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Chirped cosh–Gaussian Optical Pulses with Kudryashov's Form of Self-Phase Modulation by Variational Principle

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Abstract. This research work brings forth the understanding of optical cosh-Gaussian dynamics by incorporating the newly formulated Kudryashov equation. The method used is Anderson's variational approach. We show that an appropriate choice of the trial wavefunction allows highlighting several varieties of stable and unstable solutions corresponding to non-dissipative or dissipative spatiotemporal solitons, propagating without deformation and with deformation.

Keywords: solitons, Kudryashov equation, variational approach

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1. Introduction

In telecommunications, the different models of the Schrödinger equation have succeeded in solving the challenges of the propagation dynamics of ultrashort pulses in fibers for the needs of high-speed transmission, which has thus led to a considerable improvement in the technology. Fiber optic telecommunications networks have become an indispensable and integral part of society. The ability of fiber optic transmission systems to handle short-circuit drops is not attributed to attenuation, dispersion, noise, or non-linear phenomena during signal propagation. Instead, it stems from their robust design and redundancy measures, which mitigate the impact of diverse catastrophic effects. Fiber optic systems are engineered to withstand such disruptions, leveraging their significant capacity and resilience to ensure uninterrupted signal transmission. Optical solitons emerged in the 1990s as an extremely promising technique for transmitting very high data rates over very long distances over optical fibers. One of the pioneering research areas in telecommunications and nonlinear optics is the theory of optical solitons, extensively covered in references [1-11]. The study of optical solitons in metamaterials and optical fibers has been intensely explored, and innovative concepts have emerged. New forms of nonlinear refractive indices have been introduced [1-5]. As detailed in references [3-5], the Kudryashov equation is employed to model a newly proposed refractive index law governing soliton propagation over intercontinental distances. While this proposition is theoretical, there is an anticipation of pending experimental results for a refractive index following this form. This model has so far enjoyed sustained success from a theoretical point of view. The research focused on the Kudryashov equation serves a dual purpose, contributing to fundamental knowledge and the advancement of components dedicated to processing optical pulses for ultra-high-speed telecommunications using purely optical means. Soliton-type solutions have been intensively studied in recent years.

In comparison, the study of Kudryashov equation is still in its infancy. The present study focuses on this equation to obtain cosh-Gaussian soliton solutions. We show that a variational approach built on a suitably chosen trial function can quickly achieve our goals. This approach consists of reducing the problem of the evolution of the impulse field to a simple problem of the dynamics of a mechanical system governed by a restricted number of degrees of freedom, which are associated with the fundamental parameters of the impulse.

2. Governing model

The formulation of the Kudryashov model in its dimensionless form is presented in references [1-5]:

$$iq_z + aq_{tt} + \left(\frac{b_1}{|q|^{2n}} + \frac{b_2}{|q|^n} + b_3|q|^n + b_4|q|^{2n} \right) q = 0, \quad \forall n \in \mathbb{N}. \tag{1}$$

In Eq. (1), the initial term represents linear temporal evolution, where a denotes the coefficient of group velocity dispersion. The subsequent four terms introduce nonlinearity, originating from the refractive index law of an optical fiber and contributing to self-phase modulation within the model. Specifically, when b_1 and b_2 are set to zero, the model simplifies to the dual-power law of refractive index. Also, when $b_1 = b_2 = b_3 = 0$, the model reverts to the power law. Therefore, Kudryashov model serves as an extension encompassing these well-known forms of refractive index. The instances where n=1 in the described cases are commonly recognized as parabolic and fundamental Kerr laws, respectively. These designations further characterize and distinguish the behavior of the refractive index in the context of the model, highlighting the versatility and broad applicability of Kudryashov formulation in capturing various optical phenomena.

3. Solving the Kudryashov's equation by the variational approach

By extending Euler-Lagrange's least-action principles to dissipative systems, Anderson's method serves as the basis for expressing the generalized nonlinear Schrödinger equation (NLSE) in terms of essential parameters, often denoted as collective variables. This application is elucidated in references [6-8].

Turning our attention to Eq. (1), we will explore the complex form in the search for the solution q:

$$q(z,t)=u(z,t)+iv(z,t). \tag{2}$$

Let u and v denote real functions. Substituting the expression from (2) into Eq. (1) yields:

$$-v_z+au_{tt}+u[b_1(u^2+v^2)^{-n}+b_2(u^2+v^2)^{-\frac{n}{2}}+b_3(u^2+v^2)^{\frac{n}{2}}+b_4(u^2+v^2)^n]=0, \tag{3}$$

$$u_z+av_{tt}+v[b_1(u^2+v^2)^{-n}+b_2(u^2+v^2)^{-\frac{n}{2}}+b_3(u^2+v^2)^{\frac{n}{2}}+b_4(u^2+v^2)^n]=0. \tag{4}$$

By using the variational method described in [10], the Lagrangian L_0 can be rewritten as follows:

$$\begin{aligned} L_0=\frac{i}{4}(q_zq^*-q_x^*q)+\frac{b_1}{-2n+2}\Big|q\,|^{-2n+2}+\frac{b_2}{-n+2}\Big|q\,|^{-n+2} \\ +\frac{b_3}{n+2}\Big|q\,|^{n+2}+\frac{b_4}{2n+2}\Big|q\,|^{2n+2}-\frac{a}{2}|\,q_t\,|^2,\ \forall n\notin\{1;2\}. \end{aligned} \tag{5}$$

3.1. cosh-Gaussian soliton parameters dynamics

Therefore, the averaged Lagrangian of Eq. (1) is defined as:

$$L_g=\int_{-\infty}^{+\infty}L_0dt. \tag{6}$$

The variational method involves an optimization procedure that requires a suitable trial function. Consider the soliton solution of (1) represented as follows:

$$\begin{aligned} q(X_J,t)=X_1f\left(\frac{t-X_2}{X_3}\right) \\ \times\exp\left\{i\left[\frac{X_4}{2}(t-X_2)^2+X_5(t-X_2)+X_6\right]\right\},\ J=(1,2\ldots6), \end{aligned} \tag{7}$$

Where X_1 stands for the amplitude of the pulse, X_2 for the center, X_3 for the width, X_4 for the chirp, X_5 for the frequency, and X_6 for the phase, all of which are functions of z. The function $f(\tau)=\cosh\left(\frac{\tau}{\Omega}\right)\exp(-\tau^2)$ describes the pulse shape in our calculations, where Ω is a parameter. According to [6-8], the integral is defined in the Lagrangian variational algorithm:

$$\varphi_{i,j,k}=\int_{-\infty}^{+\infty}\tau^if^i(\tau)\left(\frac{\partial f}{\partial\tau}\right)^kdt. \tag{8}$$

By substituting (7) in (5); (6) becomes:

$$\begin{aligned} L_g=\frac{1}{2}X_1^2(sX_3^2X_4+pX_3X_5)\dot{X}_2-\frac{1}{4}uX_1^2X_3^3\dot{X}_4-\frac{1}{2}sX_1^2X_3^2\dot{X}_5 \\ -\frac{1}{2}pX_1^2X_3\dot{X}_6+\frac{b_1c_{2n}}{-2n+2}X_1^{-2n+2}X_3 \\ +\frac{b_2c_n}{-n+2}X_1^{-n+2}X_3+\frac{b_3a_n}{n+2}X_1^{n+2}X_3+\frac{b_4a_{2n}}{2n+2}X_1^{2n+2}X_3 \\ -\frac{1}{2}\frac{aX_1^2(r+pX_3^2X_5^2+2sX_3^3X_4X_5+uX_3^4X_4^2)}{X_3} \end{aligned} \tag{9}$$

where $s=\varphi_{1,2,0}$, $p=\varphi_{0,2,0}$, $u=\varphi_{2,2,0}$, $r=\varphi_{0,0,2}$, $c_{2n}=\varphi_{0,-2n+2,0}$, $c_n=\varphi_{0,-n+2,0}$, $a_n=\varphi_{0,n+2,0}$, $a_{2n}=\varphi_{0,2n+2,0}$. A modification is made to the Euler-Lagrange equation, yielding:

$$\frac{\partial L_g}{\partial X_J(z)}-\frac{d}{dz}\frac{\partial L_g}{\partial \dot{X}_J(z)}=0. \tag{10}$$

Inserting (9) in (10) allowed us to have the X_J derivatives with respect to z. The obtained variational equations are applicable only when $\forall n \notin \{1;2\}$

$$\left[\dot{X}_j(n)\right]=\left[B(n)\right]\text{with} n \notin \{1;2\} \tag{11}$$

$\left[\dot{X}_j(n)\right]$ and $\left[B(n)\right]$ are column vectors shown in Appendix A. The different expressions are shown in Appendix A.

3.2. Study of particular cases for $n=1$ and $n=2$

3.2.1. Particular case n=1. In this case, the Eq. (1) becomes:

$$iq_z + aq_{tt} + \left(\frac{b_1}{|q|^2}+\frac{b_2}{|q|}+b_3|q|+b_4|q|^2\right)q=0. \tag{12}$$

By using the variational method described in [5], and taken up in sections 3 and 3.1, the average Lagrangian is given by the following expression:

$$\begin{aligned} L_g &= \frac{1}{2}X_1^2\left(sX_3^2X_4+pX_3X_5\right)\dot{X}_2-\frac{1}{4}uX_1^2X_3^3\dot{X}_4 \\ &-\frac{1}{2}sX_1^2X_3^2\dot{X}_5-\frac{1}{2}pX_1^2X_3\dot{X}_6+b_1\left(v\ln\left(X_1\right)+hX_3\right) \\ &+b_2cX_1X_3+\frac{1}{3}b_3dX_1^3X_3+\frac{1}{4}b_4kX_1^4X_3 \\ &-\frac{1}{2}\frac{aX_1^2\left(r+pX_3^2X_5^2+2sX_3^3X_4X_5+uX_3^4X_4^2\right)}{X_3}, \end{aligned} \tag{13}$$

where $s=\varphi_{1,2,0}$, $p=\varphi_{0,2,0}$, $u=\varphi_{2,2,0}$, $r=\varphi_{0,0,2}$, $v=\varphi_{0,0,0}$, $c=\varphi_{0,1,0}$, $d=\varphi_{0,3,0}$, $k=\varphi_{0,4,0}$ and $h=\int_{-\infty}^{+\infty}\ln\left(f(\tau)\right)dt$. Therefore, we derive the following set of variational equations, applicable only when $n=1$.

$$\left[\dot{X}_j(1)\right]=\left[B(1)\right]\text{with} n=1. \tag{14}$$

$\left[\dot{X}_j(1)\right]$ and $\left[B(1)\right]$ are column vectors shown in Appendix B. Appendix B presents the variational equations associated with this particular case.

3.2.2. Particular case n=2. In this case, the Eq. (1) becomes:

$$iq_z + aq_{tt} + \left(\frac{b_1}{|q|^4}+\frac{b_2}{|q|^2}+b_3|q|^2+b_4|q|^4\right)q=0. \tag{15}$$

The Lagrangian $\int_{-\infty}^{+\infty}\ln\left(f(\tau)\right)dt$ is subject to a rewriting process through the use of the variational method described in [5]:

$$\begin{aligned} L_0 &= \frac{i}{4}\left(q_zq^*-q_x^*q\right)-\frac{b_1}{2}\frac{1}{|q|^2}+b_2 ln\left(|q|\right) \\ &+\frac{b_3}{4}\left|q\right|^4+\frac{b_4}{6}\left|q\right|^4-\frac{a}{2}|q_t|^2. \end{aligned} \tag{16}$$

The average Lagrangian gives:

$$\begin{aligned} L_g =& \frac{1}{2}X_1^2\left(sX_3^2X_4 + pX_3X_5\right)\dot{X}_2 - \frac{1}{4}uX_1^2X_3^3\dot{X}_4 \\ &-\frac{1}{2}sX_1^2X_3^2\dot{X}_5 - \frac{1}{2}pX_1^2X_3\dot{X}_6 - \frac{1}{2}\frac{b_1c_2X_3}{X_1^2} \\ &+b_2\left(v\ln\left(X_1\right) + hX_3\right) + \frac{1}{4}b_3kX_1^4X_3 + \frac{1}{6}b_4mX_1^6X_3 \\ &-\frac{1}{2}\frac{aX_1^2\left(r + pX_3^2X_5^2 + 2sX_3^3X_4X_5 + uX_3^4X_4^2\right)}{X_3}, \end{aligned} \tag{17}$$

where $s=\varphi_{1,2,0}$, $p=\varphi_{0,2,0}$, $u=\varphi_{2,2,0}$, $r=\varphi_{0,0,2}$, $v=\varphi_{0,0,0}$, $c_2=\varphi_{0,-2,0}$, $k=\varphi_{0,4,0}$, $m=\varphi_{0,6,0}$ and $\int_{-\infty}^{+\infty}\ln\left(f\left(\tau\right)\right)dt$. The Euler-Lagrange equation is modified to:

$$\frac{\partial L_g}{\partial X_J\left(z\right)} - \frac{d}{dz}\frac{\partial L_g}{\partial \dot{X}_J\left(z\right)} = 0. \tag{18}$$

Inserting (17) into (18) allowed us to have the X_J derivatives with respect to z. For the condition $n=2$, the resulting set of variational equations is applicable.

$$\left[\dot{X}_j(2)\right]=\left[B(2)\right] \text{with} n=2. \tag{19}$$

$\left[\dot{X}_j(2)\right]$ and $\left[B(2)\right]$ are column vectors shown in Appendix C. Appendix C presents the variational equations associated with this particular case.

4. Numerical simulation of cosh-Gaussian soliton dynamics for the particular case n=2

In this section, the model is developed using a cosh-Gaussian pulse defined as follows:

$$f\left(\tau\right) = \cosh\left(\frac{\tau}{\Omega}\right)\exp\left(-\tau^2\right) \quad \text{with} \quad \Omega > 0. \tag{20}$$

Numerical results are presented for specific cases, and the corresponding integrals are detailed in Table 1 in Appendix D. The nonlinear terms arising from the refractive index law of an optical fiber, contributing to self-phase modulation in the model, are highlighted in Eq. (1). If $b_1=b_2=b_3=0$, the model reverts to the power law in this scenario. A numerical study was conducted to examine the dynamics of the cosh-Gaussian within a metamaterial. This study focused on the evolution of various parameters under conditions $b_1=b_2=b_3=0$ and $b_1=b_2=0$, aiming to understand the influence of the generally parabolic law's nonlinearity terms. The integration of systems of ordinary differential equations was carried out using the standard fourth-order Runge-Kutta method, resulting in various outcomes. Figs. 1 and 2 display the system dynamics for the hyperbolic cosine, with initial parameters specified as $\left[X_1, X_2, X_3, X_4, X_5, X_6\right]=\left[0.18; 0; 4.45; 2; 1; 0\right]$.

Under condition $b_1=b_2=0$, the model transforms into the double power law of the refractive index. A detailed analysis of this curve in these circumstances indicates periodic variations in amplitude, pulse width, and chirp relative to *z*. The significance of the initial condition choice should not be overlooked in such studies. The selected parameters are tailored to enable the unattenuated propagation of the cosh pulse. In Fig. 1, the representation spans 400 m of propagation, as the behavior remains constant beyond that distance. For the condition $b_1 = b_2 = b_3 = 0$, the model gives way to the power law. This positions Kudryashov model as an extension of these established forms of refractive index. Under these circumstances, the space exhibits aperiodic characteristics, and the term $b_3 = 0$ functions as a dissipative factor. In Fig. 2, a single period of propagation is evident, and the representation covers 1000 m to emphasize the non-existence of periodicity. The variational equations X_1, X_3, X_4, X_5 obtained are identical to those in [5]. We notice a periodicity in the shape of these parameters [5].

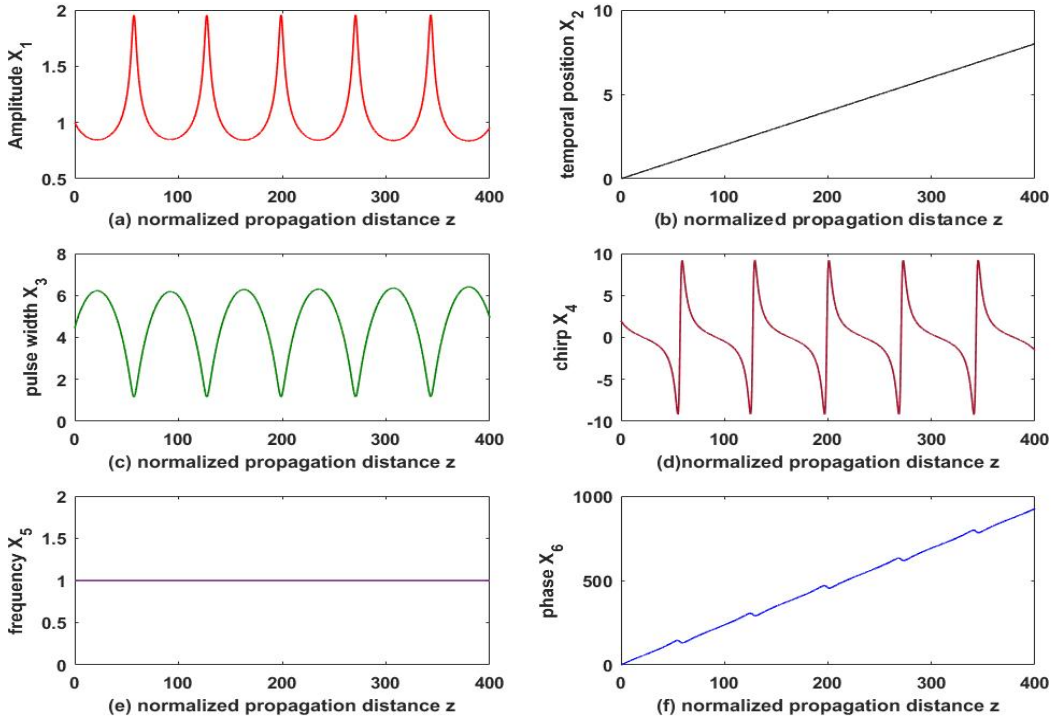

Fig. 1. Variation of normalized pulse parameters (X_1-amplitude of cosh, X_2-center position of the cosh, X_3- cosh pulse width, X_4-cosh chirp, X_5-cosh frequency, X_6-cosh phase) with propagation distance with $n = 2$, $a = 0{,}1$, $b_1 = 0$, $b_2 = 0$, $b_3 = 7$, $b_4 = -1$.

Let's study the particular case where $n \neq \{1;2\}$ and consider without chirped pulse so $X_4 = 0$. The solution (7) is given by:

$$q\left(X_J, t\right) = X_1 f\left(\frac{t - X_2}{X_3}\right) \exp\left\{i\left[X_5\left(t - X_2\right) + X_6\right]\right\}, J = \left(1,2 ... 6\right), \tag{21}$$

and its modulus is given by:

$$\left|q\left(X_J, t\right)\right| = \left|X_1 f\left(\frac{t - X_2}{X_3}\right)\right|. \tag{22}$$

This case, under the conditions of the nonlinearity power law, i.e. $b_3 = b_2 = b_1 = 0$, the variational equations thus associated are deduced from Appendix D and are as follows for the parameters so we need for the representation of $\left|q\left(X_J,t\right)\right|$

$$\begin{cases}\dot{X}_1=0\\ \dot{X}_2=2aX_5\\ \dot{X}_3=0\\ \dot{X}_4=0\\ \dot{X}_5=0\end{cases}\Rightarrow\begin{cases}X_1=cst\\ X_2=2aX_5{*}z\\ X_3=cst\\ X_4=0\\ X_5=cst\end{cases}\qquad(23)$$

Appendix D is obtained from variational equations presented in appendice A when $b_1=b_2=b_3=0$.

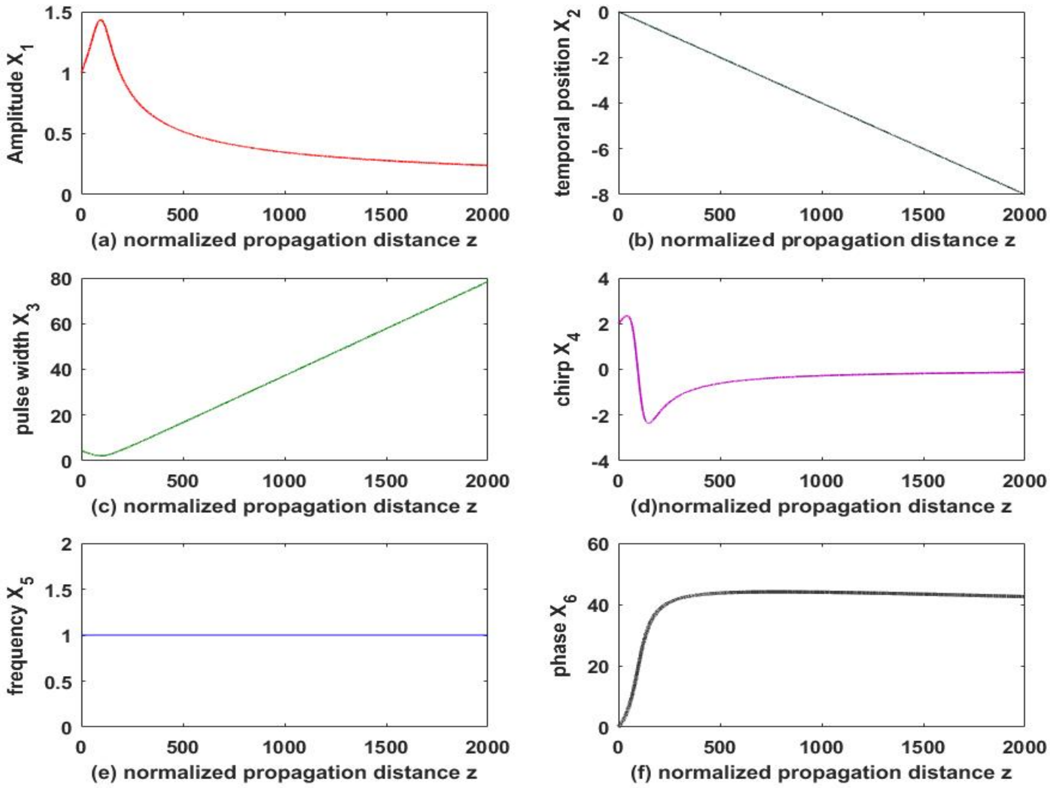

Fig. 2. Examining the impact of propagation distance, the normalized pulse parameters (X_1 - amplitude of cosh, X_2 - center position of cosh, X_3 - cosh pulse width, X_4 - cosh chirp, X_5 - cosh frequency, X_6 - cosh phase) change with respect to the given parameters: $n=2$, $a=-0.01$, $b_1=b_2=b_3=0$, and $b_4=0.2$.

Fig. 2 provides a few plots of this cosh-Gaussian without chirp pulse with the governing model (1) under the conditions of the non-linearity power law, i.e., $b_1=b_2=b_3=0$. This case has been studied by Elsayed et al. [4]. The important information on the behavior and the characteristics of the envelope of the studied pulse are presented on the curves below. Using the parameters: $a=1$, $X_1=0.18$; $X_2=2z$; $X_3=4.48$; $X_5=1$. Fig. 2 shows the typical soliton propagation along an optical fiber. Fig. 2c shows that during propagation, the energy contained in the wave is concentrated in the fiber's core. We find a good agreement between

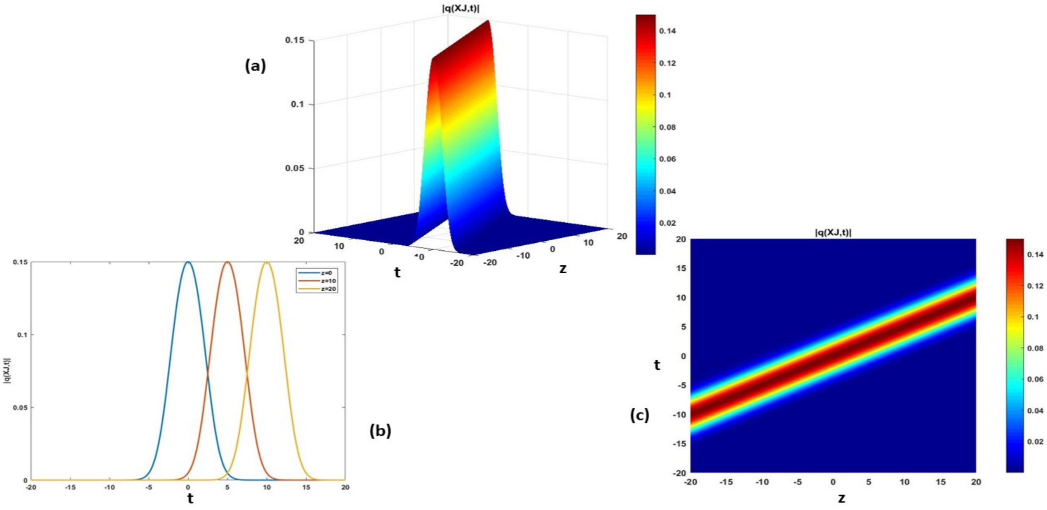

Fig. 3. A depiction of a cosh-Gaussian pulse profile using: (a) a surface plot, (b) 2D plots changing over time, and (c) a contour plot.

our results and those obtained by Elsayed et al. [4]. At the end of our study, we highlighted two types of solutions for the Kudryashov equation using a simple method that requires little calculation time. The first solution (Fig. 1) is a typical example of a non-dissipative system. The cosh-Gaussian pulse propagates without deformation in the fiber. It is a soliton-type solution. Fig. 2 highlights a very dissipative solution. The system exhibits aperiodic behavior. Our study made it possible to highlight the impact of the nonlinear term b_3 responsible for the energy loss observed at the level of the second solution. The curves in Fig. 3 allowed us to validate our method.

5. Conclusion

This study introduces a Lagrangian approach to analyze the dynamics of optical cosh-Gaussian solitons within optical metamaterials. The behavior of these solitons is described by the dimensionless form of the Kudryashov model, which is then solved using a Lagrangian approach. This method employed a six-parameter trial wavefunction for optical cosh-Gaussian solitons, encompassing amplitude, center position, pulse width, chirp, frequency, and phase to approximate the precise solution. The potential applications of this work extend to the field of telecommunications, where it could be employed to optimize information transmission. Numerical simulations were conducted to visually represent the evolution of these soliton parameters over varying propagation distances. The critical finding of this investigation underscores the pivotal role of the initial conditions in shaping the study's outcomes. Remarkably, a comparative analysis with other results demonstrates excellent agreement.

Appendix A.

$$\dot{X}_1 = -aX_1X_4\,,\ \dot{X}_2 = 2aX_5\,,\ \dot{X}_3 = 2aX_3X_4\,, \tag{A1}$$

$$\dot{X}_4 = \frac{-p}{\left(pu - s^2\right)X_1X_3^3}$$
$$\times[b_1c_{2n}X_1^{-2n+1}X_3 + b_2c_nX_1^{-n+1}X_3 + b_3a_{2n}X_1^{n+1}X_3 + b_4a_{2n}X_1^{2n+1}X_3$$

$$-\frac{aX_1\left(r+pX_3^2X_5^2+2sX_3^3X_4X_5+uX_3^4X_4^2\right)}{X_3}\Bigg]+\frac{2p}{\left(pu-s^2\right)X_1^2X_3^2}$$

$$\times\left[\begin{array}{l}\dfrac{b_1c_{2n}X_1^{-2n+2}}{-2n+2}+\dfrac{b_2c_nX_1^{-n+2}}{-n+2}+\dfrac{b_3a_nX_1^{n+2}}{n+2}+\dfrac{b_4a_{2n}X_1^{2n+2}}{2n+2}\\ -\dfrac{a}{2}\dfrac{X_1^2\left(2pX_3X_5^2+6sX_3^2X_4X_5+4uX_3^4X_4^2\right)}{X_3}\\ +\dfrac{a}{2}\dfrac{X_1^2\left(r+pX_3^2X_5^2+2sX_3^3X_4X_5+uX_3^4X_4^2\right)}{X_3^2}\end{array}\right] \tag{A2}$$

$$+\frac{2as\left(pu-s^2\right)}{p^2u^2-2pus^2+s^4}\frac{\left(sX_3^3X_4+pX_3^2X_5\right)X_4}{X_3^3},$$

$$\dot{X}_5=\frac{s}{\left(pu-s^2\right)X_1X_3^2}\times[b_1c_{2n}X_1^{-2n+1}X_3+b_2c_nX_1^{-n+1}X_3+b_3a_{2n}X_1^{n+1}X_3$$

$$+b_4a_{2n}X_1^{2n+1}X_3-\frac{aX_1\left(r+pX_3^2X_5^2+2sX_3^3X_4X_5+uX_3^4X_4^2\right)}{X_3}\Bigg]-\frac{2s}{\left(pu-s^2\right)X_1^2X_3}$$

$$\times\left[\begin{array}{l}\dfrac{b_1c_{2n}X_1^{-2n+2}}{-2n+2}+\dfrac{b_2c_nX_1^{-n+2}}{-n+2}+\dfrac{b_3a_nX_1^{n+2}}{n+2}+\dfrac{b_4a_{2n}X_1^{2n+2}}{2n+2}\\ -\dfrac{a}{2}\dfrac{X_1^2\left(2pX_3X_5^2+6sX_3^2X_4X_5+4uX_3^3X_4^2\right)}{X_3}\\ +\dfrac{a}{2}\dfrac{X_1^2\left(r+pX_3^2X_5^2+2sX_3^3X_4X_5+uX_3^4X_4^2\right)}{X_3^2}\end{array}\right] \tag{A3}$$

$$+\frac{2as\left(pu-s^2\right)}{p^2u^2-2pus^2+s^4}\frac{\left(uX_3^4X_4+sX_3^3X_5\right)X_4}{X_3^2},$$

$$\dot{X}_6=\frac{1}{2}\frac{3pu-4s^2}{p\left(pu-s^2\right)X_1X_3}$$

$$\times\left[\begin{array}{l}b_1c_{2n}X_1^{-2n+1}X_3+b_2c_nX_1^{-n+1}X_3+b_3a_{2n}X_1^{n+1}X_3\\ +b_4a_{2n}X_1^{2n+1}X_3-\dfrac{aX_1\left(r+pX_3^2X_5^2+2sX_3^3X_4X_5+uX_3^4X_4^2\right)}{X_3}\end{array}\right]$$

$$-\frac{2as\left(pu-s^2\right)}{p^2u^2-2pus^2+s^4}\frac{\left(uX_3^4X_4+sX_3^3X_5\right)X_5}{X_3^3}+\frac{2as\left(pu-s^2\right)}{\left(p^2u^2-2pus^2+s^4\right)}\frac{\left(sX_3^3X_4+pX_3^2X_5\right)X_4}{X_3^2}$$

$$\times\left[\left(psu-s^3\right)X_3X_4+2\left(p^2u-ps^2\right)X_5\right]-\frac{pu-2s^2}{p\left(pu-s^2\right)X_1^2X_3} \tag{A4}$$

$$\times\left[\begin{array}{l}\dfrac{b_1c_{2n}X_1^{-2n+2}}{-2n+2}+\dfrac{b_2c_nX_1^{-n+2}}{-n+2}+\dfrac{b_3a_nX_1^{n+2}}{n+2}+\dfrac{b_4a_{2n}X_1^{2n+2}}{2n+2}\\ -\dfrac{a}{2}\dfrac{X_1^2\left(2pX_3X_5^2+6sX_3^2X_4X_5+4uX_3^3X_4^2\right)}{X_3}\\ +\dfrac{a}{2}\dfrac{X_1^2\left(r+pX_3^2X_5^2+2sX_3^3X_4X_5+uX_3^4X_4^2\right)}{X_3^2}\Bigg]\end{array}\right].$$

Appendix B. $n=1$

$$\dot{X}_1=-aX_1X_4\,,\ \dot{X}_2=2aX_5\,,\ \dot{X}_3=2aX_3X_4\,, \tag{B1}$$

$$\dot{X}_4=\frac{1}{6}\frac{1}{(pu-s^2)X_1^2X_3^4}[-12a(pu-s^2)X_1^2X_3^4X_4^3-3b_4kpX_1^4X_3^2$$
$$-2b_3dpX_1^3X_3^2+6b_2cpX_1X_3^2+12aprX_1^2+12b_1phX_3^2-6b_1pvX_3], \tag{B2}$$
$$\dot{X}_5=-\frac{1}{6}\frac{s}{(pu-s^2)X_1^2X_3^3}\begin{bmatrix}-3b_4kX_1^4X_3^2-2b_3dX_1^3X_3^2+6b_2cX_1X_3^2\\+12arX_1^2+12b_1hX_3^2-6b_1vX_3\end{bmatrix},$$

$$\dot{X}_6=\frac{1}{12}\frac{1}{p(pu-s^2)X_1^2X_3^2}$$
$$\times[-12ap(pu-s^2)X_1^2X_3^2X_5^2+3(5pu-6s^2)b_4kX_1^4X_3^2+2(7pu-8s^2)b_3dX_1^3X_3^2 \tag{B3}$$
$$+6b_2cpuX_1X_3^2-12(2pu-3s^2)arX_1^2-12(pu-2s^2)b_1hX_3^2+6(3pu-4s^2)b_1vX_3].$$

Appendix C. $n=2$

$$\dot{X}_1=-aX_1X_4\,,\ \dot{X}_2=2aX_5\,,\ \dot{X}_3=2aX_3X_4\,, \tag{C1}$$

$$\dot{X}_4=-\frac{1}{6}\frac{1}{(pu-s^2)X_1^4X_3^4}$$
$$\times[12a(pu-s^2)X_1^4X_3^4X_4^2+3b_3kpX_1^6X_3^2+4b_4mpX_1^8X_3^2 \tag{C2}$$
$$+12b_1c_2pX_3^2-12aprX_1^4-12b_2hpX_1^2X_3^2+6b_2vpX_1^2X_3],$$

$$\dot{X}_5=\frac{1}{6}\frac{s}{(pu-s^2)X_1^4X_3^3}$$
$$\times\left[4b_4mX_1^8X_3^2+3b_3kX_1^6X_3^2+12b_1c_2X_3^2-12arX_1^4-12b_2hX_1^2X_3^2+6b_2vX_1^2X_3\right], \tag{C3}$$

$$\dot{X}_6=\frac{1}{12}\frac{1}{p(pu-s^2)X_1^4X_3^2}[12ap(pu-s^2)X_1^4X_3^2X_5^2+4(4pu-5s^2)b_4mX_1^8X_3^2$$
$$+3(5pu-6s^2)b_3kX_1^6X_3^2+24b_1c_2puX_3^2 \tag{C4}$$
$$-12(2pu-3s^2)arX_1^4-12(pu-2s^2)b_2hX_1^2X_3^2$$
$$+6(3pu-4s^2)b_2vX_1^2X_3].$$

Appendix D. $n\neq\{1;2\}$, $b_1=b_2=b_3=0$

$$\dot{X}_1=-aX_1X_4\,,\ \dot{X}_2=2aX_5\,,\ \dot{X}_3=2aX_3X_4\,, \tag{D1}$$

$$\dot{X}_4=\frac{p}{(pu-s^2)X_1X_3^3}$$
$$\times\left[-\frac{b_4a_{2n2}X_1^{2n+2}X_3}{X_1}+\frac{aX_1\left(uX_3^4X_4^2+2sX_3^3X_4X_5+pX_3^2X_5^2+r\right)}{X_3}\right]$$
$$+\frac{asX_4\left(2sX_3^3X_4+2pX_3^2X_5\right)}{(pu-s^2)X_3^3}-\frac{2p}{(pu-s^2)X_1^2X_3^2} \tag{D2}$$
$$\times\begin{bmatrix}-\frac{b_4a_{2n2}X_1^{2n+2}}{2n+2}+\frac{1}{2}\frac{aX_1^2\left(4uX_3^3X_4^2+6sX_3^2X_4X_5+2pX_3X_5^2\right)}{X_3}\\-\frac{1}{2}\frac{aX_1^2\left(uX_3^4X_4^2+2sX_3^3X_4X_5+pX_3^2X_5^2+r\right)}{X_3^2}\end{bmatrix},$$

$$\dot{X}_5 = -\frac{s}{\left(pu - s^2\right) X_1 X_3^2} \times \left[-\frac{b_4 a_{2n2} X_1^{2n+2} X_3}{X_1} + \frac{aX_1\left(uX_3^4 X_4^2 + 2sX_3^3 X_4 X_5 + pX_3^2 X_5^2 + r\right)}{X_3} \right] + \frac{asX_4\left(2sX_3^3 X_4 + 2pX_3^2 X_5\right)}{\left(pu - s^2\right) X_3^3} - \frac{2p}{\left(pu - s^2\right) X_1^2 X_3^2} \times \left[\begin{array}{l} -\frac{b_4 a_{2n2} X_1^{2n+2}}{2n+2} + \frac{1}{2}\frac{aX_1^2\left(4uX_3^3 X_4^2 + 6sX_3^2 X_4 X_5 + 2pX_3 X_5^2\right)}{X_3} \\ -\frac{1}{2}\frac{aX_1^2\left(uX_3^4 X_4^2 + 2sX_3^3 X_4 X_5 + pX_3^2 X_5^2 + r\right)}{X_3^2} \end{array} \right]. \quad \text{(D3)}$$

$$\dot{X}_6 = -\frac{1}{2}\frac{\left(3pu - 4s^2\right)}{p\left(pu - s^2\right) X_1 X_3} \times \left[-\frac{b_4 a_{2n+2} X_1^{2n+2} X_3}{X_1} + \frac{aX_1\left(uX_3^4 X_4^2 + 2sX_3^3 X_4 X_5 + pX_3^2 X_5^2 + r\right)}{X_3} \right] + \frac{\left(pu - 2s^2\right)}{p\left(pu - s^2\right) X_1^2} \times \left[\begin{array}{l} -\frac{b_4 a_{2n+2} X_1^{2n+2}}{2n+2} + \frac{1}{2}\frac{aX_1^2\left(4uX_3^3 X_4^2 + 6sX_3^2 X_4 X_5 + 2pX_3 X_5^2\right)}{X_3} \\ -\frac{1}{2}\frac{aX_1^2\left(uX_3^4 X_4^2 + 2sX_3^3 X_4 X_5 + pX_3^2 X_5^2 + r\right)}{X_3^2} \end{array} \right] - \frac{asX_5\left(2uX_3^4 X_4^2 + 2sX_3^3 X_5\right)}{\left(pu - s^2\right) X_3^3} + \frac{1}{2}\frac{au\left(sX_3 X_4 + 2pX_5\right)\left(2sX_3^3 X_4 + 2pX_3^2 X_5\right)}{p\left(pu - s^2\right) X_3^2}. \quad \text{(D4)}$$

Appendix E.

Table E1. Values of different integrals $\varphi_{i,j,k}$ for cosh pulses from $\forall n \in \mathbb{N}$

$\varphi_{i,j,k}$	cosh
1	2
$\varphi_{0,1,0}$	$\sqrt{\pi}\exp\left(\frac{1}{4\Omega^2}\right)$
$\varphi_{0,2,0}$	$\frac{1}{4}\exp\left(\frac{1}{2\Omega^2}\right)\sqrt{2\pi} + \frac{1}{4}\sqrt{2\pi}$
$\varphi_{0,3,0}$	$\frac{1}{12}\exp\left(\frac{3}{4\Omega^2}\right)\sqrt{3\pi} + \frac{1}{4}\exp\left(\frac{1}{12\Omega^2}\right)\sqrt{3\pi}$
$\varphi_{0,4,0}$	$\frac{1}{16}\exp\left(\frac{1}{\Omega^2}\right)\sqrt{\pi} + \frac{1}{4}\exp\left(\frac{1}{4\Omega^2}\right)\sqrt{\pi} + \frac{3}{16}\sqrt{\pi}$
$\varphi_{0,6,0}$	$\frac{1}{192}\exp\left(\frac{3}{2\Omega^2}\right)\sqrt{6\pi} + \frac{1}{32}\exp\left(\frac{2}{3\Omega^2}\right)\sqrt{6\pi} + \frac{5}{64}\exp\left(\frac{1}{6\Omega^2}\right)\sqrt{6\pi} + \frac{5}{96}\sqrt{6\pi}$
$\varphi_{0,-2,0}$	Divergent integral
$\varphi_{0,0,2}$	$\frac{1}{4}\frac{\sqrt{2\pi}}{\Omega^2}\left(\Omega^2\exp\left(\frac{1}{2\Omega^2}\right) + \Omega^2 - 1\right)$

1	2
$\varphi_{1,2,0}$	0
$\varphi_{2,2,0}$	$\frac{1}{16}\frac{\sqrt{2\pi}}{\Omega^2}\left(\Omega^2\exp\left(\frac{1}{2\Omega^2}\right)+\Omega^2+\exp\left(\frac{1}{2\Omega^2}\right)\right)$
$\varphi_{0,0,0}$	Divergent integral
$\varphi_{1,0,0}$	0
$\varphi_{2,0,0}$	Divergent integral
$\varphi_{0,2n+2,0}$	$\int_{-\infty}^{+\infty}\left(\cosh\left(\frac{\tau}{\Omega}\right)\exp\left(-z^2\right)\right)^{2n+2}d\tau$
$\varphi_{0,n+2,0}$	$\int_{-\infty}^{+\infty}\left(\cosh\left(\frac{\tau}{\Omega}\right)\exp\left(-z^2\right)\right)^{n+2}d\tau$
$\varphi_{0,-2n+2,0}$	Divergent integral
$\varphi_{0,-n+2,0}$	Divergent integral

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Анотація. *Ця робота відкриває розуміння оптичної динаміки cosh-Гаусса шляхом розв'язування нещодавно сформульованого рівняння Кудряшова. Використовується метод варіаційного підходу Андерсона. Показано, що відповідний вибір пробної хвильової функції дозволяє виділити декілька різновидів стійких і нестійких розв'язків, що відповідають недисипативним або дисипативним просторово-часовим солітонам, які поширюються без деформації та з деформацією,.*

Ключові слова: *солітони, рівняння Кудряшова, варіаційний підхід*