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Optical Solitons for the Concatenation Model with Kerr Law Nonlinearity by Lie Symmetry

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Abstract. This paper employs the Lie symmetry analysis to integrate the concatenation model studied with the Kerr law of self-phase modulation. The reduced ordinary differential equation is integrated using two approaches, which are the extended tanh method and the generalized Arnous' approach. These yielded dark and singular solitons for the model.

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1. Introduction

The concatenation model was first proposed exactly a decade ago by conjoining three of the well–known preexisting equations from nonlinear optics. They are the nonlinear Schrodinger's equation (NLSE), the Lakshmanan–Porsezian–Daniel (LPD) model, and the Sasa–Satsuma equation (SSE) [1, 2]. Recently, this model has gained enormous popularity and has its presence across a wide range of journals. Several features of this model have been studied. These include the numerical analysis of the model, Painleve analysis, application of the method of undetermined coefficients, quiescent solitons, bifurcation analysis, utilization of Kudryashov's approach, trial equation approach, solitons in magneto-optic waveguides, application to internet traffic control and several many other features [3–10]. Very recently, the model has been studied with differential group delay and the soliton solution from such a model has also been recovered. Thus, a wide range of features of this model and a plethora of applications to various optoelectronic devices has been uncovered.

The current paper will cover a new ground that has not been featured in the past. The powerful method of Lie symmetry will be implemented to address the model. This approach will first reduce the governing partial differential equation to a pair of ordinary differential equations (ODEs), just as the method of traveling wave hypothesis does. The difference is that Lie symmetry is a fancy approach while the traveling wave hypothesis is a fairly simple and straightforward approach. These ODEs are next going to be integrated using two approaches, namely the extended tanh method and the generalized Arnous' scheme. The details are jotted in the rest of the paper after a succinct introduction to the governing model.

2. Lie symmetry analysis

The concatenation model is structured as [1, 2]

$$\begin{gathered} iq_t + aq_{xx} + b|q|^2 q + c_1[\delta_1 q_{xxxx} + \delta_2 (q_x)^2 q^* + \delta_3 |q_x|^2 q + \delta_4 |q|^2 q_{xx} + \delta_5 q^2 q^*_{xx} + \\ \delta_6 |q|^4 q] + ic_2[\delta_7 q_{xxx} + \delta_8 |q|^2 q_x + \delta_9 q^2 q^*_x] = 0, \quad i = \sqrt{-1}. \end{gathered} \tag{1}$$

Eq. (1) is the dimensionless structure of the model that is considered with the Kerr law of nonlinearity. The first term is the linear temporal evolution while a is the coefficient of chromatic dispersion, b and δ_6 are the coefficients of self-phase modulation (SPM) that stems from Kerr's law of nonlinear refractive index change. Next δ_1 and δ_7 are the coefficients of third–order dispersion and fourth–order dispersion, respectively. Finally, the coefficients δ_2, δ_3, δ_8 and δ_9 imply the additional nonlinear effects, while the coefficients δ_4 and δ_5 give the nonlinear dispersive effects. The independent variables are x and t respectively and represent the spatial and temporal co-ordinates while the dependent variable *q(x,t)* accounts for the wave amplitude.

The three individual models that Eq. (1) comprises of are embedded in it. The first three terms are from NLSE. The coefficient of c_1 is from the LPD model while the coefficient of c_2 is due to SSE. Thus, Eq. (1) is the desired and the newly proposed concatenation model with the Kerr law of SPM. This model (1) will be first addressed by Lie symmetry analysis and subsequently by the two aforesaid integration schemes that will reveal dark and singular solitons and their combination thereof.

In this section, we will apply the Lie classical method on Eq. (1) in order to obtain the infinitesimals. Now, we will assume

$$q(x,t) = u(x,t) + iv(x,t), \tag{2}$$

where u and v are real-valued functions. Eq. (2) transform Eq. (1) into real and imaginary portions as

$$\begin{gathered} -v_t + au_{xx} + b(u^2+v^2)u + c_1((\delta_1 u_{xxxx}) + \delta_2(-(v_x)^2 + (u_x)^2)u + 2\delta_2(u_x)(v_x)v \\ + \frac{\delta_3(uu_x + vv_x)u}{\sqrt{u^2+v^2}} + \delta_4(u^2+v^2)u_{xx} + \delta_5(-v^2+u^2)u_{xx} + 2\delta_5 uvv_{xx} + \delta_6(u^2+v^2)^2 u) \\ -c_2(\delta_7 v_{xxx} + 2\delta_9 uvu_x + v_x((\delta_8-\delta_9)u^2 + v^2(\delta_8+\delta_9))) = 0, \\ \\ u_t + av_{xx} + b(u^2+v^2)v + c_1((\delta_1 v_{xxxx}) - \delta_2(-(v_x)^2 + (u_x)^2)v + 2\delta_2(u_x)(v_x)u \\ + \frac{\delta_3(uu_x + vv_x)v}{\sqrt{u^2+v^2}} + \delta_4(u^2+v^2)v_{xx} - \delta_5(-v^2+u^2)v_{xx} + 2\delta_5 uvu_{xx} + \delta_6(u^2+v^2)^2 v) \\ -c_2(\delta_7 u_{xxx} + 2\delta_9 uvv_x + u_x((\delta_8-\delta_9)v^2 + u^2(\delta_8+\delta_9))) = 0. \end{gathered} \tag{3}$$

The basic methodology for obtaining the infinitesimal generators of the system (3), which is derived from Eq. (1) is presented in this section. Consider a one-parameter (ϱ) Lie group of point transformations, which leave system (3) invariant, as follows:

$$
\begin{aligned}
x^* &= x + \varrho\xi(x,t,u,v) + O(\varrho^2),\\
t^* &= t + \varrho\tau(x,t,u,v) + O(\varrho^2),\\
u^* &= u + \varrho\eta(x,t,u,v) + O(\varrho^2),\\
v^* &= v + \varrho\phi(x,t,u,v) + O(\varrho^2),
\end{aligned}
\tag{4}
$$

where ξ, τ, η and ϕ are known as infinitesimal symmetries relying on x, t, u and υ. The infinitesimal generator V, also known as a vector field, associated with the preceding transformation is given by

$$V = \xi\frac{\partial}{\partial x} + \tau\frac{\partial}{\partial t} + \eta\frac{\partial}{\partial u} + \phi\frac{\partial}{\partial \upsilon}. \tag{5}$$

If symmetries of system (3) are generated by infinitesimal generator V, then it must satisfy the invariance condition:

$$Pr^{(4)}V(\Delta) = 0, \tag{6}$$

whenever $\Delta \equiv 0$ in system (3). Here, $Pr^{(4)}$ represents the fourth-order prolongation, which can be written as

$$
\begin{aligned}
Pr^{(4)} = V &+ \eta^t\frac{\partial}{\partial u_t} + \eta^x\frac{\partial}{\partial u_x} + \eta^{xx}\frac{\partial}{\partial u_{xx}} + \eta^{xxx}\frac{\partial}{\partial u_{xxx}} + \eta^{xxxx}\frac{\partial}{\partial u_{xxxx}}\\
&+ \phi^t\frac{\partial}{\partial \upsilon_t} + \phi^x\frac{\partial}{\partial \upsilon_x} + \phi^{xx}\frac{\partial}{\partial \upsilon_{xx}} + \phi^{xxx}\frac{\partial}{\partial \upsilon_{xxx}} + \phi^{xxxx}\frac{\partial}{\partial \upsilon_{xxxx}},
\end{aligned}
\tag{7}
$$

where η^t, η^x, η^{xx}, η^{xxx}, η^{xxxx}, ϕ^t, ϕ^x, ϕ^{xx}, ϕ^{xxx} and ϕ^{xxxx} are known as extended infinitesimals (for more details, see [1, 2]). By applying the prolongation formula (7) to the system (3), a system of determining equations has been obtained. The following infinitesimals were obtained after solving those determining equations:

$$\xi = c_1,\ \tau = c_2, \eta = c_3 v,\ \phi = \ \ -c_3 u. \tag{8}$$

Thus, the Lie algebra of symmetries of the system (3) can be spanned by the following vector fields:

$$V_1 = \frac{\partial}{\partial t},\ \ V_2 = \frac{\partial}{\partial x},\ \ V_3 = v\frac{\partial}{\partial u} - u\frac{\partial}{\partial v}. \tag{9}$$

Using the similarity variables, now we reduce the governing model (1) into non-linear ordinary differential equations. To do so, we must solve the characteristic equation

$$\frac{dx}{\xi(x,t,u,\upsilon)} = \frac{dt}{\tau(x,t,u,\upsilon)} = \frac{du}{\eta(x,t,u,\upsilon)} = \frac{dv}{\phi(x,t,u,\upsilon)}, \tag{10}$$

where ξ, τ, ϕ and η are given by Eq. (8). On solving Lagrange's (10), by taking the generator $V_3 + \mu V_1 + \lambda V_2$, where μ and λ are arbitrary real numbers, we have the following similarity variables:

$$
\begin{aligned}
&\xi = h(x - vt),\ \ u(x,t) = Q(\xi)\cos(\phi(x,t)),\\
&\upsilon(x,t) = Q(\xi)\sin(\phi(x,t)),\ \ \phi(x,t) = -kx + \omega t + \theta.
\end{aligned}
\tag{11}
$$

where k, ω, θ are the wave number, the wave frequency, and the phase constant, respectively.

Hence, from Eq. (2), we derive

$$q(x,t) = Q(\xi)e^{i(-kx+\omega t+\theta)}. \tag{12}$$

Using Eq. (11) in Eq. (1), we obtain the real part as

$$\begin{aligned} &c_1\delta_1 h^4 Q'''' + (c_1\delta_4 h^2 + c_1\delta_5 h^2)Q^2 Q'' + (c_1\delta_2 h^2 + c_1\delta_3 h^2)Q(Q')^2 \\ &+(ah^2 - 6c_1\delta_1 h^2 k^2 + 3c_2\delta_7 kh^2)Q'' + c_1\delta_6 Q^5 + (b - c_1\delta_2 k^2 + c_1\delta_3 k^2 \\ &-c_1\delta_4 k^2 + c_2\delta_8 k - c_2\delta_9 k)Q^3 + (-\omega - ak^2 + c_1\delta_1 k^4 - c_2\delta_7 k^3)Q = 0, \end{aligned} \tag{13}$$

and the imaginary part as

$$\begin{aligned} &\left(-4c_1\delta_1 kh^3 + c_2\delta_7 h^3\right)Q''' + \left(-hv - 2ahk + 4c_1\delta_1 k^3 h - 3c_2\delta_7 hk^2\right)Q' \\ &+\left(-2c_1\delta_2 hk - 2c_1\delta_4 hk + 2c_1\delta_5 kh + c_2\delta_8 h + c_2\delta_9 h\right)Q^2 Q' = 0, \end{aligned} \tag{14}$$

where $Q' = \dfrac{dQ(\xi)}{d\xi}$, $Q'' = \dfrac{d^2Q(\xi)}{d\xi^2}$, $Q''' = \dfrac{d^3Q(\xi)}{d\xi^3}$, and $Q'''' = \dfrac{d^4Q(\xi)}{d\xi^4}$. By employing the following constraints

$$\begin{aligned} &-4c_1\delta_1 kh^3 + c_2\delta_7 h^3 = 0, \\ &-2c_1\delta_2 hk - 2c_1\delta_4 hk + 2c_1\delta_5 kh + c_2\delta_8 h + c_2\delta_9 h = 0, \end{aligned} \tag{15}$$

one can obtain the velocity of the soliton from Eq. (14) as

$$v = -2ak + 4c_1\delta_1 k^3 - 3c_2\delta_7 k^2. \tag{16}$$

Now, Eq. (13) can be rewritten as

$$A_1 Q'''' + A_2 Q^2 Q'' + A_3 Q(Q')^2 + A_4 Q'' + A_5 Q^5 + A_6 Q^3 + A_7 Q = 0, \tag{17}$$

where,

$$\begin{aligned} &A_1 = c_1\delta_1 h^4, \\ &A_2 = c_1\delta_4 h^2 + c_1\delta_5 h^2, \\ &A_3 = c_1\delta_2 h^2 + c_1\delta_3 h^2, \\ &A_4 = ah^2 - 6c_1\delta_1 h^2 k^2 + 3c_2\delta_7 kh^2, \\ &A_5 = c_1\delta_6, \\ &A_6 = b - c_1\delta_2 k^2 + c_1\delta_3 k^2 - c_1\delta_4 k^2 + c_2\delta_8 k - c_2\delta_9 k, \\ &A_7 = -\omega - ak^2 + c_1\delta_1 k^4 - c_2\delta_7 k^3. \end{aligned} \tag{18}$$

3. Extended tanh method

In this section, we will derive the solutions of Eq. (17) by employing the extended tanh method [15]. The extended tanh scheme suggests the solution of Eq. (17) in the following form

$$Q(\xi) = \sum_{i=0}^{n} B_j \tanh^i(m\xi) + \sum_{j=1}^{n} C_j \tanh^{-j}(m\xi), \tag{19}$$

where B_i and C_j are arbitrary constants and at least one of them should be non-zero, m is the wave width. By balancing the terms Q'''' and Q^5 from Eq. (17), we have $n = 1$. Therefore, solution of Eq. (17), takes the form

$$Q(\xi) = B_0 + B_1 \tanh(m\xi) + \frac{C_1}{\tanh(m\xi)}. \tag{20}$$

Now substituting Eq. (20) into Eq. (17) and by equating terms of the same power of tanh function to zero, we get the following system of equations:

$$
\begin{aligned}
&0=24A_1C_1m^4+2A_2C_1^3m^2+A_3C_1^3m^2+A_5C_1^5,\\
&0=4B_0A_2C_1^2m^2+B_0A_3C_1^2m^2+5B_0A_5C_1^4,\\
&0=-40A_1C_1m^4-2A_2C_1^3m^2+2A_2\left(B_0^2+2B_1C_1\right)C_1m^2\\
&\quad-2A_3C_1^2m\left(B_1m+C_1m\right)+A_3B_1C_1^2m^2+2A_4C_1m^2\\
&\quad+A_5\left(C_1\left(2C_1^2\left(B_0^2+2B_1C_1\right)+4C_1^2B_0^2\right)+4B_0^2C_1^3+B_1C_1^4\right)+A_6C_1^3,\\
&0=4A_2B_0B_1C_1m^2-4B_0A_2C_1^2m^2-2A_3B_0C_1m\left(B_1m+C_1m\right)\\
&\quad+A_5\left\{C_1\left(4C_1^2B_0B_1+4C_1B_0\left(B_0^2+2B_1C_1\right)\right)\right.\\
&\quad\left.+B_0\left(2C_1^2\left(B_0^2+2B_1C_1\right)+4C_1^2B_0^2\right)+4B_1C_1^3B_0\right\}+3A_6C_1^2B_0,\\
&0=16A_1C_1m^4-2A_2B_1C_1^2m^2-2A_2\left(B_0^2+2B_1C_1\right)C_1m^2\\
&\quad+2A_2B_1^2C_1m^2+A_3C_1\left[2C_1m^2B_1+\left(B_1m+C_1m\right)^2\right]\\
&\quad-2A_3B_1C_1m\left(B_1m+C_1m\right)\\
&\quad-2A_4C_1m^2+A_5\left(C_1\left\{2C_1^2B_1^2+8C_1B_0^2B_1+\left(B_0^2+2B_1C_1\right)^2\right\}\right.\\
&\quad\left.+B_0\left[4C_1^2B_0B_1+4C_1B_0\left(B_0^2+2B_1C_1\right)\right]+B_1\left(2C_1^2\left(B_0^2+2B_1C_1\right)+4C_1^2B_0^2\right)\right)\\
&\quad+A_6\left(C_1\left(B_0^2+2B_1C_1t\right)+2B_0^2C_1+B_1C_1^2\right)+A_7C_1,\\
&0=-8A_2B_0B_1C_1m^2+A_3B_0\left(2C_1m^2B_1+\left(B_1m+C_1m\right)^2\right)\\
&\quad+A_5\left\{C_1\left(4C_1B_0B_1^2+4\left(B_0^2+2B_1C_1\right)B_0B_1\right)\right.\\
&\quad+B_0\left(2C_1^2B_1^2+8C_1B_0^2B_1+\left(B_0^2+2B_1C_1\right)^2\right)\\
&\quad\left.+B_1\left(4C_1^2B_0B_1+4C_1B_0\left(B_0^2+2B_1C_1\right)\right)\right\}\\
&\quad+A_6\left(4C_1B_0B_1+\left(B_0^2+2B_1C_1\right)B_0\right)+B_0A_7,\\
&0=16A_1B_1m^4+2A_2B_1C_1^2m^2-2A_2\left(B_0^2+2B_1C_1\right)B_1m^2\\
&\quad-2A_2B_1^2C_1m^2-2A_3B_1C_1m\left(B_1m+C_1m\right)+A_3B_1\left(2C_1m^2B_1+\left(B_1m+C_1m\right)^2\right)\\
&\quad-2A_4B_1m^2+A_5(C_1\left(2\left(B_0^2+2B_1C_1\right)B_1^2+4B_0^2B_1^2\right)\\
&\quad+B_0\left(4C_1B_0B_1^2+4\left(B_0^2+2B_1C_1\right)B_0B_1\right)+B_1\left(2C_1^2B_1^2+8C_1B_0^2B_1+\left(B_0^2+2B_1C_1\right)^2\right)\\
&\quad+A_6\left(2B_0^2B_1+C_1B_1^2+B_1\left(B_0^2+2B_1C_1\right)\right)+A_7B_1,\\
&0=-4A_2B_0B_1^2m^2+4A_2B_0B_1C_1m^2-2A_3B_0\left(B_1m+C_1m\right)B_1m+A_5\left\{4C_1B_0B_1^3\right.\\
&\quad\left.+B_0\left(2\left(B_0^2+2B_1C_1\right)B_1^2+4B_0^2B_1^2\right)+B_1\left(4C_1B_0B_1^2+4\left(B_0^2+2B_1C_1\right)B_0B_1\right)\right\}+3A_6B_0B_1^2,\\
&0=-40A_1B_1m^4+2A_2\left(B_0^2+2B_1C_1\right)B_1m^2-2A_2B_1^3m^2+A_3C_1B_1^2m^2\\
&\quad-2A_3B_1^2\left(B_1m+C_1m\right)m+2A_4B_1m^2\\
&\quad+A_5\left(C_1B_1^4+4B_0^2B_1^3+B_1\left(2\left(B_0^2+2B_1C_1\right)B_1^2+4B_0^2B_1^2\right)\right)+A_6B_1^3,\\
&0=4A_2B_0B_1^2m^2+A_3B_0B_1^2m^2+5B_0A_5B_1^4,\\
&0=24A_1B_1m^4+2A_2B_1^3m^2+A_3B_1^3m^2+A_5B_1^5.
\end{aligned}
\tag{21}
$$

After solving system Eq. (21), we have four sets of solutions:

Set-1.

$$B_1 = 0, C_1 = \frac{\sqrt{2}\sqrt{-A_5\left(A_6 - \sqrt{-4A_7A_5 + A_6^2}\right)}}{2A_5}, B_0 = 0,$$

$$A_4 = \frac{\left(-16A_1m^4 + 4A_7\right)A_5 - 2\left(A_2m^2 - \frac{A_6}{2}\right)\left(\sqrt{-4A_7A_5 + A_6^2} - A_6\right)}{4A_5m^2}, \quad (22)$$

$$A_3 = \frac{48A_1m^4A_5 - 2A_2m^2A_6 + 2A_2m^2\sqrt{-4A_7A_5 + A_6^2} - 2A_7A_5 + A_6^2 - A_6\sqrt{-4A_7A_5 + A_6^2}}{\left(A_6 - \sqrt{-4A_7A_5 + A_6^2}\right)m^2}.$$

Inserting Eq. (22) together with Eq. (11) into Eq. (2), we get a singular soliton solution

$$q(x,t) = C_1\coth\left(mh(x - vt)\right)e^{i(-kx + \omega t + \theta)}. \quad (23)$$

Set-2.

$$B_1 = -\frac{\sqrt{-2A_5\left(A_6 + \sqrt{-4A_7A_5 + A_6^2}\right)}}{2A_5}, C_1 = 0, B_0 = 0,$$

$$A_4 = \frac{\left(-16A_1m^4 + 4A_7\right)A_5 - 2\left(A_2m^2 - \frac{A_6}{2}\right)\left(\sqrt{-4A_7A_5 + A_6^2} - A_6\right)}{4A_5m^2}, \quad (24)$$

$$A_3 = \frac{48A_1m^4A_5 - 2A_2m^2A_6 + 2A_2m^2\sqrt{-4A_7A_5 + A_6^2} - 2A_7A_5 + A_6^2 - A_6\sqrt{-4A_7A_5 + A_6^2}}{\left(A_6 - \sqrt{-4A_7A_5 + A_6^2}\right)m^2}.$$

Inserting Eq. (24) together with Eq. (11) into Eq. (2), we get a dark soliton solution

$$q(x,t) = B_1\tanh\left(mh(x - vt)\right)e^{i(-kx + \omega t + \theta)}. \quad (25)$$

Fig. 1 showcases several plots illustrating the dark soliton solution (25) within the context of model Eq. (1), and the specific parameter values employed are as follows: $m = 1$, $h = 1$, $a = 1$, $k = 1$, $c_1 = 1$, $c_2 = 1$, $\delta_1 = 1$, $\delta_2 = 1$, $\delta_3 = 1$, $\delta_4 = 1$, $\delta_6 = 1$, $\delta_7 = 1$, $\delta_8 = 1$, $\delta_9 = 1$, $\omega = 1$, and $b = 1$.

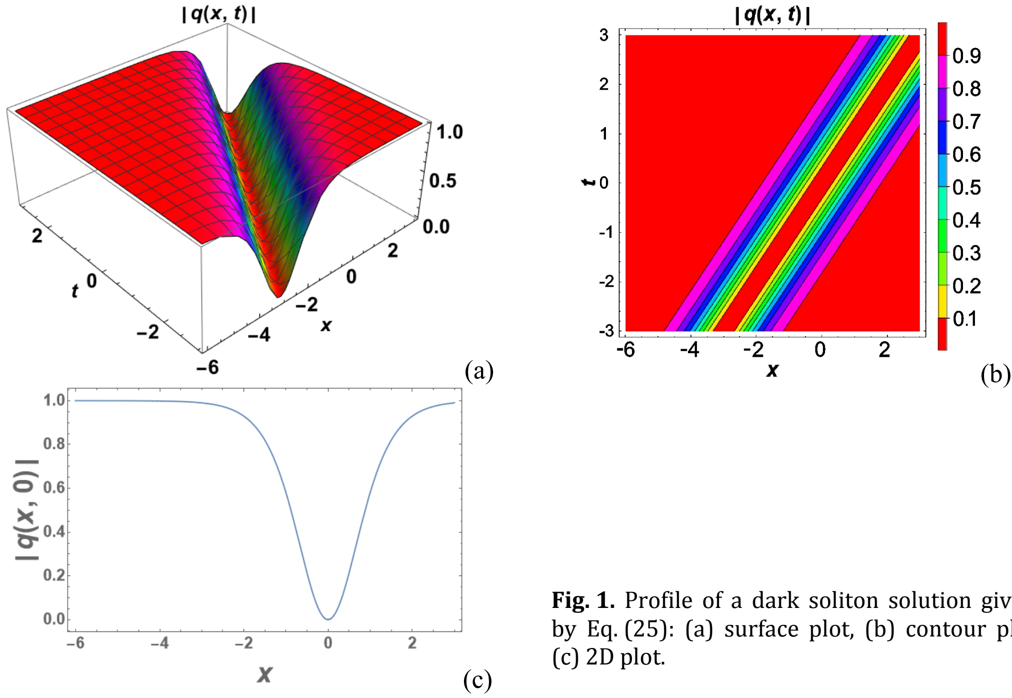

Fig. 1. Profile of a dark soliton solution given by Eq. (25): (a) surface plot, (b) contour plot (c) 2D plot.

Set-3.

$$B_1=-\frac{\sqrt{-2A_5\left(A_6+\sqrt{-4A_5A_7+A_6^2}\right)}}{4A_5},\ C_1=-\frac{\sqrt{-2A_5\left(A_6-\sqrt{-4A_7A_5+A_6^2}\right)}}{4A_5},$$

$$B_0=0,\ m=m,\ A_4=\frac{\left(-256A_1m^4+4A_7\right)A_5+8\left(A_6+\sqrt{-4A_5A_7+A_6^2}\right)\left(A_2m^2-\frac{A_6}{8}\right)}{16A_5m^2},\tag{26}$$

$$A_3=\frac{768A_1m^4A_5-8A_2m^2A_6-8A_2m^2\sqrt{-4A_5A_7+A_6^2}-2A_5A_7+A_6^2+A_6\sqrt{-4A_5A_7+A_6^2}}{4\left(A_6+\sqrt{-4A_5A_7+A_6^2}\right)m^2}.$$

Inserting Eq. (26) together with Eq. (11) into Eq. (2), we get

$$q(x,t)=\ \left[B_1\tanh\left(mh(x-vt)\right)+C_1\coth\left(mh(x-vt)\right)\right]e^{i(-kx+\omega t+\theta)}.\tag{27}$$

For $B_1\neq 0$ and $C_1=0$, Eq. (27) reduces to a dark soliton solution, while and for $B_1=0$ and $C_1\neq 0$, Eq. (27) reduces to a singular soliton solution. Therefore (27) is the structure of a dark-singular straddled soliton

Set-4.

$$B_1=-C_1, C_1=C_1, B_0=0,$$

$$m=-\frac{\sqrt{-3A_1\left(A_2C_1^2+\sqrt{-24A_1A_5C_1^4+A_2^2C_1^4+6A_1A_6C_1^2-3A_1A_7}\right)}}{6A_1},$$

$$A_4=\frac{\begin{pmatrix}-24A_1A_5C_1^4+2A_2^2C_1^4+6A_1A_6C_1^2\\ +2A_2C_1^2\sqrt{\left(-24A_5C_1^4+6A_6C_1^2-3A_7\right)A_1+A_2^2C_1^4}+6A_1A_7\end{pmatrix}}{3A_2C_1^2+3\sqrt{\left(-24A_5C_1^4+6A_6C_1^2-3A_7\right)A_1+A_2^2C_1^4}},\tag{28}$$

$$A_3=\frac{\begin{pmatrix}-36A_1A_5C_1^4+2A_2^2C_1^4+12A_1A_6C_1^2\\ +2A_2C_1^2\sqrt{\left(-24A_5C_1^4+6A_6C_1^2-3A_7\right)A_1+A_2^2C_1^4}-6A_1A_7\end{pmatrix}}{C_1^2\left(A_2C_1^2+\sqrt{\left(-24A_5C_1^4+6A_6C_1^2-3A_7\right)A_1+A_2^2C_1^4}\right)}.$$

Inserting Eq. (28) together with Eq. (11) into Eq. (2), we get a dark-singular straddeld solution

$$q(x,t)=C_1\left[-\tanh\left(mh(x-vt)\right)+\coth\left(mh(x-vt)\right)\right]e^{i(-kx+\omega t+\theta)}.\tag{29}$$

4. Applications of generalized Arnous' method

In this section, we drive the solutions of Eq. (1) by performing a generalized Arnous' method [14]. First, we have to derive the positive integer N, by balancing the terms Q'''' and Q^5, we have $N=1$. Consequently, one gets

$$Q(\xi)=\alpha_0+\frac{\alpha_1+\alpha_2\phi'(\xi)}{\phi(\xi)},\tag{30}$$

as a solution of Eq. (17). Here, the constants α_0, α_1, and α_2 are determined later and the function $\phi(\xi)$ satisfies the relation

$$[\phi'(\xi)^2]=[\phi(\xi)^2-\chi]\ln(a)^2,\tag{31}$$

with

$$\phi^{(n)}\xi=\begin{cases}\phi(\xi)\ln(a)^2, & n \text{ is even}, \quad n\geq 2,\\ \phi(\xi)'\ln(a)^2, & n \text{ is odd}, \quad n\geq 2.\end{cases}\tag{32}$$

Eq. (31) holds the following solution

$$\phi(\xi)=k\ln(a)e^{\xi\ln(a)}+\frac{\chi}{4k\ln(a)e^{\xi\ln(a)}}, \tag{33}$$

where k is the wavevector, and χ are arbitrary parameters. Substituting Eq. (30) into Eq. (17), we get an over-determined system of algebraic equations; by collecting all terms of the same power and equating them to zero, we get the following system of equations

$$\begin{aligned}
&0=\alpha_0\left(5A_5\ln(a)^4\alpha_2^4+10\left(A_5\alpha_0^2+\frac{3A_6}{10}\right)\alpha_2^2\ln(a)^2+A_5\alpha_0^4+A_6\alpha_0^2+A_7\right),\\
&0=\alpha_1\left(\begin{aligned}&5A_5\ln(a)^4\alpha_2^4+A_2\ln(a)^4\alpha_2^2+30A_5\ln(a)^2\alpha_0^2\alpha_2^2+A_1\ln(a)^4\\&+A_2\ln(a)^2\alpha_0^2+5A_5\alpha_0^4+3A_6\ln(a)^2\alpha_2^2+A_4\ln(a)^2+3A_6\alpha_0^2+A_7\end{aligned}\right),\\
&0=-2\left(5A_5\alpha_0\alpha_2^4+\alpha_0\alpha_2^2A_3\right)\chi\ln(a)^4+\ln(a)^2(-4\alpha_0\alpha_2^2A_2\chi\ln(a)^2+\alpha_0\alpha_1^2A_3+30A_5\alpha_0\alpha_1^2\alpha_2^2\\
&\quad-(-2\alpha_0\alpha_2^2\ln(a)^2A_3+10A_5\alpha_0^3\alpha_2^2+3A_6\alpha_0\alpha_2^2)\chi)+2\alpha_1^2\alpha_0A_2\ln(a)^2+10A_5\alpha_1^2\alpha_0^3+3A_6\alpha_0\alpha_1^2,\\
&0=-2\left(5A_5\alpha_1\alpha_2^4+3\alpha_2^2\alpha_1A_3\right)\chi\ln(a)^4+(-6\alpha_1\alpha_2^2A_2\chi\ln(a)^2+\alpha_1^3A_3+10A_5\alpha_1^3\alpha_2^2\\
&\quad-(8A_1\alpha_1\ln(a)^2+\alpha_2^2A_2\alpha_1\ln(a)^2-4\alpha_1\alpha_2^2\ln(a)^2A_3+30A_5\alpha_1\alpha_0^2\alpha_2^2+3A_6\alpha_1\alpha_2^2)\chi)\ln(a)^2\\
&\quad-12A_1\alpha_1\chi\ln(a)^4+\alpha_1^3A_2\ln(a)^2-2\alpha_0^2A_2\alpha_1\chi\ln(a)^2-2\alpha_1\chi\ln(a)^2A_4+10A_5\alpha_1^3\alpha_0^2+A_6\alpha_1^3,\\
&0=\left(5A_5\alpha_0\alpha_2^4+\alpha_0\alpha_2^2A_3\right)\chi^2\ln(a)^4-\left(-4\alpha_0\alpha_2^2A_2\chi\ln(a)^2+\alpha_0\alpha_1^2A_3+30A_5\alpha_0\alpha_1^2\alpha_2^2\right)\chi\ln(a)^2\\
&\quad-4\alpha_1^2\alpha_0A_2\chi\ln(a)^2+5A_5\alpha_1^4\alpha_0,\\
&0=\left(5A_5\alpha_1\alpha_2^4+3\alpha_2^2\alpha_1A_3\right)\chi^2\ln(a)^4-\left(-6\alpha_1\alpha_2^2A_2\chi\ln(a)^2+\alpha_1^3A_3+10A_5\alpha_1^3\alpha_2^2\right)\chi\ln(a)^2\\
&\quad+24A_1\alpha_1\chi^2\ln(a)^4-2\alpha_1^3A_2\chi\ln(a)^2+A_5\alpha_1^5,\\
&0=\left(A_5\alpha_2^5+\alpha_2^3A_3\right)\ln(a)^4+\left(20A_1\alpha_2\ln(a)^2-2\alpha_2^3\ln(a)^2A_3+10A_5\alpha_0^2\alpha_2^3+A_6\alpha_2^3\right)\ln(a)^2\\
&\quad-20A_1\alpha_2\ln(a)^4+\alpha_2^3\ln(a)^4A_3+5A_5\alpha_0^4\alpha_2+3A_6\alpha_0^2\alpha_2+A_7\alpha_2,\\
&0=\left(20A_5\alpha_1\alpha_0\alpha_2^3+2\alpha_1\alpha_0\alpha_2A_3\right)\ln(a)^2+2\alpha_0\alpha_2A_2\alpha_1\ln(a)^2\\
&\quad-2\alpha_0\alpha_2\ln(a)^2\alpha_1A_3+20A_5\alpha_1\alpha_0^3\alpha_2+6A_6\alpha_1\alpha_0\alpha_2,\\
&0=-2(A_5\alpha_2^5+\alpha_2^3A_3)\chi\ln(a)^4+(-2\alpha_2^3A_2\chi\ln(a)^2+3\alpha_1^2\alpha_2A_3+10A_5\alpha_1^2\alpha_2^3\\
&\quad-(20A_1\alpha_2\ln(a)^2-2\alpha_2^3\ln(a)^2A_3+10A_5\alpha_0^2\alpha_2^3+A_6\alpha_2^3)\chi)\ln(a)^2\\
&\quad+12A_1\alpha_2\chi\ln(a)^4-2\alpha_0^2A_2\alpha_2\chi\ln(a)^2+2\alpha_1^2\alpha_2A_2\ln(a)^2\\
&\quad-2\alpha_1^2\alpha_2\ln(a)^2A_3-2\alpha_2\chi\ln(a)^2A_4+30A_5\alpha_1^2\alpha_0^2\alpha_2+3A_6\alpha_1^2\alpha_2,\\
&0=-\left(20A_5\alpha_1\alpha_0\alpha_2^3+2\alpha_1\alpha_0\alpha_2A_3\right)\chi\ln(a)^2-8\alpha_1\alpha_0A_2\alpha_2\chi\ln(a)^2+20A_5\alpha_1^3\alpha_0\alpha_2,\\
&0=\left(A_5\alpha_2^5+\alpha_2^3A_3\right)\chi^2\ln(a)^4-\left(-2\alpha_2^3A_2\chi\ln(a)^2+3\alpha_1^2\alpha_2A_3+10A_5\alpha_1^2\alpha_2^3\right)\chi\ln(a)^2\\
&\quad+24A_1\alpha_2\chi^2\ln(a)^4-6\alpha_1^2A_2\alpha_2\chi\ln(a)^2+5A_5\alpha_1^4\alpha_2.
\end{aligned} \tag{34}$$

Solving the above system by using the software Maple, we get the result as follows

Set-1.

$$\begin{aligned}
&\chi=0,\ \alpha_1=0,\ A_2=-\frac{4A_5\left(\ln(a)^2\alpha_2^2+\alpha_0^2\right)}{\ln(a)^2},\\
&A_6=-2A_5\ln(a)^2\alpha_2^2-2A_5\alpha_0^2,\ A_7=A_5\left(\ln(a)^4\alpha_2^4-2\ln(a)^2\alpha_0^2\alpha_2^2+\alpha_0^4\right).
\end{aligned} \tag{35}$$

Therefore, by putting Eq. (35) along with Eq. (30) into Eq. (11) and Eq. (12), we have the solution of the governing model defined by Eq. (1) as

$$q(x,t)=\left[\alpha_0+\alpha_2\left\{\frac{\ln(a)\begin{pmatrix}4k^2(\ln(a))^2(\cosh(\xi\ln(a))+\sinh(\xi\ln(a)))\\-\chi(\cosh(\xi\ln(a))-\sinh(\xi\ln(a)))\end{pmatrix}}{\begin{matrix}4k^2(\ln(a))^2(\cosh(\xi\ln(a))+\sinh(\xi\ln(a)))\\+\chi(\cosh(\xi\ln(a))-\sinh(\xi\ln(a)))\end{matrix}}\right\}\right]\times e^{i(-kx+\omega t+\theta)}, \tag{36}$$

where $\xi=h(x-vt)$, $a>0$, α_0, A_1, A_4, A_3 and α_2 are arbitrary constants. If $\chi=4k^2(\ln(a))^2$, from (36), we have a dark soliton solution as follows

$$q(x,t)=\left[\alpha_0+\alpha_2\{\ln(a)\tanh(\xi\ln(a))\}\right]e^{i(-kx+\omega t+\theta)}. \tag{37}$$

If $\chi=-4k^2(\ln(a))^2$, the solution (36) reduces to a singular soliton solution as follows

$$q(x,t)=\left[\alpha_0+\alpha_2\{\ln(a)\coth(\xi\ln(a))\}\right]e^{i(-kx+\omega t+\theta)}. \tag{38}$$

Set-2.

$$\begin{aligned}&\alpha_1=\sqrt{-\chi}\,\alpha_2\ln(a),\\&A_1=-\frac{4A_5\alpha_2^2\left(-\ln(a)^2\alpha_2^2+\alpha_0^2\right)}{3\ln(a)^2},\ A_2=-\frac{4A_5\left(\ln(a)^2\alpha_2^2+\alpha_0^2\right)}{\ln(a)^2},\\&A_3=-\frac{4A_5\left(\ln(a)^2\alpha_2^2-4\alpha_0^2\right)}{\ln(a)^2},A_4=\frac{4A_5\left(2\ln(a)^4\alpha_2^4-5\ln(a)^2\alpha_0^2\alpha_2^2+3\alpha_0^4\right)}{3\ln(a)^2},\\&A_6=-2A_5\ln(a)^2\alpha_2^2-2A_5\alpha_0^2,\\&A_7=A_5\left(\ln(a)^4\alpha_2^4-2\ln(a)^2\alpha_0^2\alpha_2^2+\alpha_0^4\right).\end{aligned} \tag{39}$$

Therefore, by putting Eq. (39) along with Eq. (30) into Eq. (11) and Eq. (12), we have the solution of the governing model defined by Eq. (1) as

$$q(x,t)=\left[\alpha_0+\frac{\begin{matrix}\ln(a)\left(\sqrt{-\chi}\,\alpha_2\ln(a)+\alpha_2\right)\\\times\left(4k^2(\ln(a))^2\begin{pmatrix}\cosh(\xi\ln(a))\\+\sinh(\xi\ln(a))\end{pmatrix}-\chi\begin{pmatrix}\cosh(\xi\ln(a))\\-\sinh(\xi\ln(a))\end{pmatrix}\right)\end{matrix}}{\left(4k^2(\ln(a))^2\begin{pmatrix}\cosh(\xi\ln(a))\\+\sinh(\xi\ln(a))\end{pmatrix}+\chi\begin{pmatrix}\cosh(\xi\ln(a))\\-\sinh(\xi\ln(a))\end{pmatrix}\right)}\right]\times e^{i(-kx+\omega t+\theta)}, \tag{40}$$

where $\xi=h(x-vt)$, $a>0$, $\chi<0$, α_0 and α_2 are arbitrary constants. If $\chi=4k^2(\ln(a))^2$, from (40), we have a dark soliton solution as follows

$$q(x,t)=\left[\alpha_0+\ln(a)\alpha_2\left(\sqrt{-\chi}\ln(a)+1\right)\tanh(\xi\ln(a))\right]e^{i(-kx+\omega t+\theta)}. \tag{41}$$

If $\chi=-4k^2(\ln(a))^2$, the solution (40) reduces to a sigular soliton solution as follows

$$q(x,t)=\left[\alpha_0+\ln(a)\alpha_2\left(\sqrt{-\chi}\ln(a)+1\right)\coth\left(\xi\ln(a)\right)\right]e^{i(-kx+\omega t+\theta)}. \tag{42}$$

Set-3. $\alpha_1=0,$

$$A_1=-\frac{A_5\alpha_2^2\left(-\ln(a)^2\alpha_2^2+\alpha_0^2\right)}{3\ln(a)^2},\ A_2=-\frac{A_5\left(\ln(a)^2\alpha_2^2+\alpha_0^2\right)}{\ln(a)^2},$$

$$A_3=-\frac{A_5\left(\ln(a)^2\alpha_2^2-4\alpha_0^2\right)}{\ln(a)^2},\ A_4=\frac{A_5\left(2\ln(a)^4\alpha_2^4-5\ln(a)^2\alpha_0^2\alpha_2^2+3\alpha_0^4\right)}{3\ln(a)^2}, \tag{43}$$

$$A_6=-2A_5\ln(a)^2\alpha_2^2-2A_5\alpha_0^2,$$

$$A_7=A_5\left(\ln(a)^4\alpha_2^4-2\ln(a)^2\alpha_0^2\alpha_2^2+\alpha_0^4\right).$$

Therefore, by putting Eq. (43) along with Eq. (30) into Eq. (11) and Eq. (12), we have the solution of the governing model (1) as

$$q(x,t)=\left[\alpha_0+\alpha_2\left\{\frac{\ln(a)\left(4k^2(\ln(a))^2\begin{pmatrix}\cosh(\xi\ln(a))\\+\sinh(\xi\ln(a))\end{pmatrix}-\chi\left(\cosh(\xi\ln(a))-\sinh(\xi\ln(a))\right)\right)}{\left(4k^2(\ln(a))^2\left(\cosh(\xi\ln(a))+\sinh(\xi\ln(a))\right)+\chi\left(\cosh(\xi\ln(a))-\sinh(\xi\ln(a))\right)\right)}\right\}\right]e^{i(-kx+\omega t+\theta)}, \tag{44}$$

where $\xi=h(x-vt)$, $a>0$, α_0, χ and α_2 are arbitrary constants. If $\chi=4k^2(\ln(a))^2$, the solution (44) give us a dark soliton solution as follows

$$q(x,t)=\left[\alpha_0+\alpha_2\left\{\ln(a)\tanh(\xi\ln(a))\right\}\right]e^{i(-kx+\omega t+\theta)}. \tag{45}$$

If $\chi=-4k^2(\ln(a))^2$, the solution (44) reduces to a singular soliton solution given as

$$q(x,t)=\left[\alpha_0+\alpha_2\left\{\ln(a)\coth(\xi\ln(a))\right\}\right]e^{i(-kx+\omega t+\theta)}. \tag{46}$$

5. Conclusions

The current paper studied the concatenation model with the aid of Lie symmetry analysis. Thereafter, the reduced ODE was integrated using a couple of approaches. These are the extended tanh method and Arnous' algorithm. These algorithms revealed dark and singular soliton solutions to the model. Unfortunately, the two adopted approaches failed to recover bright soliton solutions to the model that are the primary information carrier bits across intercontinental distances. Nevertheless, the Lie symmetry is an advanced mathematical technique to reduce the given model to the necessary ODE, which can be addressed using additional integration methodologies such as the enhanced Kudryashov's approach and many others.

There are wide open possibilities that lay ahead to handle this equation from different perspectives. One of the very many things that need to be covered is the study of the concatenation model with fractional temporal evolution. This can address and mitigate the Internet bottleneck effect. Additionally, the soliton perturbation theory, as well as the establishment of the quasi-particle theory in order to suppress the intra-channel collision of optical solitons, is imperative. Subsequently, the application of the semi-inverse variational principle to recover the analytical soliton solutions to the perturbed version of the concatenation model, with arbitrary intensity, is also on the table. These are just the tip of the iceberg. The results will be recovered and reported with time once they are aligned with the pre-existing ones [1-15].

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The authors claim there is no conflict of interest.

References

1. Ankiewicz, A., & Akhmediev, N. (2014). Higher-order integrable evolution equation and its soliton solutions. *Physics Letters A*, *378*(4), 358-361.
2. Ankiewicz, A., Wang, Y., Wabnitz, S., & Akhmediev, N. (2014). Extended nonlinear Schrödinger equation with higher-order odd and even terms and its rogue wave solutions. *Physical Review E*, *89*(1), 012907.
3. Arnous, A. H., Biswas, A., Kara, A. H., Yildirim, Y., Moraru, L., Iticescu, C., Moldovanu, S., & Alghamdi, A. A. (2023). Optical solitons and conservation laws for the concatenation model with spatio-temporal dispersion (internet traffic regulation). *Journal of the European Optical Society-Rapid Publications*, *19*(2), 35.
4. Biswas, A., Vega—Guzman, J. M., Yildirim, Y., Moshokoa, S. P., Aphane, M. & Alghamdi, A. A. (2023). Optical solitons for the concatenation model with power-law nonlinearity: undetermined coefficients. *Ukrainian Journal of Physical Optics*, *24*(3), 185-192.
5. González-Gaxiola, O., Biswas, A., Chavez, J. R. D., & Asiri, A. (2023). Bright and dark optical solitons for the concatenation model by the Laplace-Adomian decomposition scheme. *Ukrainian Journal of Physical Optics*, *24*(3), 222-234.
6. Kudryashov, N. A., Biswas, A., Borodina, A. G., Yıldırım, Y., & Alshehri, H. M. (2023). Painlevé analysis and optical solitons for a concatenated model. *Optik*, *272*, 170255.
7. Kukkar, A., Kumar, S., Malik, S., Biswas, A., Yıldırım, Y., Moshokoa, S. P., ... & Alghamdi, A. A. (2023). Optical solitons for the concatenation model with Kurdryashov's approaches. *Ukrainian Journal of Physical Optics*, *24*(2), 155 - 160.
8. Tang, L., Biswas, A., Yıldırım, Y., & Alghamdi, A. A. (2023). Bifurcation analysis and optical solitons for the concatenation model. *Physics Letters A*, 128943.
9. Wang, M. Y., Biswas, A., Yıldırım, Y., Moraru, L., Moldovanu, S., & Alshehri, H. M. (2022). Optical solitons for a concatenation model by trial equation approach. *Electronics*, *12*(1), 19.
10. Yıldırım, Y., Biswas, A., Moraru, L., & Alghamdi, A. A. (2023). Quiescent optical solitons for the concatenation model with nonlinear chromatic dispersion. *Mathematics*, *11*(7), 1709.
11. Bluman, G., & Anco, S. (2008). *Symmetry and integration methods for differential equations* (Vol. 154). Springer Science & Business Media.
12. Olver, P. J. (1993). *Applications of Lie groups to differential equations* (Vol. 107). Springer Science & Business Media.
13. Kumar, M., & Tanwar, D. V. (2019). On Lie symmetries and invariant solutions of (2+1)–dimensional Gardner equation. *Communications in Nonlinear Science and Numerical Simulation*, *69*, 45-57.
14. Zhang, J., Wei, X., & Lu, Y. (2008). A generalized (G′/G)-expansion method and its applications. *Physics Letters A*, *372*(20), 3653-3658.
15. Abdou, M. A. (2007). The extended tanh method and its applications for solving nonlinear physical models. *Applied mathematics and computation*, *190*(1), 988-996.

Sushmita Kumari Dubey, Sachin Kumar, Sandeep Malik, Anjan Biswas, Anwar Jafaar Mohamad Jawad, Yakup Yıldırım, Luminita Moraru & Ali Saleh Alshomrani. (2024). Optical Solitons for the Concatenation Model with Kerr Law Nonlinearity by Lie Symmetry. *Ukrainian Journal of Physical Optics*, *25*(1), 01073 – 01084.
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Анотація. *У статті використовується аналіз симетрії Лі для інтеграції вивченої моделі конкатенації із законом Керра автофазової автомодуляції. Наведене звичайне диференціальне рівняння інтегрується з допомогою двох підходів: розширеного* tanh-*методу гіперболічного тангенсу і узагальненого підходу Арнуса. Це дало темні та сингулярні солітони для моделі.*

Ключові слова: *солітони, метод гіперболічного тангенсу, метод Арнуса*