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Quescent Optical Solitons for the Dispersive Concatenation Model with Kerr Law Nonlinearity Having Nonlinear Chromatic Dispersion

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Abstract. The current work obtains the quiescent optical solitons to the dispersive concatenation model that is considered with the Kerr law of nonlinear refractive index and nonlinear chromatic dispersion. Two integration schemes reveal this full spectrum of quiescent optical solitons. Their existence criteria are also presented in the work.

Keywords: quiescent optical solitons, Kudryashov scheme, Riccati equation algorithm, dispersive concatenation model
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1. Introduction

The concept of the concatenation model was conceived a decade ago in 2014, by Ankiewicz et al. This is obtained through the conjunction of the well–known models that describe the propagation of solitons through optical fibers. They are the nonlinear Schrödinger's equation (NLSE), the Lakshmanan–Porsezian–Daniel (LPD) model, and the Sasa–Satsuma equation [1, 2]. Thereafter, a dispersive version of this concatenation model was proposed and reported during 2014 and 2015 [3–5]. This form of the concatenation model is obtained by conjoining the Schrödinger–Hirota equation (SHE), the LPD model, and the dispersive NLSE with fifth–order dispersion in it. A plethora of results have been recently reported for the concatenation model, which ranges from retrieval of optical solitons and conservation laws, numerical studies of solitons with the Laplace–Adomian decomposition scheme, application of the model to magneto–optic waveguides, and many more [6–10]. The current paper will focus on retrieving the quiescent optical solitons for the dispersive concatenation model that is considered with nonlinear chromatic dispersion (CD) and having Kerr law of self–phase modulation (SPM). The theory of quiescent optical solitons is becoming increasingly important due to their potential impact on soliton propagation dynamics [7]. These solitons, which remain stationary rather than propagating as expected, are primarily caused by the nonlinear effects of chromatic dispersion. Researchers have studied this phenomenon to understand why solitons stall in optical fibers, a problem that has significant implications for the telecommunications industry. Previous investigations sought to uncover the mechanisms behind soliton stalling and develop solutions to mitigate this undesirable feature. This background provides essential context for our current work, where we aim to contribute to addressing the challenges associated with quiescent optical solitons.

Two integration approaches are employed in order to retrieve these quiescent solitons to the model. They are the newly enhanced Kyudryashov's scheme and the projective Riccati's equation algorithm. These two approaches collectively reveal a full spectrum of optical solitons that are enumerated in the paper. The paper first revisits the dispersive concatenation model that is with nonlinear CD and Kerr law of SPM. The integration algorithms are recapitulated in a succinct manner and subsequently implemented to recover the quiescent solitons. The details are exhibited in the rest of the paper after revisiting the governing model.

2. An overview over the integration algorithms

The dimensionless form of the dispersive concatenation model is given as:

$$\begin{aligned} & iq_t + a\left(|q|^n q\right)_{xx} + b|q|^2 q - i\delta_1\left[\sigma_1 q_{xxx} + \sigma_2 |q|^2 q_x\right] \\ & +\delta_2\left[\sigma_3 q_{xxxx} + \sigma_4 |q|^2 q_{xx} + \sigma_5 |q|^4 q + \sigma_6 |q_x|^2 q + \sigma_7 q_x^2 q^* + \sigma_8 q_{xx}^* q^2\right] \\ & -i\delta_3\left[\sigma_9 q_{xxxxx} + \sigma_{10} |q|^2 q_{xxx} + \sigma_{11} |q|^4 q_x + \sigma_{12} q q_x q_{xx}^* \right. \\ & \left. +\sigma_{13} q^* q_x q_{xx} + \sigma_{14} q q_x^* q_{xx} + \sigma_{15} q_x^2 q_x^*\right] = 0. \end{aligned} \tag{1}$$

In Eq. (1), the dependent variable $q(x,t)$ is the wave amplitude and is a complex-valued function. The independent variables are x and t that account for the spatial and temporal variables respectively. Then a and b are the coefficients of nonlinear CD and Kerr law of SPM, respectively, while $i = \sqrt{-1}$. The first term accounts for the linear temporal evolution of the pulses, and the first five terms of (1) formulate the SHE, while the coefficient of δ_2 is from the LPD model, and, finally, the coefficient of δ_3 is from the Kerr law of SPM. Lastly, $\delta_1, \sigma_1 .... \sigma_{15}$ are the coefficients of nonlinear dispersion effects.

Assume the solution structure of Eq. (1) is:

$$q(x,t) = U(Kx)e^{i(\omega t + \theta_0)}. \tag{2}$$

Here, $U(Kx)$ represents the amplitude component of the soliton solution where K is the wave width, while ω represents the frequency, and θ_0 is the phase constant. After substituting of Eq. (2) into Eq. (1) and then decomposing it into real and imaginary parts, one obtains:

$$aK^2(n+1)U^{n+1}U''+aK^2n(n+1)U^nU'^2+\delta_2K^4\sigma_3U^{(4)}U+\delta_2K^2(\sigma_4+\sigma_8)U^3U''$$
$$+\delta_2K^2(\sigma_6+\sigma_7)U^2U'^2+(\delta_2\kappa^2\sigma_6+1)U^4+\delta_2\sigma_5U(\xi)^6-\omega U^2=0, \quad (3)$$

and

$$-\delta_3K^5\sigma_9UU^{(5)}-\delta_3K^3\sigma_{10}U^3U^{(3)}-\delta_1K^3\sigma_1UU^{(3)}-\delta_3K^3\sigma_{15}UU'^3$$
$$-\delta_3K^3(\sigma_{12}+\sigma_{13}+\sigma_{14})U^2U'U''-\delta_3K\sigma_{11}U^5U'-\delta_1K\sigma_2U^3U'=0. \quad (4)$$

From the imaginary part, we get the parametric restrictions

$$\sigma_2=\sigma_{11}=\sigma_{15}=\sigma_1=\sigma_{10}=\sigma_9=0, \quad (5)$$

and

$$\sigma_{12}+\sigma_{13}+\sigma_{14}=0. \quad (6)$$

Provided that $n=2$ for integrability. Then Eq. (1) reduces to

$$iq_t+a\left(|q|^2q\right)_{xx}+b|q|^2q$$
$$+\delta_2\left[\sigma_3q_{xxxx}+\sigma_4|q|^2q_{xx}+\sigma_5|q|^4q+\sigma_6|q_x|^2q+\sigma_7q_x^2q^*+\sigma_8q_{xx}^*q^2\right]$$
$$-i\delta_3\left[\sigma_{12}qq_xq_{xx}^*+\sigma_{13}q^*q_xq_{xx}+\sigma_{14}qq_x^*q_{xx}\right]=0, \quad (7)$$

and Eq. (3) reaches

$$K^2(3a+\delta_2(\sigma_4+\sigma_8))U^2U''+K^2(6a+\delta_2(\sigma_6+\sigma_7))UU'^2+\delta_2K^4\sigma_3U^{(4)}$$
$$+(\delta_2\kappa^2\sigma_6+1)U^3+\delta_2\sigma_5U^5-\omega U=0, \quad (8)$$

which can be simplified as

$$d_5U^2U''+d_4UU'^2+d_3U^5+d_2U^3+d_1U+K^2U^{(4)}=0, \quad (9)$$

where

$$d_1=-\frac{\omega}{\delta_2k^2\sigma_3},\ d_2=\frac{\delta_2\kappa^2\sigma_6+1}{\delta_2k^2\sigma_3},\ d_3=\frac{\sigma_5}{k^2\sigma_3},$$
$$d_4=\frac{6a+\delta_2(\sigma_6+\sigma_7)}{\delta_2\sigma_3},\ d_5=\frac{3a+\delta_2(\sigma_4+\sigma_8)}{\delta_2\sigma_3}. \quad (10)$$

Consider a governing model

$$F(u,u_x,u_t,u_{xt},u_{xx},...)=0, \quad (11)$$

where $u=u(x,t)$ denotes a wave profile, while t and x depict the time and space variables in sequence.
The relations

$$u(x,t)=U(\xi),\quad \xi=k(x-\upsilon t), \quad (12)$$

condense Eq. (9) to

$$P(U,-k\upsilon U',kU',k^2U'',...)=0, \quad (13)$$

where k is the wave width, ξ is the wave variable, and υ is the wave velocity.

2.1. The enhanced Kudryashov's method

Step–1: Assuming the solution of Eq. (13) is as follows:

$$U(\xi)=\lambda_0+\sum_{l=1}^{N}\sum_{i+j=l}\lambda_{ij}Q^i(\xi)R^j(\xi)\ , \quad (14)$$

where the constants $\lambda_0,\lambda_{ij}(i,j=0,1,...,N)$ will be computed later, the functions $Q(\xi)$ and $R(\xi)$ satisfy the following ordinary differential equations (ODEs):

$$R'(\xi)^2 = R(\xi)^2 - \chi R(\xi)^4, \tag{15}$$

and

$$Q'(\xi) = \eta Q(\xi)^2 - Q(\xi). \tag{16}$$

Step–2: From Eqs (15) and (16), $R(\xi)$ and $Q(\xi)$ can be written in the form:

$$R(\xi) = \frac{4a}{4\beta^2 e^{\xi} + \chi e^{-\xi}}, \tag{17}$$

and

$$Q(\xi) = \frac{1}{\eta + \mu e^{\xi}}, \tag{18}$$

where β, χ, η and μ are constant arbitrary parameters.

Step–3: The balance number N can be determined in Eq. (14) by balancing between the nonlinear term and the highest order derivative term in Eq. (13).

Step–4: Without disregarding Eqs. (15) and (16), substituting by Eq. (14) into Eq. (13), results in a polynomial of $Q(\xi), R(\xi)$ and $R'(\xi)$. A system of algebraic equations that are overdetermined is created if all terms with the same powers are gathered and equalized to zero. We can obtain the precise solutions to Eq. (11) if we solve the system using Maple or Mathematica software.

2.2. New projective Riccati's equation approach

The central proceedings of the new projective Riccati equations method are as follows.

Step–1: Assume Eq. (13) has the formal solution

$$U(\xi) = \alpha_0 + \sum_{i=1}^{N} F^{i-1}(\xi)\left(\alpha_i F(\xi) + \varrho_i G(\xi)\right), \tag{19}$$

where $F(\xi)$ and $G(\xi)$ satisfy the following ODEs:

$$F'(\xi) = -F(\xi)G(\xi),\ G'(\xi) = 1 - G^2(\xi) - rF(\xi), \tag{20}$$

with

$$G(\xi)^2 = 1 - 2rF(\xi) + R(r)F(\xi)^2. \tag{21}$$

Here r is constant and N is a positive integer derived from the balancing principle in Eq. (13), where α_0, α_i and $\varrho_i\,(i = 0,1,...,N)$ are constants.

Step–2: The solutions of Eq. (20) are listed as follows:

Case-1: $R(r) = 0$

$$F(\xi) = \frac{1}{2r}\operatorname{sech}^2\left[\frac{\xi}{2}\right], \text{ and } G(\xi) = \tanh\left[\frac{\xi}{2}\right], \tag{22}$$

or

$$F(\xi) = -\frac{1}{2r}\operatorname{csch}^2\left[\frac{\xi}{2}\right], \text{ and } G(\xi) = \coth\left[\frac{\xi}{2}\right]. \tag{23}$$

Case-2: $R(r) = \frac{24}{25}r^2$

$$F(\xi) = \frac{1}{r}\frac{5\operatorname{sech}[\xi]}{5\operatorname{sech}[\xi] \pm 1}, \text{ and } G(\xi) = \frac{\tanh[\xi]}{1 \pm 5\operatorname{sech}[\xi]}. \tag{24}$$

Case-3: $R(r)=\frac{5}{9}r^2$

$$F(\xi)=\frac{1}{r}\frac{3\,\text{sech}[\xi]}{3\,\text{sech}[\xi]\pm 2},\quad \text{and}\quad G(\xi)=\frac{2}{2\coth[\xi]\pm 3\,\text{csch}[\xi]}. \tag{25}$$

Case-4: $R(r)=r^2+1$

$$F(\xi)=\frac{\text{csch}[\xi]}{r\,\text{csch}[\xi]+1},\quad \text{and}\quad G(\xi)=\frac{\coth[\xi]}{r\,\text{csch}[\xi]+1}. \tag{26}$$

Case-5: $R(r)=r^2-1$

$$F(\xi)=\frac{4\,\text{sech}[\xi]}{3\tanh[\xi]+4r\,\text{sech}[\xi]+5},\quad \text{and}\quad G(\xi)=\frac{5\tanh[\xi]+3}{3\tanh[\xi]+4r\,\text{sech}[\xi]+5}, \tag{27}$$

or

$$F(\xi)=\frac{\text{sech}[\xi]}{r\,\text{sech}[\xi]+1},\quad \text{and}\quad G(\xi)=\frac{\tanh[\xi]}{r\,\text{sech}[\xi]+1}. \tag{28}$$

Step–3: Inserting Eq. (19) along with Eqs. (20) and (21) into Eq. (13), we get a polynomial of $F(\xi)$ and $G(\xi)$ which equals to zero. The obtained coefficients of this polynomial give the needed parameters in Eqs. (12) and (19).

3. Solitons for the governing model

3.1. The new enhanced Kudryashov's method

Balancing U^5 with $U^{(4)}$ in Eq. (9) gives $N=1$, accordingly the solution takes the form:

$$U(\xi)=\lambda_0+\lambda_{01}R(\xi)+\lambda_{10}Q(\xi). \tag{29}$$

Putting Eq. (29) into Eq. (9) along with Eqs. (15) and (16), leads to the following system of algebraic equations:

$$d_4\eta^2\lambda_{10}^3+2d_5\eta^2\lambda_{10}^3+d_3\lambda_{10}^5+24\eta^4\lambda_{10}K^2=0, \tag{30}$$

$$d_4\eta^2\lambda_{01}\lambda_{10}^2+4d_5\eta^2\lambda_{01}\lambda_{10}^2+5d_3\lambda_{01}\lambda_{10}^4=0, \tag{31}$$

$$d_4\eta^2\lambda_0\lambda_{10}^2+4d_5\eta^2\lambda_0\lambda_{10}^2-2d_4\eta\lambda_{10}^3-3d_5\eta\lambda_{10}^3+5d_3\lambda_0\lambda_{10}^4-60\eta^3\lambda_{10}K^2=0, \tag{32}$$

$$2d_5\eta^2\lambda_{01}^2\lambda_{10}+10d_3\lambda_{01}^2\lambda_{10}^3=0, \tag{33}$$

$$4d_5\eta^2\lambda_0\lambda_{01}\lambda_{10}-2d_4\eta\lambda_{01}\lambda_{10}^2-6d_5\eta\lambda_{01}\lambda_{10}^2+20d_3\lambda_0\lambda_{01}\lambda_{10}^3=0, \tag{34}$$

$$2d_4\eta\lambda_{01}\lambda_{10}^2=0, \tag{35}$$

$$\begin{aligned}&2d_5\eta^2\lambda_0^2\lambda_{10}-2d_4\eta\lambda_0\lambda_{10}^2-6d_5\eta\lambda_0\lambda_{10}^2+10d_3\lambda_0^2\lambda_{10}^3\\&+d_2\lambda_{10}^3+d_4\lambda_{10}^3+d_5\lambda_{10}^3+50\eta^2\lambda_{10}K^2=0,\end{aligned} \tag{36}$$

$$10d_3\lambda_{01}^3\lambda_{10}^2-2d_5\lambda_{01}\lambda_{10}^2\chi=0, \tag{37}$$

$$30d_3\lambda_0\lambda_{01}^2\lambda_{10}^2-3d_5\eta\lambda_{01}^2\lambda_{10}=0, \tag{38}$$

$$2d_4\eta\lambda_{01}^2\lambda_{10}=0, \tag{39}$$

$$-6d_5\eta\lambda_0\lambda_{01}\lambda_{10}+30d_3\lambda_0^2\lambda_{01}\lambda_{10}^2+3d_2\lambda_{01}\lambda_{10}^2+d_4\lambda_{01}\lambda_{10}^2+3d_5\lambda_{01}\lambda_{10}^2=0, \tag{40}$$

$$2d_4\eta\lambda_0\lambda_{01}\lambda_{10}-2d_4\lambda_{01}\lambda_{10}^2=0, \tag{41}$$

$$-3d_5\eta\lambda_{10}\lambda_0^2+10d_3\lambda_{10}^2\lambda_0^3+3d_2\lambda_{10}^2\lambda_0+d_4\lambda_{10}^2\lambda_0+2d_5\lambda_{10}^2\lambda_0-15\eta\lambda_{10}K^2=0, \quad (42)$$

$$-d_4\lambda_{10}\lambda_{01}^2\chi-4d_5\lambda_{10}\lambda_{01}^2\chi+5d_3\lambda_{10}\lambda_{01}^4=0, \quad (43)$$

$$20d_3\lambda_0\lambda_{01}^3\lambda_{10}-4d_5\lambda_0\lambda_{01}\lambda_{10}\chi=0, \quad (44)$$

$$30d_3\lambda_0^2\lambda_{10}\lambda_{01}^2+3d_2\lambda_{10}\lambda_{01}^2+d_4\lambda_{10}\lambda_{01}^2+3d_5\lambda_{10}\lambda_{01}^2=0, \quad (45)$$

$$-2d_4\lambda_{01}^2\lambda_{10}=0, \quad (46)$$

$$20d_3\lambda_{01}\lambda_{10}\lambda_0^3+6d_2\lambda_{01}\lambda_{10}\lambda_0+4d_5\lambda_{01}\lambda_{10}\lambda_0=0, \quad (47)$$

$$-2d_4\lambda_0\lambda_{01}\lambda_{10}=0, \quad (48)$$

$$5d_3\lambda_{10}\lambda_0^4+3d_2\lambda_{10}\lambda_0^2+d_5\lambda_{10}\lambda_0^2+d_1\lambda_{10}+\lambda_{10}K^2=0, \quad (49)$$

$$-d_4\lambda_{01}^3\chi-2d_5\lambda_{01}^3\chi+d_3\lambda_{01}^5+24\lambda_{01}K^2\chi^2=0, \quad (50)$$

$$-d_4\lambda_0\lambda_{01}^2\chi-4d_5\lambda_0\lambda_{01}^2\chi+5d_3\lambda_0\lambda_{01}^4=0, \quad (51)$$

$$-2d_5\lambda_0^2\lambda_{01}\chi+10d_3\lambda_0^2\lambda_{01}^3+d_2\lambda_{01}^3+d_4\lambda_{01}^3+d_5\lambda_{01}^3-20\lambda_{01}K^2\chi=0, \quad (52)$$

$$10d_3\lambda_{01}^2\lambda_0^3+3d_2\lambda_{01}^2\lambda_0+d_4\lambda_{01}^2\lambda_0+2d_5\lambda_{01}^2\lambda_0=0 \quad (53)$$

$$5d_3\lambda_{01}\lambda_0^4+3d_2\lambda_{01}\lambda_0^2+d_5\lambda_{01}\lambda_0^2+d_1\lambda_{01}+\lambda_{01}K^2=0, \quad (54)$$

$$d_3\lambda_0^5+d_2\lambda_0^3+d_1\lambda_0=0. \quad (55)$$

Here $d_1 \ldots d_5$ are given by Eq. (10).

Solving these equations together yields the following results:

Result-1:

$$K=2\sqrt{-\frac{d_1(2d_2-d_4-d_5)}{8d_2+d_4-4d_5}},\ \lambda_0=\pm2\sqrt{\frac{5d_1}{-8d_2-d_4+4d_5}},\ \lambda_{01}=0,$$
$$\lambda_{10}=\mp4\eta\sqrt{\frac{5d_1}{-8d_2-d_4+4d_5}}, \quad (56)$$
$$d_3=\frac{96d_2^2+4d_4d_2-16d_5d_2-d_4^2-16d_5^2+8d_4d_5}{400d_1}.$$

Plugging the obtained parameters in Eq. (56) with Eqs. (17) and (18) into Eq. (29), as a consequence, we get:

$$q(x,t)=\pm2\sqrt{\frac{5d_1}{-8d_2-d_4+4d_5}}\left(1-\frac{2\eta}{\mu e^{2\sqrt{-\frac{d_1(2d_2-d_4-d_5)}{8d_2+d_4-4d_5}}x}+\eta}\right)e^{i(\omega t+\theta_0)}. \quad (57)$$

Setting $\eta=\pm\mu$, we get dark and singular solitons with $d_1(2d_2-d_4-d_5)(8d_2+d_4-4d_5)<0$ and $d_1(-8d_2-d_4+4d_5)>0$,

$$q(x,t)=\pm2\sqrt{\frac{5d_1}{-8d_2-d_4+4d_5}}\tanh\left[\sqrt{-\frac{d_1(2d_2-d_4-d_5)}{8d_2+d_4-4d_5}}x\right]e^{i(\omega t+\theta_0)}, \quad (58)$$

and

$$q(x,t)=\pm2\sqrt{\frac{5d_1}{-8d_2-d_4+4d_5}}\coth\left[\sqrt{-\frac{d_1(2d_2-d_4-d_5)}{8d_2+d_4-4d_5}}x\right]e^{i(\omega t+\theta_0)}, \quad (59)$$

respectively.

Result-2:

$$K=\pm\sqrt{-d_1},\ \lambda_0=0,\ \lambda_{01}=\pm 2\sqrt{-\frac{5d_1\chi}{d_2+d_4+d_5}},$$
$$\lambda_{10}=0,\ d_3=\frac{6d_2^2+7d_4d_2+2d_5d_2+d_4^2-4d_5^2-3d_4d_5}{100d_1}. \tag{60}$$

Plugging the obtained parameters in Eq. (60) with Eqs. (17) and (18) into Eq. (29), as a consequence, we get:

$$q(x,t)=\frac{\pm 8\beta\sqrt{-\frac{5d_1\chi}{d_2+d_4+d_5}}}{4\beta^2e^{\sqrt{-d_1}x}+\chi e^{-\sqrt{-d_1}x}}e^{i(\omega t+\theta_0)}. \tag{61}$$

Setting $\chi=\pm 4\beta^2$, we get bright soliton with $d_1<0$ and $d_2+d_4+d_5>0$,

$$q(x,t)=\pm 2\sqrt{-\frac{5d_1}{d_2+d_4+d_5}}\operatorname{sech}\left[\sqrt{-d_1}x\right]e^{i(\omega t+\theta_0)}, \tag{62}$$

and singular soliton with $d_1<0$ and $d_2+d_4+d_5<0$,

$$q(x,t)=\mp 2\sqrt{\frac{5d_1}{d_2+d_4+d_5}}\operatorname{csch}\left[\sqrt{-d_1}x\right]e^{i(\omega t+\theta_0)}. \tag{63}$$

3.2. New projective equation method

Balancing U^5 with $U^{(4)}$ in Eq. (9) gives $N=1$, accordingly the solution takes the form:

$$U(\xi)=\alpha_0+\alpha_1F(\xi)+\varrho_1G(\xi). \tag{64}$$

Plugging Eq. (64) together with Eqs. (20) and (21) into Eq. (9), we get a system of algebraic equations:

$$5\alpha_1^4d_3\varrho_1+10\alpha_1^2d_3\varrho_1^3R(\tau)+3\alpha_1^2d_4\varrho_1R(\tau)+6\alpha_1^2d_5\varrho_1R(\tau)+d_3\varrho_1^5R(\tau)^2$$
$$+d_4\varrho_1^3R(\tau)^2+2d_5\varrho_1^3R(\tau)^2+24K^2\varrho_1R(\tau)^2=0, \tag{65}$$

$$-20\alpha_1^2d_3\tau\varrho_1^3-4\alpha_1^2d_4\tau\varrho_1-7\alpha_1^2d_5\tau\varrho_1+20\alpha_0\alpha_1^3d_3\varrho_1+20\alpha_0\alpha_1d_3\varrho_1^3R(\tau)$$
$$+2\alpha_0\alpha_1d_4\varrho_1R(\tau)+8\alpha_0\alpha_1d_5\varrho_1R(\tau)-4d_3\tau\varrho_1^5R(\tau)-2d_4\tau\varrho_1^3R(\tau)$$
$$-5d_5\tau\varrho_1^3R(\tau)-36K^2\tau\varrho_1R(\tau)=0, \tag{66}$$

$$-40\alpha_0\alpha_1d_3\tau\varrho_1^3-2\alpha_0\alpha_1d_4\tau\varrho_1-8\alpha_0\alpha_1d_5\tau\varrho_1+10\alpha_1^2d_3\varrho_1^3+30\alpha_0^2\alpha_1^2d_3\varrho_1+3\alpha_1^2d_2\varrho_1$$
$$+2\alpha_1^2d_5\varrho_1+10\alpha_0^2d_3\varrho_1^3R(\tau)+2\alpha_0^2d_5\varrho_1R(\tau)+2d_3\varrho_1^5R(\tau)+d_2\varrho_1^3R(\tau)+2d_5\varrho_1^3R(\tau)$$
$$+4d_3\tau^2\varrho_1^5+d_4\tau^2\varrho_1^3+2d_5\tau^2\varrho_1^3+8K^2\varrho_1R(\tau)+6K^2\tau^2\varrho_1=0, \tag{67}$$

$$-20\alpha_0^2d_3\tau\varrho_1^3-\alpha_0^2d_5\tau\varrho_1+20\alpha_0\alpha_1d_3\varrho_1^3+20\alpha_0^3\alpha_1d_3\varrho_1+6\alpha_0\alpha_1d_2\varrho_1$$
$$+2\alpha_0\alpha_1d_5\varrho_1-4d_3\tau\varrho_1^5-2d_2\tau\varrho_1^3-d_5\tau\varrho_1^3-K^2\tau\varrho_1=0, \tag{68}$$

$$10\alpha_0^2d_3\varrho_1^3+5\alpha_0^4d_3\varrho_1+3\alpha_0^2d_2\varrho_1+d_3\varrho_1^5+d_2\varrho_1^3+d_1\varrho_1=0, \tag{69}$$

$$\alpha_1^5d_3+\alpha_1^3d_4R(\tau)+2\alpha_1^3d_5R(\tau)+10\alpha_1^3d_3\varrho_1^2R(\tau)+5\alpha_1d_3\varrho_1^4R(\tau)^2$$
$$+3\alpha_1d_4\varrho_1^2R(\tau)^2+6\alpha_1d_5\varrho_1^2R(\tau)^2+24\alpha_1K^2R(\tau)^2=0, \tag{70}$$

$$-2\alpha_1^3d_4\tau-3\alpha_1^3d_5\tau-20\alpha_1^3d_3\tau\varrho_1^2+5\alpha_0\alpha_1^4d_3+\alpha_0\alpha_1^2d_4R(\tau)+4\alpha_0\alpha_1^2d_5R(\tau)$$
$$+30\alpha_0\alpha_1^2d_3\varrho_1^2R(\tau)-20\alpha_1d_3\tau\varrho_1^4R(\tau)-8\alpha_1d_4\tau\varrho_1^2R(\tau)-17\alpha_1d_5\tau\varrho_1^2R(\tau)$$
$$+5\alpha_0d_3\varrho_1^4R(\tau)^2+\alpha_0d_4\varrho_1^2R(\tau)^2+4\alpha_0d_5\varrho_1^2R(\tau)^2-60\alpha_1K^2\tau R(\tau)=0, \tag{71}$$

$$-2\alpha_0\alpha_1^2d_4\tau-6\alpha_0\alpha_1^2d_5\tau+20\alpha_1d_3\tau^2\varrho_1^4+5\alpha_1d_4\tau^2\varrho_1^2+10\alpha_1d_5\tau^2\varrho_1^2$$
$$-60\alpha_0\alpha_1^2d_3\tau\varrho_1^2+10\alpha_1^3d_3\varrho_1^2+10\alpha_0^2\alpha_1^3d_3+\alpha_1^3d_2+\alpha_1^3d_4+\alpha_1^3d_5$$
$$+2\alpha_0^2\alpha_1d_5R(\tau)-20\alpha_0d_3\tau\varrho_1^4R(\tau)+10\alpha_1d_3\varrho_1^4R(\tau)-2\alpha_0d_4\tau\varrho_1^2R(\tau) \quad (72)$$
$$-10\alpha_0d_5\tau\varrho_1^2R(\tau)+30\alpha_0^2\alpha_1d_3\varrho_1^2R(\tau)+3\alpha_1d_2\varrho_1^2R(\tau)$$
$$+2\alpha_1d_4\varrho_1^2R(\tau)+7\alpha_1d_5\varrho_1^2R(\tau)+30\alpha_1K^2\tau^2+20\alpha_1K^2R(\tau)=0,$$

$$-3\alpha_0^2\alpha_1d_5\tau+20\alpha_0d_3\tau^2\varrho_1^4+\alpha_0d_4\tau^2\varrho_1^2+4\alpha_0d_5\tau^2\varrho_1^2-20\alpha_1d_3\tau\varrho_1^4$$
$$-60\alpha_0^2\alpha_1d_3\tau\varrho_1^2-6\alpha_1d_2\tau\varrho_1^2-2\alpha_1d_4\tau\varrho_1^2-7\alpha_1d_5\tau\varrho_1^2+30\alpha_0\alpha_1^2d_3\varrho_1^2 \quad (73)$$
$$+10\alpha_0^3\alpha_1^2d_3+3\alpha_0\alpha_1^2d_2+\alpha_0\alpha_1^2d_4+2\alpha_0\alpha_1^2d_5+10\alpha_0d_3\varrho_1^4R(\tau)$$
$$+10\alpha_0^3d_3\varrho_1^2R(\tau)+3\alpha_0d_2\varrho_1^2R(\tau)+4\alpha_0d_5\varrho_1^2R(\tau)-15\alpha_1K^2\tau=0,$$

$$-20\alpha_0^3d_3\tau\varrho_1^2-20\alpha_0d_3\tau\varrho_1^4-6\alpha_0d_2\tau\varrho_1^2-2\alpha_0d_5\tau\varrho_1^2$$
$$+30\alpha_1\alpha_0^2d_3\varrho_1^2+5\alpha_1d_3\varrho_1^4+3\alpha_1d_2\varrho_1^2+\alpha_1d_5\varrho_1^2 \quad (74)$$
$$+5\alpha_1\alpha_0^4d_3+3\alpha_1\alpha_0^2d_2+\alpha_1\alpha_0^2d_5+\alpha_1d_1+\alpha_1K^2=0,$$

$$10\alpha_0^3d_3\varrho_1^2+5\alpha_0d_3\varrho_1^4+3\alpha_0d_2\varrho_1^2+\alpha_0^5d_3+\alpha_0^3d_2+\alpha_0d_1=0. \quad (75)$$

Solving these equations together yields the following results:

Case-1: $R(\tau)=0$

$$\alpha_0=0,\ \alpha_1=0,\ \varrho_1=\pm2\sqrt{\frac{5d_1}{-8d_2-d_4+4d_5}},\ K=2\sqrt{-\frac{d_1(2d_2-d_4-d_5)}{8d_2+d_4-4d_5}},$$
$$d_3=\frac{96d_2^2+4d_4d_2-16d_5d_2-d_4^2-16d_5^2+8d_4d_5}{400d_1}. \quad (76)$$

Plugging the obtained parameters in Eq. (76) with Eqs. (22) and (23) into Eq. (64), as a consequence, we get dark and singular solitons with $d_1(2d_2-d_4-d_5)(8d_2+d_4-4d_5)<0$ and $d_1(-8d_2-d_4+4d_5)>0$,

$$q(x,t)=\pm2\sqrt{\frac{5d_1}{-8d_2-d_4+4d_5}}\tanh\left[\sqrt{-\frac{d_1(2d_2-d_4-d_5)}{8d_2+d_4-4d_5}}x\right]e^{i(\omega t+\theta_0)}, \quad (77)$$

and

$$q(x,t)=\pm2\sqrt{\frac{5d_1}{-8d_2-d_4+4d_5}}\coth\left[\sqrt{-\frac{d_1(2d_2-d_4-d_5)}{8d_2+d_4-4d_5}}x\right]e^{i(\omega t+\theta_0)}, \quad (78)$$

respectively.

Case-2: $R(\tau)=\frac{24}{25}\tau^2$

$$\alpha_0=0,\ \alpha_1=\pm4\tau\sqrt{\frac{6d_1}{5(-8d_2-d_4+4d_5)}},\ \varrho_1=\pm2\sqrt{\frac{5d_1}{-8d_2-d_4+4d_5}},$$
$$K=2\sqrt{-\frac{d_1(2d_2-d_4-d_5)}{8d_2+d_4-4d_5}},\ d_3=\frac{96d_2^2+4d_4d_2-16d_5d_2-d_4^2-16d_5^2+8d_4d_5}{400d_1}. \quad (79)$$

Plugging the obtained parameters in Eq. (79) with Eq. (24) into Eq. (64), as a consequence, we get straddled bright–dark solitons with $d_1(2d_2-d_4-d_5)(8d_2+d_4-4d_5)<0$ and $d_1(-8d_2-d_4+4d_5)>0$,

$$q(x,t)=\pm 2\sqrt{\frac{5d_1}{-8d_2-d_4+4d_5}}\times\left(\frac{2\sqrt{6}\operatorname{sech}\left[2\sqrt{-\frac{d_1(2d_2-d_4-d_5)}{8d_2+d_4-4d_5}}x\right]\pm\tanh\left[2\sqrt{-\frac{d_1(2d_2-d_4-d_5)}{8d_2+d_4-4d_5}}x\right]}{5\operatorname{sech}\left[2\sqrt{-\frac{d_1(2d_2-d_4-d_5)}{8d_2+d_4-4d_5}}x\right]\pm 1}\right)\times e^{i(\omega t+\theta_0)}. \quad (80)$$

Case-3: $R(\tau)=\frac{5}{9}\tau^2$

$$\alpha_0=0,\ \alpha_1=\pm\frac{10}{3}\tau\sqrt{\frac{d_1}{-8d_2-d_4+4d_5}},\ \varrho_1=\pm 10\sqrt{\frac{d_1}{-8d_2-d_4+4d_5}},$$
$$K=2\sqrt{-\frac{d_1(2d_2-d_4-d_5)}{8d_2+d_4-4d_5}},\ d_3=\frac{96d_2^2+4d_4d_2-16d_5d_2-d_4^2-16d_5^2+8d_4d_5}{400d_1}. \quad (81)$$

Plugging the obtained parameters in Eq. (81) with Eq. (25) into Eq. (64), as a consequence, we get straddled bright–singular solitons with $d_1(2d_2-d_4-d_5)(8d_2+d_4-4d_5)<0$ and $d_1(-8d_2-d_4+4d_5)>0$,

$$q(x,t)=\pm\sqrt{\frac{100d_1}{-8d_2-d_4+4d_5}}\left\{\frac{\operatorname{sech}\left[2\sqrt{-\frac{d_1(2d_2-d_4-d_5)}{8d_2+d_4-4d_5}}x\right]}{3\operatorname{sech}\left[2\sqrt{-\frac{d_1(2d_2-d_4-d_5)}{8d_2+d_4-4d_5}}x\right]\pm 2}+\frac{2}{2\coth\left[2\sqrt{-\frac{d_1(2d_2-d_4-d_5)}{8d_2+d_4-4d_5}}x\right]\pm 3\operatorname{csch}\left[2\sqrt{-\frac{d_1(2d_2-d_4-d_5)}{8d_2+d_4-4d_5}}x\right]}\right\}e^{i(\omega t+\theta_0)}. \quad (82)$$

Case-4: $R(\tau)=\tau^2+1$

$$\alpha_0=0,\ \alpha_1=\pm 2\sqrt{\frac{5d_1(\tau^2+1)}{-8d_2-d_4+4d_5}},\ \varrho_1=\pm 2\sqrt{\frac{5d_1}{-8d_2-d_4+4d_5}},$$
$$K=2\sqrt{-\frac{d_1(2d_2-d_4-d_5)}{8d_2+d_4-4d_5}},\ d_3=\frac{96d_2^2+4d_4d_2-16d_5d_2-d_4^2-16d_5^2+8d_4d_5}{400d_1}. \quad (83)$$

Plugging the obtained parameters in Eq. (83) with Eq. (26) into Eq. (64), as a consequence, we get straddled singular–singular solitons with $d_1(2d_2-d_4-d_5)(8d_2+d_4-4d_5)<0$ and $d_1(-8d_2-d_4+4d_5)>0$,

$$q(x,t)=\pm 2\sqrt{\frac{5d_1}{-8d_2-d_4+4d_5}}\times\left(\frac{\sqrt{\tau^2+1}\operatorname{csch}\left[2\sqrt{-\frac{d_1(2d_2-d_4-d_5)}{8d_2+d_4-4d_5}}x\right]+\coth\left[2\sqrt{-\frac{d_1(2d_2-d_4-d_5)}{8d_2+d_4-4d_5}}x\right]}{\tau\operatorname{csch}\left[2\sqrt{-\frac{d_1(2d_2-d_4-d_5)}{8d_2+d_4-4d_5}}x\right]+1}\right)\times e^{i(\omega t+\theta_0)}. \quad (84)$$

Case-5: $R(\tau)=\tau^2-1$

$$\alpha_0=0,\ \alpha_1=\pm 2\sqrt{\frac{5d_1(1-\tau^2)}{8d_2+d_4-4d_5}},\ \varrho_1=\pm 2\sqrt{\frac{5d_1}{-8d_2-d_4+4d_5}},$$
$$K=2\sqrt{-\frac{d_1(2d_2-d_4-d_5)}{8d_2+d_4-4d_5}},\ d_3=\frac{96d_2^2+4d_4d_2-16d_5d_2-d_4^2-16d_5^2+8d_4d_5}{400d_1}. \tag{85}$$

Plugging the obtained parameters in Eq. (85) with Eqs. (27) and (28) into Eq. (64), as a consequence, we get straddeled bright–dark solitons with $d_1(2d_2-d_4-d_5)(8d_2+d_4-4d_5)<0$ and $d_1(-8d_2-d_4+4d_5)>0$, respectively.

$$q(x,t)=\pm 2\sqrt{\frac{5d_1}{-8d_2-d_4+4d_5}}$$
$$\times\left(\frac{4\sqrt{\tau^2-1}\,\mathrm{sech}\left[2\sqrt{-\frac{d_1(2d_2-d_4-d_5)}{8d_2+d_4-4d_5}}x\right]+5\tanh\left[2\sqrt{-\frac{d_1(2d_2-d_4-d_5)}{8d_2+d_4-4d_5}}x\right]+3}{4\tau\,\mathrm{sech}\left[2\sqrt{-\frac{d_1(2d_2-d_4-d_5)}{8d_2+d_4-4d_5}}x\right]+3\tanh\left[2\sqrt{-\frac{d_1(2d_2-d_4-d_5)}{8d_2+d_4-4d_5}}x\right]+5}\right) \tag{86}$$
$$\times e^{i(\omega t+\theta_0)},$$

and

$$q(x,t)=\pm 2\sqrt{\frac{5d_1}{-8d_2-d_4+4d_5}}$$
$$\times\left(\frac{\sqrt{\tau^2-1}\,\mathrm{sech}\left[2\sqrt{-\frac{d_1(2d_2-d_4-d_5)}{8d_2+d_4-4d_5}}x\right]+\tanh\left[2\sqrt{-\frac{d_1(2d_2-d_4-d_5)}{8d_2+d_4-4d_5}}x\right]}{\tau\,\mathrm{sech}\left[2\sqrt{-\frac{d_1(2d_2-d_4-d_5)}{8d_2+d_4-4d_5}}x\right]+1}\right) \tag{87}$$
$$\times e^{i(\omega t+\theta_0)}.$$

4. Conclusions

The paper retrieved quiescent optical solitons for the dispersive concatenation model with Kerr's law of nonlinear refractive index change and with nonlinear CD. The two approaches collectively revealed the full spectrum of quiescent optical solitons. The parameter restrictions for the existence of such solitons are also indicated. The paper results are thus promising to pursue additional avenues from the model. The model will be next studied with the power–law of SPM. Later, it is imperative to address this model with differential group delay. Subsequently, the model must be additionally considered with fractional temporal evolution as opposed to linear temporal evolution. This would give an even wider perspective to the present work. Such studies are underway, and the results of these research activities will be sequentially reported with time.

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Анотація. *У поточній роботі отримано стаціонарні оптичні солітони для дисперсійної моделі конкатенації, яка розглядається з врахуванням закону Керра для нелінійного показника заломлення і нелінійної хроматичної дисперсії. З використанням двох методів інтегрування виявлено повний спектр стаціонарних оптичних солітонів. У роботі також наведені критерії їхнього існування.*

Ключові слова: *стаціонарні оптичні солітони, схема Кудряшова, алгоритм рівняння Ріккаті, модель дисперсійної конкатенації*