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Implicit Quiescent Optical Solitons with Generalized Quadratic–Cubic Form of Self–Phase Modulation and Nonlinear Chromatic Dispersion by Lie Symmetry

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Abstract. The current paper extracts the implicit form of quiescent optical solitons that emerge from the nonlinear Schrödinger's equation with the generalized form of quadratic–cubic nonlinear refractive index change. The work is with linear temporal evolution as well as with generalized temporal evolution. The results are in terms of Appell hypergeometric functions as in the case of the quadratic–cubic form of nonlinear refractive index, reported earlier. Lie symmetry analysis has made this retrieval possible.

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1. Introduction

The sustainment of optical soliton propagation through transcontinental and transoceanic distances is based on the maintenance of a delicate balance between chromatic dispersion (CD) and self–phase modulation (SPM). If the balance, however, gets compromised during the course of propagation of pulses through an optical fiber, the solitons get stalled, and thus, a catastrophic situation occurs. Therefore, it is imperative to maintain this balance throughout the pulse propagation. One of the sources to lose this balance is when the CD is rendered to be nonlinear during the pulse propagation. This can happen due to many sources, such as rough handling of fibers, random injection of pulses at the initial end of the fiber, random variation of the fiber diameter, and other such unwanted sources. The current paper will obtain the structure of a quiescent optical soliton that is yielded when the CD is nonlinear for the generalized quadratic–cubic form of SPM. This paper is a sequel to a previously studied work [1] where the governing nonlinear Schrödinger's equation (NLSE) was addressed with quadratic–cubic form of SPM.

The formation of such quiescent solitons and stalling the soliton propagation in its tracks were studied for a wide variety of models using Lie symmetry analysis. These are the concatenation model and the dispersive concatenation model, Sasa–Satsuma equation, NLSE with Kudryashov's form of SPM, complex Ginzburg–Landau equation, Lakshmanan–Porsezian–Daniel model, NLSE with a few of the non–Kerr laws of nonlinear refractive index including the logarithmic law [1–11]. The prequel of this work, which is with a quadratic–cubic form of nonlinear refractive index, was addressed in 2020 [1]. The results of this work are a generalized version of the previously reported one during 2020, where the quadratic–cubic nonlinearity was addressed. The concept of quiescent optical solitons and solitary waves was first studied in 2006, and subsequently, a deluge of results has started pouring in [12, 13]. The current paper applies Lie symmetry analysis and recovers the implicit quiescent optical solitons for the NLSE with nonlinear CD and the generalized version of the quadratic–cubic form of SPM. The temporal evolutions are taken to be linear and its generalized version as well. The details are exhibited in the rest of the paper.

2. Linear temporal evolution

The dimensionless form of the governing NLSE with nonlinear CD and generalized quadratic–cubic form of SPM is given as:

$$iq_t + a\left(|q|^n q\right)_{xx} + \left(b_1|q|^m + b_2|q|^{2m}\right)q = 0. \tag{1}$$

Here, in Eq. (1) $q(x,t)$ is the wave amplitude while the independent variables are x and t which account for the spatial and temporal variables respectively. The coefficients a and b_i (j=1, 2) are the coefficients of nonlinear CD and SPM, respectively, $i=\sqrt{-1}$ and iq_t represents the linear temporal evolution. The parameter n is the nonlinearity parameter for CD while m represents the nonlinearity parameter for SPM. If $m=1$, one recovers the usual quadratic–cubic form of SPM which has been studied during 2020 [1]. The cubic dependence of the self-modulated refractive index does not exist due to centrosymmetric crystals or waveguides. Thus, the present work is considered for non-centrosymmetric crystals or waveguides as well.

However, if $n=0$, one recovers the usual linear CD. In order to proceed with the solution structure of Eq. (1), one selects the substitution

$$q(x,t) = \phi(x)e^{i\omega t}. \tag{2}$$

Here, $\phi(x)$ represents the amplitude function, which depends on the variable *x* alone and ω is the soliton frequency. Substituting Eq. (2) into Eq. (1) gives:

$$\begin{aligned} &b_1\phi^{m+2}(x) + b_2\phi^{2m+2}(x) + a(n+1)\phi^{n+1}(x)\phi''(x) \\ &+an(n+1)\phi^n(x)\{\phi'(x)\}^2 - \omega\phi^2(x) = 0. \end{aligned} \tag{3}$$

Eq. (3) admits a single Lie point symmetry, namely $\partial/\partial x$. Implementing this symmetry in the integration process of Eq. (3) yields the following implicit solution in terms of the Appell hypergeometric function of two variables:

$$x = \pm\frac{\phi^{\frac{n}{2}}}{n}\sqrt{\frac{2(n+1)(n+2)a}{\omega}}F_1\left(\frac{n}{2m};\frac{1}{2},\frac{1}{2};1+\frac{n}{2m};A_1,A_2\right), \tag{4}$$

where

$$A_1=-\frac{2(2+n)(2+m+n)\phi^m b_2}{\left[\begin{array}{l}(2+n)(2+2m+n)b_1\\ -\sqrt{(2+n)(2+2m+n)\left((2+n)(2+2m+n)b_1^2+4(2+m+n)^2\omega b_2\right)}\end{array}\right]},\qquad(5)$$

$$A_2=-\frac{2(2+n)(2+m+n)\phi^m b_2}{\left[\begin{array}{l}(2+n)(2+2m+n)b_1\\ +\sqrt{(2+n)(2+2m+n)\left((2+n)(2+2m+n)b_1^2+4(2+m+n)^2\omega b_2\right)}\end{array}\right]}.\qquad(6)$$

The Appell hypergeometric function of two variables denoted by $F_1(a;b_1,b_2;c;x,y)$ is defined by the infinite series $x^m y^n\left(\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(a)_{m+n}(b_1)_m(b_2)_n}{(c)_{m+n}m!n!}\right)$ which is convergent inside the region $\max(|x|,|y|)<1$. Thus, for Eq. (4), this amounts to saying that the convergence criteria translates to:

$$\max(|A_1|,|A_2|)<1.\qquad(7)$$

3. Generalized temporal evolution

Eq. (1) with generalized temporal evolution reads:

$$i(q^l)_t+a\left(|q|^n q^l\right)_{xx}+\left(b_1|q|^m+b_2|q|^{2m}\right)q^l=0,\qquad(8)$$

where l represents the generalized temporal evolution parameter. For the special case when $l=1$, Eq. (8) collapses to Eq. (1).

Implementing the same substitution of Eq. (2) into Eq. (8) gives:

$$\begin{array}{l}a\left\{l^2+l(2n-1)+n(n-1)\right\}\phi^n(x)\left\{\phi'(x)\right\}^2+a(l+n)\phi^{n+1}(x)\phi''(x)\\ -\omega l\phi^2(x)+b_1\phi^{m+2}(x)+b_2\phi^{2m+2}(x)=0.\end{array}\qquad(9)$$

Eq. (9) admits a single Lie point symmetry given by $\partial/\partial x$. When this is implemented into Eq. (9) one recovers the implicit solution in terms of the Appell hypergeometric function as:

$$x=\pm\frac{\phi^{\frac{n}{2}}}{n}\sqrt{\frac{2(l+n)(2l+n)a}{l\omega}}F_1\left(\frac{n}{2m};\frac{1}{2},\frac{1}{2};1+\frac{n}{2m};A_1,A_2\right).\qquad(10)$$

The convergence criteria for this series are again given by Eq. (7) where A_j ($j=1,2$), for this case, are:

$$A_1=-\frac{2b_2(2l+n)\phi^m(2l+m+n)}{\left[\begin{array}{l}b_1(2l+n)(2l+2m+n)\\ -\sqrt{(2l+n)(2l+2m+n)\left(4\omega lb_2(2l+m+n)^2+b_1^2(2l+n)(2l+2m+n)\right)}\end{array}\right]},\qquad(11)$$

$$A_2=-\frac{2b_2(2l+n)\phi^m(2l+m+n)}{\left[\begin{array}{l}\sqrt{(2l+n)(2l+2m+n)\left(4\omega lb_2(2l+m+n)^2+b_1^2(2l+n)(2l+2m+n)\right)}\\ +b_1(2l+n)(2l+2m+n)\end{array}\right]}.\qquad(12)$$

By applying the substitution of Eq. (2) to Eq. (8), we derive the corresponding ordinary differential equation (ODE) (9). What's noteworthy is that this particular ODE possesses a unique Lie point symmetry. Upon incorporating this symmetry into (9), we recover the implicit solution expressed in the form of the Appell hypergeometric function, denoted as

Eq. (10). Delving deeper into the mathematical intricacies, it's crucial to highlight that the convergence criteria for this series are once again dictated by Eq. (7). However, in this specific case, the expressions for the convergence criteria are explicitly outlined in Eqs. (11) and (12). These Eqs. (11) and (12) provide a detailed insight into the conditions under which the series associated with the Appell hypergeometric function converges. In essence, this analytical progression elucidates the systematic approach from substitution to the emergence of a singular Lie point symmetry, ultimately leading to the implicit solution represented by the Appell hypergeometric function. The subsequent discussion of convergence criteria further refines our understanding of the mathematical framework underpinning these analytical developments.

4. Conclusions

The current work is about the retrieval of implicit quiescent optical solitons to the NLSE with nonlinear CD and generalized quadratic–cubic form of nonlinear refractive index. Both linear temporal evolution and generalized temporal evolution are considered. The integration methodology that is implemented is the Lie symmetry analysis. The results of this work are thus a generalized version of the previously reported ones during 2020, where the quadratic–cubic nonlinearity was addressed. Thus, upon setting m=1, the results collapse to the ones that were achieved in the past.

The results of the current work and its prequel paper thus form a strong foundation to extend the results further along and recover additional answers when the SPM is further extended and/or generalized. Subsequently, additional forms of optoelectronic devices will be considered where such laws of nonlinearity are applicable, and these would include fibers with differential group delay and dispersion–flattened fibers. Additionally, the application of this study would be in magneto–optic waveguides, Bragg gratings, optical couplers, and other such devices would be handled. The results will be gradually and sequentially reported in a wide range of journals after aligning them with the pre–existing concepts [14–25]. The current paper addresses the generalized quadratic–cubic nonlinearity provided by parameter m. Moreover, there are no restrictions imposed on the variable m. Therefore, it is not constrained by any specific limitations or conditions. The results presented in this work offer a more generalized perspective compared to the study reported in [1]. The previous work [1] specifically addressed the quadratic–cubic nonlinearity by setting m=1. However, in our present study, the variable m is not fixed and can vary without constraints. This lack of restriction on m in the current work provides a broader exploration of the parameter space, offering insights into the system behavior beyond the specific case considered in [1].

Disclosure. The authors claim no conflict of interest.

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Анотація. *У поточній роботі отримано неявну форму стаціонарних оптичних солітонів, які виникають з розв'язку нелінійного рівняння Шредінгера з узагальненою формою квадратично-кубічної нелінійної зміни показника заломлення. Дослідження проводиться як з лінійним часовим еволюційним процесом, так і з узагальненим часовим еволюційним процесом. Результати виражені через гіпергеометричні функції Аппеля, так само як у випадку квадратично-кубічної форми нелінійного показника заломлення, який був описаний раніше. Розв'язки отримані завдяки використанню симетрії Лі.*

Ключові слова: *стаціонарні оптичні солітони, симетрія Лі*