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Optical Solitons for the Dispersive Concatenation Model with Differential Group Delay by the Complete Discriminant Approach

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Abstract. The current paper recovers optical solitons for the dispersive concatenation model with polarization–mode dispersion. The complete discriminant approach has made this retrieval possible. The intermediary Jacobi's elliptic functions gave way to the soliton solutions, with the limiting approach applied to such functions. These solitons are classified, and their surface and contour plots are sketched.

Keywords: concatenation model, solitons, dispersion, parameter constraints
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1. Introduction

The concept of the dispersive concatenation model was conceived just after the concept of the concatenation model was established in 2014 [1, 2]. This dispersive concatenation model was initially studied in 2015 [3–5]. The dispersive concatenation model is a conjoined version of three well–known models, and they are the Schrödinger–Hirota equation, Lakshmanan–Porsezian–Daniel model, and the fifth–order nonlinear Schrödinger's equation. Amongst these three fundamental equations, the first and last components are responsible for third–and fifth–order dispersive effects, respectively, hence the name. Later, after a long hiatus, these two versions of the concatenation model were extensively studied from various perspectives [6–20]. These include the extraction of the conservation laws, application to magneto-optic waveguides, quiescent optical solitons, gap solitons, numerical analysis of the models by the aid of Laplace–Adomian decomposition, and many others. Subsequently, the concatenation model was extended to polarization–mode dispersion. Today, the study on the dispersive concatenation model will be taken further by considering it with differential group delay. The fundamental model will be first presented in its dimensionless forms, and the coupled system will be next integrated to recover its soliton solution. The complete discriminant method is the tool of integration today. The intermediary functions that made these soliton solutions retrieval possible are Jacobi's elliptic functions, with the limiting value of the modulus of ellipticity approaching unity. Also, we disregard additional solutions, including singular periodic solutions and plane waves, that arise from alternate signs of the discriminant [22-26], as they are irrelevant to the context of optoelectronics. The results and their derivations are exhibited in the rest of the paper after a quick introduction to the model.

2. Mathematical analysis

2.1. Governing model

The governing model for the dispersive concatenation model with differential group delay is the coupled system [21]:

$$\begin{aligned}
&iu_t + a^{(1)}u_{xx} + \left(b_1^{(1)}|u|^2 + b_2^{(1)}|v|^2\right)u - i\delta_1^{(1)}\left[\sigma_1^{(1)}u_{xxx} + \left(\sigma_{21}^{(1)}|u|^2 + \sigma_{22}^{(1)}|v|^2\right)u\right]\\
&+\delta_2^{(1)}\left[\sigma_3^{(1)}u_{xxxx} + \left(\sigma_{41}^{(1)}|u|^2 + \sigma_{42}^{(1)}|v|^2\right)u_{xx} + \left(\sigma_{51}^{(1)}|u|^4 + \sigma_{52}^{(1)}|u|^2|v|^2 + \sigma_{53}^{(1)}|v|^4\right)u\right.\\
&\left.+\left(\sigma_{61}^{(1)}|u_x|^2 + \sigma_{62}^{(1)}|v_x|^2\right)u + \left(\sigma_{71}^{(1)}u_x^2 + \sigma_{72}^{(1)}v_x^2\right)u^* + \left(\sigma_{81}^{(1)}u_{xx}^* + \sigma_{82}^{(1)}v_{xx}^*\right)u^2\right]\\
&-i\delta_3^{(1)}\left[\sigma_9^{(1)}u_{xxxxx} + \left(\sigma_{101}^{(1)}|u|^2 + \sigma_{102}^{(1)}|v|^2\right)u_{xxx} + \left(\sigma_{111}^{(1)}|u|^4 + \sigma_{112}^{(1)}|u|^2|v|^2 + \sigma_{113}^{(1)}|v|^4\right)u_x\right.\\
&+\left(\sigma_{121}^{(1)}uu_x + \sigma_{122}^{(1)}vv_x\right)u_{xx}^* + \left(\sigma_{131}^{(1)}u^*u_x + \sigma_{132}^{(1)}v^*v_x\right)u_{xx} + \left(\sigma_{141}^{(1)}uu_x^* + \sigma_{142}^{(1)}vv_x^*\right)u_{xx}\\
&\left.+\left(\sigma_{151}^{(1)}|u_x|^2 + \sigma_{152}^{(1)}|v_x|^2\right)u_x\right] = 0
\end{aligned}, \quad (1)$$

and

$$\begin{aligned}
&iv_t + a^{(2)}v_{xx} + \left(b_1^{(2)}|v|^2 + b_2^{(2)}|u|^2\right)v - i\delta_1^{(2)}\left[\sigma_1^{(2)}v_{xxx} + \left(\sigma_{21}^{(2)}|v|^2 + \sigma_{22}^{(2)}|u|^2\right)v\right]\\
&+\delta_2^{(2)}\left[\sigma_3^{(2)}v_{xxxx} + \left(\sigma_{41}^{(2)}|v|^2 + \sigma_{42}^{(2)}|u|^2\right)v_{xx} + \left(\sigma_{51}^{(2)}|v|^4 + \sigma_{52}^{(2)}|v|^2|u|^2 + \sigma_{53}^{(2)}|u|^4\right)v\right.\\
&\left.+\left(\sigma_{61}^{(2)}|v_x|^2 + \sigma_{62}^{(2)}|u_x|^2\right)v + \left(\sigma_{71}^{(2)}v_x^2 + \sigma_{72}^{(2)}u_x^2\right)v^* + \left(\sigma_{81}^{(2)}v_{xx}^* + \sigma_{82}^{(2)}u_{xx}^*\right)v^2\right]\\
&-i\delta_3^{(2)}\left[\sigma_9^{(2)}v_{xxxxx} + \left(\sigma_{101}^{(2)}|v|^2 + \sigma_{102}^{(2)}|u|^2\right)v_{xxx} + \left(\sigma_{111}^{(2)}|v|^4 + \sigma_{112}^{(2)}|v|^2|u|^2 + \sigma_{113}^{(2)}|u|^4\right)v_x\right.\\
&+\left(\sigma_{121}^{(2)}vv_x + \sigma_{122}^{(2)}uu_x\right)v_{xx}^* + \left(\sigma_{131}^{(2)}v^*v_x + \sigma_{132}^{(2)}u^*u_x\right)v_{xx} + \left(\sigma_{141}^{(2)}vv_x^* + \sigma_{142}^{(2)}uu_x^*\right)v_{xx}\\
&\left.+\left(\sigma_{151}^{(2)}|v_x|^2 + \sigma_{152}^{(2)}|u_x|^2\right)v_x\right] = 0
\end{aligned}. \quad (2)$$

Here $u(x,t)$ and $v(x,t)$ are the wave amplitudes of the split pulses that have emerged due to birefringence. The first terms with coefficients i are the linear temporal evolution along the two components of the pulses. The second term $a^{(j)}$ is the coefficient of chromatic dispersion along the two components of birefringent fibers. Then $b_1^{(j)}$ and $b_2^{(j)}$ for j=1,2 are the coefficients of self–phase modulation (SPM) and cross–phase modulation (XPM) effects respectively. Next, $\sigma_{21}^{(j)}$ and $\sigma_{22}^{(j)}$ are the SPM and XPM for intermodal dispersions respectively. Now $\sigma_3^{(j)}$ is with fourth–order dispersions along the two components of a birefringent fiber. Again, $\sigma_{41}^{(j)}$ and $\sigma_{42}^{(j)}$ are the SPM and XPM for second–order dispersions, respectively. Also, $\sigma_{51}^{(j)}$ stands for the SPM with quintic form of nonlinearity, while $\sigma_{52}^{(j)}$ and

$\sigma_{53}^{(j)}$ account for XPM with quintic nonlinearity. However, $\sigma_{61}^{(j)}$, $\sigma_{62}^{(j)}$, $\sigma_{71}^{(j)}$, $\sigma_{72}^{(j)}$, $\sigma_{81}^{(j)}$, and $\sigma_{82}^{(j)}$ come from the radiative effect of the solitons that stem from four–wave mixing (4WM) effect and other sources of small–amplitude dispersive effects. Then, $\sigma_{9}^{(j)}$ comes from the fifth–order dispersions along the two components of birefringence. Next up, $\sigma_{101}^{(j)}$ and $\sigma_{102}^{(j)}$ are the components of SPM and XPM that stems from third–order dispersions along the two components. Again $\sigma_{111}^{(j)}$ is the SPM coefficient due to intermodal dispersion along the two components while $\sigma_{112}^{(j)}$ and $\sigma_{113}^{(j)}$ are the coefficients of XPM due to intermodal dispersion along the two components. Finally $\sigma_{121}^{(j)}$, $\sigma_{122}^{(j)}$, $\sigma_{131}^{(j)}$, $\sigma_{132}^{(j)}$, $\sigma_{141}^{(j)}$, $\sigma_{142}^{(j)}$, $\sigma_{151}^{(j)}$ and $\sigma_{152}^{(j)}$, are additional terms from soliton radiation along the two components that emerge from multi-wave mixing and other sources. It is worth mentioning that the effect of 4WM, which emerges from XPM, is ignored in the derivation of the model Eqs. (2) and (3). This is to keep the coupled system simple without additional nonlinear effects. Apart from neglecting the 4WM effects, the remaining included terms are equally important as they directly emerge from the dispersive concatenation model.

This coupled system will now be integrated with the aid of the complete discriminant approach after some preliminary mathematical analysis. We set the traveling wave hypotheses as follows:

$$u(x,t)=Q_1(\xi)e^{i\phi(x,t)},\quad v(x,t)=Q_2(\xi)e^{i\phi(x,t)},\quad \xi=k(x-vt),\quad \phi(x,t)=-hx+\omega t+\iota. \tag{3}$$

Here, $Q_l(\xi)$ for j=1,2 represents the amplitude component of the soliton, while ξ is the wave variable, where v is the speed of the soliton and k is the width of the soliton. Also, $\phi(x,t)$ is the phase component of the soliton, where h is the wave number, ω is the frequency, and ι is the phase constant. Substituting Eq. (3) into Eqs. (1) and (2), the real parts stick out as:

$$\begin{aligned}
&\left(\delta_2^{(l)}\sigma_3^{(l)}k^4-\delta_3^{(l)}\sigma_9^{(l)}k^4h\right)Q_l''''+\delta_2^{(l)}k^2\left(\sigma_{41}^{(l)}Q_l^2+\sigma_{42}^{(l)}Q_{3-l}^2\right)Q_l''+\delta_2^{(l)}k^2Q_l^2\left(\sigma_{81}^{(l)}Q_l''+\sigma_{82}^{(l)}Q_{3-l}''\right)\\
&+\delta_3^{(l)}k^2h\left(\left(-3\sigma_{101}^{(l)}-\sigma_{121}^{(l)}-\sigma_{131}^{(l)}+\sigma_{141}^{(l)}\right)Q_l^2+\left(-3\sigma_{102}^{(l)}-\sigma_{122}^{(l)}-\sigma_{132}^{(l)}+\sigma_{142}^{(l)}\right)Q_{3-l}^2\right)Q_l''\\
&+\left(a^{(l)}k^2-3\delta_1^{(l)}\sigma_1^{(l)}k^2h-6\delta_2^{(l)}\sigma_3^{(l)}k^2h^2+10\delta_3^{(l)}\sigma_9^{(l)}k^2h^3\right)Q_l''\\
&+\delta_2^{(l)}k^2\left(\left(\sigma_{61}^{(l)}+\sigma_{71}^{(l)}\right)(Q_l')^2+\left(\sigma_{62}^{(l)}+\sigma_{72}^{(l)}\right)(Q_{3-l}')^2\right)Q_l\\
&+2\delta_3^{(l)}k^2h\left(\left(\sigma_{121}^{(l)}-\sigma_{131}^{(l)}-\sigma_{141}^{(l)}\right)Q_l(Q_l')^2+\left(\sigma_{122}^{(l)}-\sigma_{132}^{(l)}-\sigma_{142}^{(l)}\right)Q_{3-l}Q_l'Q_{3-l}'\right)\\
&-\delta_3^{(l)}kh\left(\sigma_{151}^{(l)}(Q_l')^2+\sigma_{152}^{(l)}(Q_{3-l}')^2\right)Q_l+\delta_2^{(l)}\left(\sigma_{51}^{(l)}Q_l^5+\sigma_{52}^{(l)}Q_l^3Q_{3-l}^2+\sigma_{53}^{(l)}Q_lQ_{3-l}^4\right)\\
&-\delta_3^{(l)}h\left(\sigma_{111}^{(l)}Q_l^5+\sigma_{112}^{(l)}Q_l^3Q_{3-l}^2+\sigma_{113}^{(l)}Q_lQ_{3-l}^4\right)+b_1^{(1)}Q_l^3+b_2^{(1)}Q_lQ_2^2\\
&+\delta_2^{(l)}h^2\left(\left(-\sigma_{41}^{(l)}+\sigma_{61}^{(l)}-\sigma_{71}^{(l)}\right)Q_l^3+\left(-\sigma_{42}^{(l)}+\sigma_{62}^{(l)}-\sigma_{72}^{(l)}\right)Q_lQ_{3-l}^2\right)-\delta_2^{(l)}h^2(\sigma_{81}^{(l)}Q_l^3+\sigma_{82}^{(l)}Q_l^2Q_{3-l})\\
&+\delta_3^{(l)}h^3\left(\left(\sigma_{101}^{(l)}+\sigma_{121}^{(l)}+\sigma_{131}^{(l)}-\sigma_{141}^{(l)}-\sigma_{151}^{(l)}\right)Q_l^3+\left(\sigma_{102}^{(l)}+\sigma_{122}^{(l)}+\sigma_{132}^{(l)}+\sigma_{142}^{(l)}-\sigma_{151}^{(l)}\right)Q_lQ_{3-l}^2\right)\\
&+\left(-\omega-a^{(l)}h^2+\delta_1^{(l)}\sigma_1^{(l)}h^3+\delta_2^{(l)}\sigma_3^{(l)}h^4-\delta_3^{(l)}\sigma_9^{(l)}h^5\right)Q_l=0,
\end{aligned}\tag{4}$$

and the imaginary parts evolve as:

$$
\begin{aligned}
&-\delta_3^{(l)}\sigma_9^{(l)}k^5Q_l^{'''''}-\delta_3^{(l)}k^3\left(\sigma_{101}^{(l)}Q_l^2+\sigma_{102}^{(l)}Q_{3-l}^2\right)Q_l^{'''}\\
&+\left(-\delta_1^{(l)}\sigma_1^{(l)}k^3-4\delta_2^{(l)}\sigma_3^{(l)}k^3h+10\delta_3^{(l)}\sigma_9^{(l)}k^3h^2\right)Q_l^{'''}-2\delta_2^{(l)}kh\left(\sigma_{41}^{(l)}Q_l^2+\sigma_{42}^{(l)}Q_{3-l}^2\right)Q_l^{'}\\
&-2\delta_2^{(l)}kh\left(\sigma_{71}^{(l)}Q_l^2Q_l^{'}+\sigma_{72}^{(l)}Q_lQ_{3-l}Q_{3-l}^{'}\right)+2\delta_2^{(l)}kh\left(\sigma_{81}^{(l)}Q_l^{'}+\sigma_{82}^{(l)}Q_{3-l}^{'}\right)Q_l^2\\
&+\delta_3^{(l)}kh^2\left(3\sigma_{101}^{(l)}-2\sigma_{121}^{(l)}+2\sigma_{131}^{(l)}-2\sigma_{141}^{(l)}-\sigma_{151}^{(l)}\right)Q_l^2Q_l^{'}\\
&+\left(3\sigma_{102}^{(l)}-2\sigma_{122}^{(l)}+2\sigma_{132}^{(l)}-2\sigma_{142}^{(l)}-\sigma_{152}^{(l)}\right)Q_{3-l}^2Q_l^{'}\\
&+\delta_3^{(l)}kh^2\left(\left(\sigma_{121}^{(l)}+\sigma_{131}^{(l)}+\sigma_{141}^{(l)}\right)Q_l^2Q_l^{'}+\left(\sigma_{122}^{(l)}+\sigma_{132}^{(l)}+\sigma_{142}^{(l)}\right)Q_lQ_{3-l}Q_{3-l}^{'}\right)\\
&-\delta_3^{(l)}k\Big[\sigma_{111}^{(l)}Q_l^4+\sigma_{112}^{(l)}Q_l^2Q_{3-l}^2+\sigma_{113}^{(l)}Q_{3-l}^4Q_l^{'}-\delta_3^{(l)}k^3\left(\sigma_{121}^{(l)}+\sigma_{131}^{(l)}+\sigma_{141}^{(l)}\right)Q_lQ_l^{'}Q_l^{''}\\
&+\left(\sigma_{122}^{(l)}+\sigma_{132}^{(l)}+\sigma_{142}^{(l)}\right)Q_{3-l}Q_{3-l}^{'}Q_l^{''}-\delta_3^{(l)}k^3\sigma_{151}^{(l)}\left(Q_l^{'}\right)^2+\sigma_{152}^{(l)}\left(Q_{3-l}^{'}\right)^2Q_l^{'}\\
&+\left(-kv-2a^{(l)}kh+3\delta_1^{(l)}\sigma_1^{(l)}kh^2+4\delta_2^{(l)}\sigma_3^{(l)}kh^3-5\delta_3^{(l)}\sigma_9^{(l)}kh^4\right)Q_l^{'}\\
&-\delta_1^{(l)}\sigma_{21}^{(l)}Q_l^2+\sigma_{22}^{(l)}Q_{3-l}^2\Big]Q_l=0,
\end{aligned}
\tag{5}
$$

where l=1,2. Setting:

$$Q_2=\chi Q_1, \tag{6}$$

Eqs. (4) and (5) come out as:

$$
\begin{aligned}
&\left(\delta_2^{(1)}\sigma_3^{(l)}k^4-\delta_3^{(1)}\sigma_9^{(1)}k^4h\right)Q_1^{''''}+\delta_2^{(1)}k^2\left(\sigma_{41}^{(1)}+\chi^2\sigma_{42}^{(1)}+\sigma_{81}^{(1)}+\chi\sigma_{82}^{(1)}\right)\\
&+\delta_3^{(1)}k^2h\begin{pmatrix}\left(-3\sigma_{101}^{(1)}-\sigma_{121}^{(1)}-\sigma_{131}^{(1)}+\sigma_{141}^{(1)}\right)\\ +\chi^2\left(-3\sigma_{102}^{(1)}-\sigma_{122}^{(1)}-\sigma_{132}^{(1)}+\sigma_{142}^{(1)}\right)\end{pmatrix}Q_1^2Q_1^{''}\\
&+\left(a^{(1)}k^2-3\delta_1^{(1)}\sigma_1^{(1)}k^2h-6\delta_2^{(1)}\sigma_3^{(1)}k^2h^2+10\delta_3^{(1)}\sigma_9^{(1)}k^2h^3\right)Q_1^{''}\\
&+\delta_2^{(1)}k^2\left(\left(\sigma_{61}^{(1)}+\sigma_{71}^{(1)}\right)+\chi^2\left(\sigma_{62}^{(1)}+\sigma_{72}^{(1)}\right)\right)\\
&+2\delta_3^{(1)}k^2h\left(\left(\sigma_{121}^{(1)}-\sigma_{131}^{(1)}-\sigma_{141}^{(1)}\right)+\chi^2\left(\sigma_{122}^{(1)}-\sigma_{132}^{(1)}-\sigma_{142}^{(1)}\right)\right)\\
&-\delta_3^{(1)}kh\left(\sigma_{151}^{(1)}+\chi^2\sigma_{152}^{(1)}\right)Q_1(Q_1^{'})^2\\
&+\left(\delta_2^{(1)}\left(\sigma_{51}^{(1)}+\chi^2\sigma_{52}^{(1)}+\chi^4\sigma_{53}^{(1)}\right)-\delta_3^{(1)}h\left(\sigma_{111}^{(1)}+\chi^2\sigma_{112}^{(1)}+\chi^4\sigma_{113}^{(1)}\right)\right)Q_1^5\\
&+b_1^{(1)}+\chi^2b_2^{(1)}+\delta_2^{(1)}h^2\left(-\sigma_{41}^{(1)}+\sigma_{61}^{(1)}-\sigma_{71}^{(1)}\right)\\
&+\chi^2\left(-\sigma_{42}^{(1)}+\sigma_{62}^{(1)}-\sigma_{72}^{(1)}\right)-\delta_2^{(1)}h^2\left(\sigma_{81}^{(1)}+\chi\sigma_{82}^{(1)}\right)\\
&+\delta_3^{(1)}h^3\begin{pmatrix}\left(\sigma_{101}^{(1)}+\sigma_{121}^{(1)}+\sigma_{131}^{(1)}-\sigma_{141}^{(1)}-\sigma_{151}^{(1)}\right)\\ +\chi^2\left(\sigma_{102}^{(1)}+\sigma_{122}^{(1)}+\sigma_{132}^{(1)}+\sigma_{142}^{(1)}-\sigma_{151}^{(1)}\right)\end{pmatrix}Q_1^3\\
&+\left(-\omega-a^{(1)}h^2+\delta_1^{(1)}\sigma_1^{(1)}h^3+\delta_2^{(1)}\sigma_3^{(1)}h^4-\delta_3^{(1)}\sigma_9^{(1)}h^5\right)Q_1=0,
\end{aligned}
\tag{7}
$$

and

$$
\begin{aligned}
&-\delta_3^{(1)}\sigma_9^{(1)}k^5Q_1''''' - \delta_3^{(1)}k^3\left(\sigma_{101}^{(1)} + \chi^2\sigma_{102}^{(1)}\right)Q_1^2Q_1''' \\
&+\left(-\delta_1^{(1)}\sigma_1^{(1)}k^3 - 4\delta_2^{(1)}\sigma_3^{(1)}k^3h + 10\delta_3^{(1)}\sigma_9^{(1)}k^3h^2\right)Q_1''' \\
&+(-2\delta_2^{(1)}kh\left(\sigma_{41}^{(1)} + \chi^2\sigma_{42}^{(1)}\right) - 2\delta_2^{(1)}kh\left(\sigma_{71}^{(1)} + \chi^2\sigma_{72}^{(1)}\right) + 2\delta_2^{(1)}kh\left(\sigma_{81}^{(1)} + \chi\sigma_{82}^{(1)}\right) \\
&+\delta_3^{(1)}kh^2\begin{pmatrix} \left(3\sigma_{101}^{(1)} - \sigma_{121}^{(1)} + 3\sigma_{131}^{(1)} - \sigma_{141}^{(1)} - \sigma_{151}^{(1)}\right) \\ +\chi^2\left(3\sigma_{102}^{(1)} - \sigma_{122}^{(1)} + 3\sigma_{132}^{(1)} - \sigma_{142}^{(1)} - \sigma_{152}^{(1)}\right)Q_1^2Q_1' \\ -\delta_3^{(1)}k\left(\sigma_{111}^{(1)} + \chi^2\sigma_{112}^{(1)} + \chi^4\sigma_{113}^{(1)}\right)Q_1^4Q_1' \end{pmatrix} \\
&-\delta_3^{(1)}k^3\left(\left(\sigma_{121}^{(1)} + \sigma_{131}^{(1)} + \sigma_{141}^{(1)}\right) + \chi^2\left(\sigma_{122}^{(1)} + \sigma_{132}^{(1)} + \sigma_{142}^{(1)}\right)\right)Q_1Q_1'Q_1'' \\
&-\delta_3^{(1)}k^3\left(\sigma_{151}^{(1)} + \chi^2\sigma_{152}^{(1)}\right)(Q_1)^2Q_1' \\
&+\left(-kv - 2a^{(1)}kh + 3\delta_1^{(1)}\sigma_1^{(1)}kh^2 + 4\delta_2^{(1)}\sigma_3^{(1)}kh^3 - 5\delta_3^{(1)}\sigma_9^{(1)}kh^4\right)Q_1' \\
&-\delta_1^{(1)}\left(\sigma_{21}^{(1)} + \chi^2\sigma_{22}^{(1)}\right)Q_1^3 = 0.
\end{aligned}
\tag{8}
$$

From Eq. (8), we obtain the velocity:

$$
v = \frac{1}{k}\left(-2a^{(1)}kh + 3\delta_1^{(1)}\sigma_1^{(1)}kh^2 + 4\delta_2^{(1)}\sigma_3^{(1)}kh^3 - 5\delta_3^{(1)}\sigma_9^{(1)}kh^4\right),
\tag{9}
$$

along with the necessary restrictive conditions:

$$
\begin{aligned}
&-\delta_3^{(1)}\sigma_9^{(1)}k^5 = 0, \\
&-\delta_3^{(1)}k^3\left(\sigma_{101}^{(1)} + \chi^2\sigma_{102}^{(1)}\right) = 0, \\
&-\delta_1^{(1)}\sigma_1^{(1)}k^3 - 4\delta_2^{(1)}\sigma_3^{(1)}k^3h + 10\delta_3^{(1)}\sigma_9^{(1)}k^3h^2 = 0, \\
&-2\delta_2^{(1)}kh\left(\sigma_{41}^{(1)} + \chi^2\sigma_{42}^{(1)}\right) - 2\delta_2^{(1)}kh\left(\sigma_{71}^{(1)} + \chi^2\sigma_{72}^{(1)}\right) + 2\delta_2^{(1)}kh\left(\sigma_{81}^{(1)} + \chi\sigma_{82}^{(1)}\right) \\
&+\delta_3^{(1)}kh^2\begin{pmatrix} \left(3\sigma_{101}^{(1)} - \sigma_{121}^{(1)} + 3\sigma_{131}^{(1)} - \sigma_{141}^{(1)} - \sigma_{151}^{(1)}\right) \\ +\chi^2\left(3\sigma_{102}^{(1)} - \sigma_{122}^{(1)} + 3\sigma_{132}^{(1)} - \sigma_{142}^{(1)} - \sigma_{152}^{(1)}\right) \end{pmatrix} = 0, \\
&-\delta_3^{(1)}k\left(\sigma_{111}^{(1)} + \chi^2\sigma_{112}^{(1)} + \chi^4\sigma_{113}^{(1)}\right) = 0, \\
&-\delta_3^{(1)}k^3\left(\left(\sigma_{121}^{(1)} + \sigma_{131}^{(1)} + \sigma_{141}^{(1)}\right) + \chi^2\left(\sigma_{122}^{(1)} + \sigma_{132}^{(1)} + \sigma_{142}^{(1)}\right)\right) = 0, \\
&-\delta_3^{(1)}k^3\left(\sigma_{151}^{(1)} + \chi^2\sigma_{152}^{(1)}\right) = 0, \\
&-\delta_1^{(1)}\left(\sigma_{21}^{(1)} + \chi^2\sigma_{22}^{(1)}\right) = 0.
\end{aligned}
\tag{10}
$$

Abbreviate the real part of Eq. (7) as follows:

$$
A_1Q_1'''' + A_2Q_1^2Q_1'' + A_3Q_1'' + A_4Q_1(Q_1')^2 + B_1Q_1^5 + B_2Q_1^3 + B_3Q_1 = 0,
\tag{11}
$$

where

$$\begin{aligned}
&A_1=\delta_2^{(1)}\sigma_3^{(l)}k^4-\delta_3^{(1)}\sigma_9^{(1)}k^4h,\\
&A_2=\delta_2^{(1)}k^2\left(\sigma_{41}^{(1)}+\chi^2\sigma_{42}^{(1)}+\sigma_{81}^{(1)}+\chi\sigma_{82}^{(1)}\right)\\
&+\delta_3^{(1)}k^2h\left(\left(-3\sigma_{101}^{(1)}-\sigma_{121}^{(1)}-\sigma_{131}^{(1)}+\sigma_{141}^{(1)}\right)+\chi^2\left(-3\sigma_{102}^{(1)}-\sigma_{122}^{(1)}-\sigma_{132}^{(1)}+\sigma_{142}^{(1)}\right)\right),\\
&A_3=a^{(1)}k^2-3\delta_1^{(1)}\sigma_1^{(1)}k^2h-6\delta_2^{(1)}\sigma_3^{(1)}k^2h^2+10\delta_3^{(1)}\sigma_9^{(1)}k^2h^3,\\
&A_4=\delta_2^{(1)}k^2\left(\left(\sigma_{61}^{(1)}+\sigma_{71}^{(1)}\right)+\chi^2\left(\sigma_{62}^{(1)}+\sigma_{72}^{(1)}\right)\right)\\
&+2\delta_3^{(1)}k^2h\left(\left(\sigma_{121}^{(1)}-\sigma_{131}^{(1)}-\sigma_{141}^{(1)}\right)+\chi^2\left(\sigma_{122}^{(1)}-\sigma_{132}^{(1)}-\sigma_{142}^{(1)}\right)\right)-\delta_3^{(1)}kh\left(\sigma_{151}^{(1)}+\chi^2\sigma_{152}^{(1)}\right),\\
&B_1=\left(\delta_2^{(1)}\left(\sigma_{51}^{(1)}+\chi^2\sigma_{52}^{(1)}+\chi^4\sigma_{53}^{(1)}\right)-\delta_3^{(1)}h\left(\sigma_{111}^{(1)}+\chi^2\sigma_{112}^{(1)}+\chi^4\sigma_{113}^{(1)}\right)\right),\\
&B_2=b_1^{(1)}+\chi^2b_2^{(1)}+\delta_2^{(1)}h^2\left(\left(-\sigma_{41}^{(1)}+\sigma_{61}^{(1)}-\sigma_{71}^{(1)}\right)+\chi^2\left(-\sigma_{42}^{(1)}+\sigma_{62}^{(1)}-\sigma_{72}^{(1)}\right)\right)\\
&-\delta_2^{(1)}h^2\left(\sigma_{81}^{(1)}+\chi\sigma_{82}^{(1)}\right),\\
&B_3=-\omega-a^{(1)}h^2+\delta_1^{(1)}\sigma_1^{(1)}h^3+\delta_2^{(1)}\sigma_3^{(1)}h^4-\delta_3^{(1)}\sigma_9^{(1)}h^5.
\end{aligned} \tag{12}$$

We take the trial equation

$$(Q_1')^2=\sum_{i=0}^{n}c_iQ_1^i(\xi). \tag{13}$$

After substituting Eq. (13) into Eq. (11), we balance Q_1'''' and Q_1^5 to determine that n=4. The trial equation can be expressed as:

$$(Q_1')^2=c_4Q_1^4+c_3Q_1^3+c_2Q_1^2+c_1Q_1+c_0, \tag{14}$$

where

$$\begin{aligned}
&c_4==\pm\frac{-2A_2-A_4-\sqrt{\left(2A_2+A_4\right)^2-96A_1B_1}}{48A_1},\quad c_3=0,\\
&c_2=\frac{-2B_2-4A_3c_4}{2\left(A_2+A_4+20A_1c_4\right)},\quad c_1=0,\quad \mathrm{c}_0=\frac{-B_3-A_3c_2-A_1c_2^2}{A_4+12A_1c_4}.
\end{aligned} \tag{15}$$

Therefore, Eq. (14) reduces to $(Q_1')^2=c_4Q_1^4+c_2Q_1^2+c_0$. Take the transformation

$$P=(4c_4)^{\frac{1}{3}}Q_1^2,\quad \xi_1=(4c_4)^{\frac{1}{3}}\xi, \tag{16}$$

which turns Eq. (14) into:

$$(P_{\xi_1})^2=P^3+d_2P^2+d_1P,\, P_{\xi_1}=\frac{dP}{d\xi_1}, \tag{17}$$

where

$$d_2=4c_2\left(4c_4\right)^{-\frac{2}{3}},\quad d_1=4c_0\left(4c_4\right)^{-\frac{1}{3}}. \tag{18}$$

Simplify Eq. (17) to the integral form:

$$\pm\left(\xi_1-\xi_0\right)=\int\frac{dP}{\sqrt{F(P)}}, \tag{19}$$

where ξ_0 is the center position of the pulse:

$$F(P)=P\left(P^2+d_2P+d_1\right). \tag{20}$$

We give the second-order polynomial discriminant system:

$$\Delta=d_2^2-4d_1. \tag{21}$$

In order to find the solutions to the original equation, we employ the second-order discriminant system to classify the roots of the polynomial $F(P)$. We can obtain optical soliton solutions by resolving the corresponding integrals [22-26].

3. Optical solitons

Case–1: $\Delta=0$. For $P>0$, if $d_2<0$, the dark and singular solitons stand as:

$$u_1(x,t)=\left\{(4c_4)^{-\frac{1}{3}}\left(-\frac{d_2}{2}\tanh^2\left(\frac{1}{2}\sqrt{-\frac{d_2}{2}}\left((4c_4)^{\frac{1}{3}}\xi-\xi_0\right)\right)\right)\right\}^{1/2}e^{i(-hx+\omega t+\iota)}, \tag{22}$$

and

$$u_2(x,t)=\left\{(4c_4)^{-\frac{1}{3}}\left(-\frac{d_2}{2}\coth^2\left(\frac{1}{2}\sqrt{-\frac{d_2}{2}}\left((4c_4)^{\frac{1}{3}}\xi-\xi_0\right)\right)\right)\right\}^{1/2}e^{i(-hx+\omega t+\iota)}, \tag{23}$$

respectively.

Case–2: $\Delta>0$ and $d_1=0$. For $P>-d_2$, if $d_2>0$, the dark and singular solitons shape up as:

$$u_3(x,t)=\left\{(4c_4)^{-\frac{1}{3}}\left(-\frac{d_2}{2}\tanh^2\left(\frac{1}{2}\sqrt{-\frac{d_2}{2}}\left((4c_4)^{\frac{1}{3}}\xi-\xi_0\right)\right)-d_2\right)\right\}^{1/2}e^{i(-hx+\omega t+\iota)}, \tag{24}$$

and

$$u_4(x,t)=\left\{(4c_4)^{-\frac{1}{3}}\left(-\frac{d_2}{2}\coth^2\left(\frac{1}{2}\sqrt{-\frac{d_2}{2}}\left((4c_4)^{\frac{1}{3}}\xi-\xi_0\right)\right)-d_2\right)\right\}^{1/2}e^{i(-hx+\omega t+\iota)}, \tag{25}$$

respectively.

Case–3: $\Delta>0$ and $d_1\neq 0$. Suppose that $\rho_1<\rho_2<\rho_3$, one of them is zero, and the other two are roots of $P^2+d_2P+d_1$. For $\rho_1<P<\rho_2$, the snoidal wave turns out to be:

$$u_5(x,t)=\left\{(4c_4)^{-\frac{1}{3}}\left[\rho_1+(\rho_2-\rho_1)\operatorname{sn}^2\frac{\sqrt{\rho_3-\rho_1}}{2}\left(\left((4c_4)^{\frac{1}{3}}\xi-\xi_0\right),m\right)\right]\right\}^{1/2} \times e^{i(-hx+\omega t+\iota)}, \tag{26}$$

and for $P>\rho_3$, the combo snoidal and cnoidal wave appears as:

$$u_6(x,t)=\left\{(4c_4)^{-\frac{1}{3}}\left[\frac{\rho_3-\rho_2\operatorname{sn}^2\frac{\sqrt{\rho_3-\rho_1}}{2}\left(\left((4c_4)^{\frac{1}{3}}\xi-\xi_0\right),m\right)}{\operatorname{cn}^2\frac{\sqrt{\rho_3-\rho_1}}{2}\left(\left((4c_4)^{\frac{1}{3}}\xi-\xi_0\right),m\right)}\right]\right\}^{1/2} \times e^{i(-hx+\omega t+\iota)}, \tag{27}$$

where $m^2=\frac{\rho_2-\rho_1}{\rho_3-\rho_1}$. For Eqs. (26) and (27), the corresponding optical solitons that emerge when $m\to 1^-$ are:

$$u_7(x,t)=\left\{(4c_4)^{-\frac{1}{3}}\left[\rho_1+(\rho_2-\rho_1)\tanh^2\frac{\sqrt{\rho_3-\rho_1}}{2}\left(\left((4c_4)^{\frac{1}{3}}\xi-\xi_0\right)\right)\right]\right\}^{1/2}, \quad (28)$$
$$\times e^{i(-hx+\omega t+\iota)}$$

and

$$u_8(x,t)=\left\{(4c_4)^{-\frac{1}{3}}\left[\frac{\rho_3-\rho_2\tanh^2\frac{\sqrt{\rho_3-\rho_1}}{2}\left(\left((4c_4)^{\frac{1}{3}}\xi-\xi_0\right)\right)}{\operatorname{sech}^2\frac{\sqrt{\rho_3-\rho_1}}{2}\left(\left((4c_4)^{\frac{1}{3}}\xi-\xi_0\right)\right)}\right]\right\}^{1/2} e^{i(-hx+\omega t+\iota)}, \quad (29)$$

which are the dark and singular-singular straddled soliton solutions, respectively.

Case–4: $\Delta<0$, for P>0, the cnoidal wave stands as:

$$u_9(x,t)=\left\{(4c_4)^{-\frac{1}{3}}\left(\frac{2\sqrt{d_1}}{1+\operatorname{cn}^2 d_1^{\frac{1}{4}}\left(\left((4c_4)^{\frac{1}{3}}\xi-\xi_0\right),m\right)}-\sqrt{d_1}\right)\right\}^{1/2} e^{i(-hx+\omega t+\iota)}, \quad (30)$$

where $m^2=\frac{1}{2}-\frac{d_2}{4\sqrt{d_1}}$. When $m\to 1^-$ one recovers the singular optical soliton as:

$$u_{10}(x,t)=\left\{(4c_4)^{-\frac{1}{3}}\left(\frac{2\sqrt{d_1}}{1+\operatorname{sech}^2 d_1^{\frac{1}{4}}\left(\left((4c_4)^{\frac{1}{3}}\xi-\xi_0\right)\right)}-\sqrt{d_1}\right)\right\}^{1/2} e^{i(-hx+\omega t+\iota)}. \quad (31)$$

In this section, we recovered the soliton solutions $u(x,t)$. Through relationship (6), we can directly obtain the solutions $v(x,t)$, so the solutions for $v(x,t)$ are omitted here. Surface, contour, and 2D plots in Figs. 1, 2, and 3 showcase the dark soliton solution (22). These visual representations offer a comprehensive and insightful overview of how the dark soliton solution behaves within the specified parameters, contributing to the understanding and analyzing these complex mathematical phenomena. The parameters used in these simulations are as follows: $\xi_0=1$, $c_4=1$, $d_2=-1$, $k=1$, $h=1$, $a^{(1)}=1$, $\delta_1^{(1)}=1$, $\delta_2^{(1)}=1$, $\delta_3^{(1)}=1$, $\sigma_1^{(1)}=1$, $\sigma_3^{(1)}=1$, and $\sigma_9^{(1)}=1$.

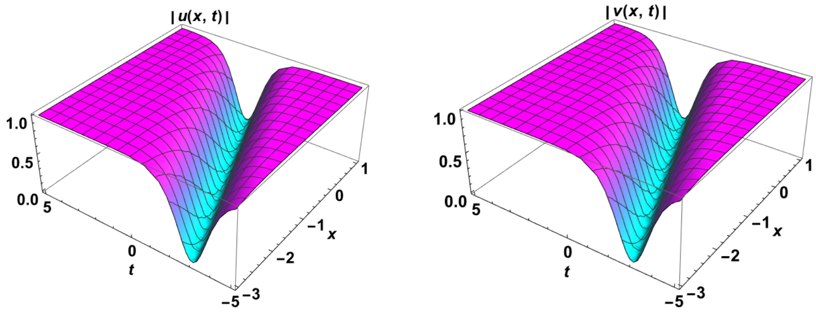

Fig. 1. Surface plots of dark solitons in birefringent fibers.

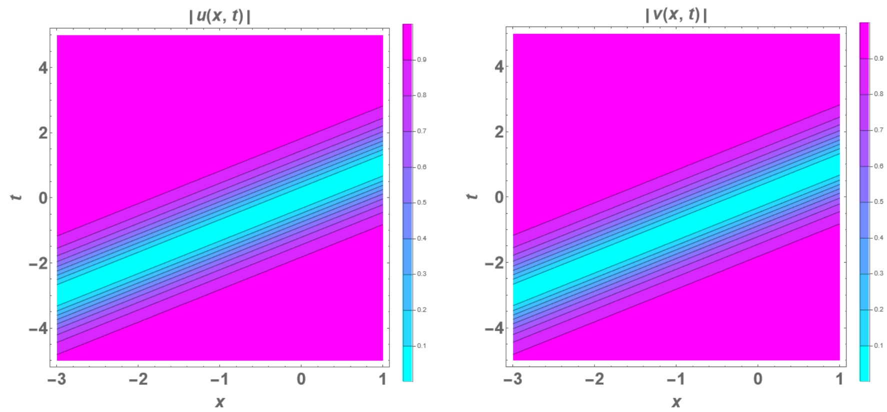

Fig. 2. Contour plots of dark solitons in birefringent fibers.

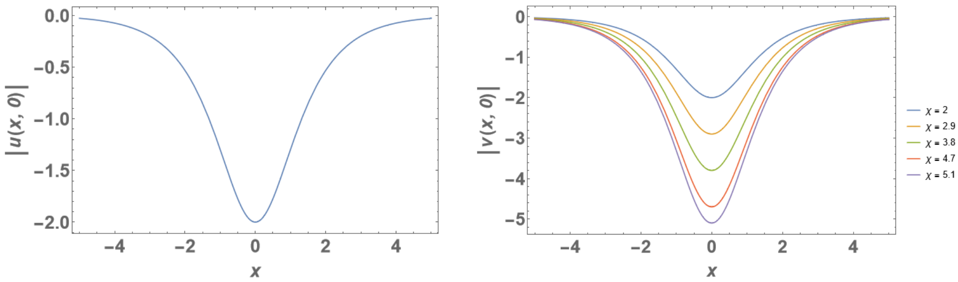

Fig. 3. 2D plots of dark solitons in birefringent fibers.

4. Conclusions

The current paper recovered optical soliton solutions to the dispersive concatenation model with polarization–mode dispersion. The complete discriminant approach was to the rescue. We ignore the emergence of singular periodic solutions and plane waves due to alterations in the discriminant's signs [22-26], as they hold no significance in optoelectronics. Thus, from the optics perspective, a complete spectrum of optical solitons has been recovered using the complete discriminant approach and is being reported in this paper. These solutions are fundamental in carrying out further future investigations of the model. Later, this model will be applied to retrieve gap solitons with fiber Bragg gratings and quiescent optical solitons for nonlinear chromatic dispersion. Later, the model will be numerically addressed using the Laplace–Adomian decomposition approach and/or variational iteration method. The results of such research activities will be sequentially disseminated after they are all connected with the pre-existing ones [22–26].

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Анотація. *У цій статті отримані розв'язки оптичних солітонів для моделі дисперсійної конкатенації з дисперсією поляризованої моди. Отримання результатів стало можливим завдяки використанню повного дискримінантного підходу. Проміжні еліптичні функції Якобі поступилися місцем солітонним розв'язкам із застосуванням до них граничних умов. В роботі класифіковані і зображені ці солітони.*

Ключові слова*: модель конкатенації, солітони, дисперсія, обмеження параметрів*