frac{1}{2m}\right)}{m(2m-1)(n+2)^{\frac{(2m+1)}{2m}}\left[5a2^{\frac{(1-m)}{m}}-\epsilon A^2(t)(2\xi+\eta+2\zeta)2^{\frac{(3-2m)}{2m}}\right]\Gamma\left(1-\frac{1}{2m}\right)} \quad (119)$$

continued on next page

$$+\frac{(2n+5)b_3A^{2n+4}(t)\Gamma\left(\frac{1}{2m}\right)}{m(2m-1)(n+3)^{\frac{(2m+1)}{2m}}\left[5a2^{\frac{(1-m)}{m}}-\epsilon A^2(t)(2\xi+\eta+2\zeta)2^{\frac{(3-2m)}{2m}}\right]\Gamma\left(1-\frac{1}{2m}\right)}$$
$$+\frac{4\epsilon\left[(\lambda+\sigma)A^{2m}(t)\kappa(t)2^{-\frac{1}{2m}}(m+1)^{-1/2m}+\kappa^2(t)A^2(t)(\xi-\eta-4\zeta)2^{-1/m}\right]\Gamma\left(\frac{1}{2m}\right)}{m(2m-1)\left[5a2^{\frac{(1-2m)}{2m}}-\epsilon A^2(t)(2\xi+\eta+2\zeta)2^{\frac{(1-m)}{m}}\right]\Gamma\left(1-\frac{1}{2m}\right)} \quad (119^*)$$
$$-\frac{\epsilon(2m+1)\left[\beta\kappa(t)B(t)-m\mu A^{2m}(t)B(t)(m+1)^{-(1+4m)/2m}\right]\Gamma\left(\frac{1}{2m}\right)}{m(2m-1)\left[5a2^{\frac{(1-2m)}{m}}-4\epsilon A^2(t)(2\xi+\eta+2\zeta)2^{\frac{(3-8m)}{2m}}\right]\Gamma\left(1-\frac{1}{2m}\right)},$$

where $\Gamma(a,x)$ is the incomplete gamma function.

3.4.2. Super-sech pulses. For the super-sech pulses, the dynamic system (110)-(114) reduce to:

$$\frac{d\theta_0(t)}{dt}=\frac{\left\{\begin{array}{l}6a\epsilon\left[\alpha\kappa(t)+-\frac{\sigma_2\kappa(t)A^{2m}(t)\mathrm{sech}^{4m^2}(x)}{2m}\left[{}_2F_1\left(\frac{1}{2},2m^2,1+2m^2,\mathrm{sech}^2(x)\right)\right]\right]\\ \times\left[{}_2F_1(1,-2m-1,2m+1,-1)-1-\frac{4m^2\sqrt{\pi}\Gamma(2m)}{(4m+1)\Gamma\left(2m+\frac{1}{2}\right)}\right]\end{array}\right\}}{\left[\begin{array}{l}10am\left[{}_2F_1(1,-2m-1,2m+1,-1)-1-\frac{4m^2\sqrt{\pi}\Gamma(2m)}{(4m+1)\Gamma\left(2m+\frac{1}{2}\right)}\right]\\ -\frac{4m\epsilon A^2(t)(2\xi+\eta+2\zeta)\left[{}_2F_1(1,-4m-1,4m+1,-1)-1\right]}{(1+4m)}\end{array}\right]}$$
$$+\frac{a\kappa^2(t)\left[\begin{array}{l}10am\left[{}_2F_1(1,-2m-1,2m+1,-1)-1-\frac{4m^2\sqrt{\pi}\Gamma(2m)}{(4m+1)\Gamma\left(2m+\frac{1}{2}\right)}\right]\\ -\frac{4m\epsilon A^2(t)(2\xi+\eta+2\zeta)\left[{}_2F_1(1,-4m-1,4m+1,-1)-1\right]}{(1+4m)}\end{array}\right]}{\left\{\begin{array}{l}10am\left[{}_2F_1(1,-2m-1,2m+1,-1)-1-\frac{4m^2\sqrt{\pi}\Gamma(2m)}{(4m+1)\Gamma\left(2m+\frac{1}{2}\right)}\right]\\ -\frac{4m\epsilon A^2(t)(2\xi+\eta+2\zeta)\left[{}_2F_1(1,-4m-1,4m+1,-1)-1\right]}{(1+4m)}\end{array}\right\}} \quad (120)$$
$$\times\frac{4^{2m}m^2\epsilon\Gamma\left(2m+\frac{1}{2}\right)B(t)\beta\kappa(t)}{\sqrt{\pi}\Gamma(2m)}$$
$$\times\left\{\begin{array}{l}\frac{{}_4F_3(2m,2m,2m,2+4m;1+2m,1+2m,1+2m;-1)}{m^3}\\ -\frac{16\left[{}_4F_3(1+2m,1+2m,1+2m,2+4m;2+2m,2+2m,2+2m;-1)\right]}{[1+2m]^3}\\ +\frac{{}_4F_3(2+2m,2+2m,2+2m,2+4m;3+2m,3+2m,3+2m;-1)}{[1+m]^3}\end{array}\right\}$$

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$$+\frac{4\epsilon m^3\mu\Gamma\left(2m+\frac{1}{2}\right)B(t)A^{2m}(t)4^{2(m+1)m-1}}{\sqrt{\pi}\Gamma(2m)}$$

$$\times\left\{\begin{aligned}&\frac{{}_4F_3\left(\begin{aligned}&2(m+1)m,2(m+1)m,2(m+1)m,2+4(m+1)m;\\&1+2(m+1)m,1+2(m+1)m,1+2(m+1)m;\\&-1\end{aligned}\right)}{m^3(m+1)^3}\\&-\frac{16\left[{}_4F_3\left(\begin{aligned}&1+2(m+1)m,1+2(m+1)m,1+2(m+1)m,2+4(m+1)m;\\&2+2(m+1)m,2+2(m+1)m,2+2(m+1)m;\\&-1\end{aligned}\right)\right]}{\left[1+2(m+1)m\right]^3}\\&+\frac{{}_4F_3\left(\begin{aligned}&2+2(m+1)m,2+2(m+1)m,2+2(m+1)m,2+4(m+1)m;\\&3+2(m+1)m,3+2(m+1)m,3+2(m+1)m;\\&-1\end{aligned}\right)}{\left[1+(m+1)m\right]^3}\end{aligned}\right\}$$

$$+\frac{\left\{\begin{aligned}&\frac{2am(3n+4)b_1A^{2n}(t)\sqrt{\pi}\Gamma(2m(n+1))\Gamma\left(2m+\frac{1}{2}\right)}{\Gamma\left(2m(n+1)+\frac{1}{2}\right)}\\&\left[{}_2F_1(1,-2m-1,2m+1,-1)-1-\frac{4m^2\sqrt{\pi}\Gamma(2m)}{(4m+1)\Gamma\left(2m+\frac{1}{2}\right)}\right]\end{aligned}\right\}}{\sqrt{\pi}\Gamma(2m)(n+1)\left[\begin{aligned}&10am\left[{}_2F_1(1,-2m-1,2m+1,-1)-1-\frac{4m^2\sqrt{\pi}\Gamma(2m)}{(4m+1)\Gamma\left(2m+\frac{1}{2}\right)}\right]\\&-4\epsilon A^2(t)(2\xi+\eta+2\zeta)\frac{m}{1+4m}\left[{}_2F_1(1,-4m-1,4m+1,-1)-1\right]\end{aligned}\right]} \tag{120*}$$

$$-\frac{\left\{\begin{aligned}&2m\epsilon A^2(t)(2\xi+\eta+2\zeta)\\&\times\left[{}_2F_1(1,-4m-1,4m+1,-1)-1\right]\\&\times\frac{b_1A^{2n}(t)\sqrt{\pi}\Gamma(2m(n+1))\Gamma\left(2m+\frac{1}{2}\right)}{\Gamma\left(2m(n+1)+\frac{1}{2}\right)}\end{aligned}\right\}}{\left\{\begin{aligned}&(n+1)(1+4m)\sqrt{\pi}\Gamma(2m)\\&\times\left[\begin{aligned}&10am\left[{}_2F_1(1,-2m-1,2m+1,-1)-1-\frac{4m^2\sqrt{\pi}\Gamma(2m)}{(4m+1)\Gamma\left(2m+\frac{1}{2}\right)}\right]\\&-4\epsilon A^2(t)(2\xi+\eta+2\zeta)\\&\times\frac{m}{1+4m}\left[{}_2F_1(1,-4m-1,4m+1,-1)-1\right]\end{aligned}\right]\end{aligned}\right\}}$$

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$$
+\frac{\left\{\begin{array}{l}\dfrac{b_2A^{2n+2}(t)\Gamma\left(2m(n+2)\right)}{\Gamma\left(2m(n+2)+\frac{1}{2}\right)}\\ \times\left[\begin{array}{l}2am(3n+7)\left[{}_2F_1(1,-2m-1,2m+1,-1)-1-\dfrac{4m^2\sqrt{\pi}\Gamma(2m)}{(4m+1)\Gamma\left(2m+\frac{1}{2}\right)}\right]\\ -\dfrac{2m\epsilon A^2(t)(2\xi+\eta+2\zeta)\left[{}_2F_1(1,-4m-1,4m+1,-1)-1\right]}{(1+4m)}\end{array}\right]\end{array}\right\}}{\left\{(n+2)\left[\begin{array}{l}10am\left[{}_2F_1(1,-2m-1,2m+1,-1)-1-\dfrac{4m^2\sqrt{\pi}\Gamma(2m)}{(4m+1)\Gamma\left(2m+\frac{1}{2}\right)}\right]\\ -4\epsilon A^2(t)(2\xi+\eta+2\zeta)\dfrac{m}{1+4m}\left[{}_2F_1(1,-4m-1,4m+1,-1)-1\right]\end{array}\right]\dfrac{\Gamma(2m)}{\Gamma\left(2m+\frac{1}{2}\right)}\right\}}
$$

$$
+\frac{\left\{\begin{array}{l}\dfrac{b_3A^{2n+4}(t)\Gamma\left(2m(n+3)\right)}{\Gamma\left(2m(n+3)+\frac{1}{2}\right)}\\ \times\left[\begin{array}{l}4am(3n+10)\left[{}_2F_1(1,-2m-1,2m+1,-1)-1-\dfrac{4m^2\sqrt{\pi}\Gamma(2m)}{(4m+1)\Gamma\left(2m+\frac{1}{2}\right)}\right]\\ -\dfrac{4\epsilon mA^2(t)(2\xi+\eta+2\zeta)\left[{}_2F_1(1,-4m-1,4m+1,-1)-1\right]}{(1+4m)}\end{array}\right]\end{array}\right\}}{\left\{\dfrac{2(n+3)\Gamma(2m)}{\Gamma\left(2m+\frac{1}{2}\right)}\left[\begin{array}{l}10am\left[{}_2F_1(1,-2m-1,2m+1,-1)-1-\dfrac{4m^2\sqrt{\pi}\Gamma(2m)}{(4m+1)\Gamma\left(2m+\frac{1}{2}\right)}\right]\\ -4\epsilon A^2(t)(2\xi+\eta+2\zeta)\dfrac{m}{1+4m}\left[{}_2F_1(1,-4m-1,4m+1,-1)-1\right]\end{array}\right]\right\}}, \tag{120*}
$$

$$
\frac{d\bar{x}(t)}{dt}=-2a\kappa(t), \tag{121}
$$

$$
\frac{d\kappa(t)}{dt}=\frac{8\epsilon\left[\begin{array}{l}2m\beta\kappa(t)\left[{}_2F_1(1,-2m-1,2m+1,-1)-1-\dfrac{4m^2\sqrt{\pi}\Gamma(2m)}{(4m+1)\Gamma\left(2m+\frac{1}{2}\right)}\right]\\ +m\mu A^{2m}(t)I_{0,2m,2}\end{array}\right]B^4(t)}{I_{0,2,0}}, \tag{122}
$$

$$
\begin{aligned}
\frac{dB(t)}{dt}=&-\frac{2B(t)}{A(t)}\frac{dA(t)}{dt}+4\epsilon\beta B(t)\kappa^2(t)\\
&-\frac{4\epsilon\delta A^{2m}(t)B(t)\Gamma\left(2m+\frac{1}{2}\right)\Gamma\left(2m(m+1)\right)}{\Gamma(2m)\Gamma\left(2m(m+1)+\frac{1}{2}\right)}\\
&+\frac{\epsilon\sigma_1A^{2m}(t)B(t)\operatorname{sech}^{4m^2}(x)}{m^2}\left[{}_2F_1\left(\frac{1}{2},2m^2,1+2m^2,\operatorname{sech}^2(x)\right)\right],
\end{aligned} \tag{123}
$$

$$
\begin{aligned}
B^4(t)=&\frac{\left\{\begin{array}{l}2\epsilon\left[\alpha\kappa(t)-\dfrac{\sigma_2\kappa(t)A^{2m}(t)\operatorname{sech}^{4m^2}(x)\left[{}_2F_1\left(\dfrac{1}{2},2m^2,1+2m^2,\operatorname{sech}^2(x)\right)\right]}{4m^2}\right]\\ \times\dfrac{\sqrt{\pi}\Gamma(2m)}{\Gamma\left(2m+\dfrac{1}{2}\right)}\end{array}\right\}}{\left\{\begin{array}{l}5am\left[{}_2F_1(1,-2m-1,2m+1,-1)-1-\dfrac{4m^2\sqrt{\pi}\Gamma(2m)}{(4m+1)\Gamma\left(2m+\dfrac{1}{2}\right)}\right]\\ -\dfrac{2\epsilon mA^2(t)(2\xi+\eta+2\zeta)\left[{}_2F_1(1,-4m-1,4m+1,-1)-1\right]}{(1+4m)}\end{array}\right\}}\\
&+\frac{\left\{\dfrac{(2n+1)b_1A^{2n}(t)\sqrt{\pi}\Gamma(2m(n+1))}{\Gamma\left(2m(n+1)+\dfrac{1}{2}\right)}\right\}}{\left\{(2n+2)\left[\begin{array}{l}am\left[{}_2F_1(1,-2m-1,2m+1,-1)-1-\dfrac{4m^2\sqrt{\pi}\Gamma(2m)}{(4m+1)\Gamma\left(2m+\dfrac{1}{2}\right)}\right]\\ -\dfrac{2\epsilon mA^2(t)(2\xi+\eta+2\zeta)\left[{}_2F_1(1,-4m-1,4m+1,-1)-1\right]}{(1+4m)}\end{array}\right]\right\}}\\
&+\frac{\left\{\dfrac{(2n+3)b_2A^{2n+2}(t)\sqrt{\pi}\Gamma(2m(n+2))}{\Gamma\left(2m(n+2)+\dfrac{1}{2}\right)}\right\}}{\left\{(2n+4)\left[\begin{array}{l}5am\left[{}_2F_1(1,-2m-1,2m+1,-1)-1-\dfrac{4m^2\sqrt{\pi}\Gamma(2m)}{(4m+1)\Gamma\left(2m+\dfrac{1}{2}\right)}\right]\\ -\dfrac{2\epsilon mA^2(t)(2\xi+\eta+2\zeta)\left[{}_2F_1(1,-4m-1,4m+1,-1)-1\right]}{(1+4m)}\end{array}\right]\right\}}\qquad(124)\\
&+\frac{\left\{\dfrac{(2n+5)b_3A^{2n+4}(t)\sqrt{\pi}\Gamma(2m(n+3))}{\Gamma\left(2m(n+3)+\dfrac{1}{2}\right)}\right\}}{\left\{(2n+6)\left[\begin{array}{l}am\left[{}_2F_1(1,-2m-1,2m+1,-1)-1-\dfrac{4m^2\sqrt{\pi}\Gamma(2m)}{(4m+1)\Gamma\left(2m+\dfrac{1}{2}\right)}\right]\\ -\dfrac{2m\epsilon A^2(t)(2\xi+\eta+2\zeta)\left[{}_2F_1(1,-4m-1,4m+1,-1)-1\right]}{(1+4m)}\end{array}\right]\right\}}\\
&+\frac{\left\{\dfrac{2\epsilon(\lambda+\sigma)A^{2m}(t)\kappa(t)\sqrt{\pi}\Gamma(2m(m+1))}{\Gamma\left(2m(m+1)+\dfrac{1}{2}\right)}+\dfrac{2\epsilon\kappa^2(t)A^2(t)(\xi-\eta-4\zeta)\sqrt{\pi}\Gamma(4m)}{\Gamma\left(4m+\dfrac{1}{2}\right)}\right\}}{\left\{\begin{array}{l}5am\left[{}_2F_1(1,-2m-1,2m+1,-1)-1-\dfrac{4m^2\sqrt{\pi}\Gamma(2m)}{(4m+1)\Gamma\left(2m+\dfrac{1}{2}\right)}\right]\\ -\dfrac{2\epsilon mA^2(t)(2\xi+\eta+2\zeta)\left[{}_2F_1(1,-4m-1,4m+1,-1)-1\right]}{(1+4m)}\end{array}\right\}}
\end{aligned}
$$

continued on next page

$$
\begin{aligned}
&-\frac{\left\{\begin{array}{l} m^{2}\epsilon\beta\kappa(t)B(t)4^{2m} \\ \times\left[\begin{array}{l} \dfrac{{}_4F_3(2m,2m,2m,2+4m;1+2m,1+2m,1+2m;-1)}{m^{3}} \\ -\dfrac{16\left[{}_4F_3(1+2m,1+2m,1+2m,2+4m;2+2m,2+2m,2+2m;-1)\right]}{[1+2m]^{3}} \\ +\dfrac{{}_4F_3(2+2m,2+2m,2+2m,2+4m;3+2m,3+2m,3+2m;-1)}{[1+m]^{3}} \end{array}\right] \end{array}\right\}}{\left\{\begin{array}{l} 5am\left[{}_2F_1(1,-2m-1,2m+1,-1)-1-\dfrac{4m^{2}\sqrt{\pi}\,\Gamma(2m)}{(4m+1)\Gamma\left(2m+\frac{1}{2}\right)}\right] \\ -\dfrac{2\epsilon mA^{2}(t)(2\xi+\eta+2\zeta)\left[{}_2F_1(1,-4m-1,4m+1,-1)-1\right]}{(1+4m)} \end{array}\right\}} \\
&+\frac{\left\{\begin{array}{l} \epsilon m^{3}\mu A^{2m}(t)B(t)4^{2(m+1)m} \\ \times\left[\begin{array}{l} \dfrac{{}_4F_3\left(\begin{array}{l} 2(m+1)m,2(m+1)m,2(m+1)m,2+4(m+1)m;1 \\ +2(m+1)m,1+2(m+1)m,1+2(m+1)m;-1 \end{array}\right)}{m^{3}(m+1)^{3}} \\ -\dfrac{16\left[{}_4F_3\left(\begin{array}{l} 1+2(m+1)m,1+2(m+1)m,1+2(m+1)m, \\ 2+4(m+1)m;2+2(m+1)m,2+2(m+1)m,2+2(m+1)m;-1 \end{array}\right)\right]}{[1+2(m+1)m]^{3}} \\ +\dfrac{{}_4F_3\left(\begin{array}{l} 2+2(m+1)m,2+2(m+1)m,2+2(m+1)m, \\ 2+4(m+1)m;3+2(m+1)m,3+2(m+1)m,3+2(m+1)m;-1 \end{array}\right)}{[1+(m+1)m]^{3}} \end{array}\right] \end{array}\right\}}{\left\{\begin{array}{l} 5am\left[{}_2F_1(1,-2m-1,2m+1,-1)-1-\dfrac{4m^{2}\sqrt{\pi}\,\Gamma(2m)}{(4m+1)\Gamma\left(2m+\frac{1}{2}\right)}\right] \\ -\dfrac{2\epsilon mA^{2}(t)(2\xi+\eta+2\zeta)\left[{}_2F_1(1,-4m-1,4m+1,-1)-1\right]}{(1+4m)} \end{array}\right\}}.
\end{aligned}
\tag{124*}
$$

4. Conclusions

This paper recovered the soliton parameter dynamics when the SPMs have polynomial and triple power-law forms. The results of the current paper are extensions of the previously reported works with power–law and dual–power law [1, 4]. The results are thus very interesting and serve as a gateway to additional research activities that can be carried out with these parameter dynamics. The study of collision-induced timing jitter, four-wave mixing, and several other phenomena can be addressed with these parameter dynamics. The stochastic perturbation of solitons can also be studied when the perturbation terms are random and their statistics are specified.

These parameter dynamics can also be recovered for additional forms of SPM. These would include anti–cubic and generalized anti–cubic forms, quadratic–cubic forms and generalized quadratic–cubic forms of SPM, and finally, Kudryashov's several proposed forms of SPM. The research activities are currently underway. The results of such activities are awaited at the moment, and when they are available, they will be disseminated after aligning them with pre-existing ones.

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Анотація. *У статті відновлено динаміку параметрів супер-гауссових і супер-секанс-гіперболічних імпульсів для збуреного нелінійного рівняння Шредінгера з поліноміальним і триступеневим законами нелінійності. Динамічна система відновлена за допомогою варіаційного принципу.*

Ключові слова*: солітони, варіаційний принцип, збурене нелінійне рівняння Шредінгера, рівняння Ейлера–Лагранжа*