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Highly Dispersive Gap Solitons in Optical Fibers with Dispersive Reflectivity Having Parabolic–Nonlocal Nonlinearity

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Abstract. The current paper studies gap solitons with parabolic–nonlocal form of self–phase modulation. The soliton solutions for this model are revealed with the successful application of the extended auxiliary equation approach. The parameter constraints ensure the existence of such gap solitons.

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1. Introduction

Optoelectronics is one of the most fascinating areas of telecommunications engineering that has made its visibility during the past few decades. The science of optical solitons is the one that is making a lasting impression in this field. The plethora of relentless impressive results visible across various journals has captured the attention of various telecommunication engineers, physicists, and mathematicians alike [1–10]. However, a few issues are yet to be addressed for the element of perfection to be embedded with the soliton transmission technology across intercontinental distances. One such issue is the low chromatic dispersion (CD) count that must be compensated and replenished to sustain the delicate balance between CD and the self–phase modulation (SPM). There are several mathematical as well as down-to-earth measures to achieve this. One such engineering marvel is implementing the grating structure around the internal walls of the core. This grating structure would lead to dispersive reflectivity. This would yield the corresponding model with such dispersive reflectivity as opposed to CD. The current paper carries out the integration of this tactful model by the usage of an extended auxiliary equation approach where the SPM stems from parabolic nonlocal nonlinearity. The soliton solutions are retrieved and presented along with the parameter constraints that ensure their existence during transmission across intercontinental distances.

The primary objective of the present paper is to introduce and formulate, for the first time, a dimensionless representation of the highly dispersive perturbed nonlinear Schrödinger's equation (NLSE) that describes the dynamics of fiber Bragg gratings. This equation incorporates dispersion effects and perturbations while considering a parabolic non-local law governing the nonlinear refractive index within the fiber grating structure. This contribution is expected to contribute to a more comprehensive and standardized approach to investigating the complex behaviors exhibited by fiber Bragg gratings under these combined influences.

The main difference between the present paper and the previously studied paper [10] lies in their respective focuses and contributions within the realm of gap solitons and fiber Bragg gratings. While both papers involve the study of gap solitons and their behavior within the context of dispersive Bragg grating fibers, they differ in their specific objectives and the aspects they emphasize. The paper [10] addresses the characteristics of gap solitons arising from the interplay between cubic-quartic dispersive reflectivity terms and a parabolic law of nonlinear refractive index. This unique combination introduces new dynamics and behaviors that have not been extensively explored before. On the other hand, the present paper emphasizes the development of a dimensionless formulation for the highly dispersive perturbed NLSE that governs the dynamics of fiber Bragg gratings. This dimensionless equation form enables a more general and standardized approach to understanding the system's behavior by removing the constraints of specific units or scales. This contribution serves as a foundational framework for investigating the collective impact of dispersion effects, perturbations, and the parabolic non-local law of nonlinear refractive index on the behavior of fiber Bragg gratings. While both papers explore the intricate behaviors of gap solitons in dispersive Bragg grating fibers, the paper [10] investigates the specific effects of cubic-quartic dispersive reflectivity and the parabolic law of nonlinear refractive index, while the present paper is concerned with a more universal and standardized formulation of the governing equation. Together, these contributions deepen the understanding of the complex phenomena inherent to fiber Bragg gratings and contribute to advancing the collective knowledge in the field.

2. Governing model

For the first time, the equation representing the highly dispersive perturbed NLSE in fiber Bragg gratings with parabolic non–local law of nonlinear refractive index is expressed in dimensionless form:

$$\begin{aligned} &iq_t + ia_1 r_x + a_2 r_{xx} + ia_3 r_{xxx} + a_4 r_{xxxx} + ia_5 r_{xxxxx} + a_6 r_{xxxxxx} \\ &+\left(c_1|q|^2 + d_1|r|^2\right)q + \left(e_1|q|^4 + f_1|q|^2|r|^2 + g_1|r|^4\right)q + \left[l_1\left(|q|^2\right)_{xx} + m_1\left(|r|^2\right)_{xx}\right]q \\ &+i\alpha_1 q_x + \beta_1 r + \sigma_1 q^* r^2 = i\left[\gamma_1\left(|q|^2 q\right)_x + \theta_1\left(|q|^2\right)_x q + \mu_1|q|^2 q_x\right], \end{aligned} \quad (1)$$

and

$$\begin{aligned}&ir_t + ib_1q_x + b_2q_{xx} + ib_3q_{xxx} + b_4q_{xxxx} + ib_5q_{xxxxx} + b_6q_{xxxxxx}\\&+\left(c_2|r|^2 + d_2|q|^2\right)r + \left(e_2|r|^4 + f_2|r|^2|q|^2 + g_2|q|^4\right)r\\&+\left[l_2\left(|r|^2\right)_{xx} + m_2\left(|q|^2\right)_{xx}\right]r\\&+i\alpha_2r_x + \beta_2q + \sigma_2r^*q^2 = i\left[\gamma_2\left(|r|^2r\right)_x + \theta_2\left(|r|^2\right)_x r + \mu_2|r|^2r_x\right],\end{aligned} \tag{2}$$

where $q(x,t)$ and $r(x,t)$ are complex–valued functions that represent the wave profiles for the two components in fiber Bragg gratings, while $q^*(x,t)$ and $r^*(x,t)$ are complex-valued conjugate functions, $i=\sqrt{-1}$. The first terms in Eqs. (1) and (2) represent the linear-temporal evolution. The constants a_1 and b_1 are the coefficients of inter–modal dispersion, a_2 and b_2 are the coefficients of CD, a_3 and b_3 are the coefficients of third–order dispersion, a_4 and b_4 are the coefficients of fourth–order dispersion, a_5 and b_5 are the coefficients of fifth–order dispersion and a_6 and b_6 are the coefficients of sixth–order dispersion. The constants c_j and e_j, $(j=1,2)$ are the coefficients of SPM, while d_j and g_j, $(j=1,2)$ are the coefficients of cross–phase modulation. The constants $f_j,(j=1,2)$ are the coefficients of nonlinear terms, the constants l_j and $m_j,(j=1,2)$ are the coefficients of non-local law terms, α_j, $(j=1,2)$ are the coefficients of IMD, β_j, $(j=1,2)$ are the coefficients of detuning parameters and σ_j $(j=1,2)$ are the coefficients of the four-wave mixing (4WM) parameters. Finally, γ_j, $(j=1,2)$ are the coefficients of self-steeping (SS) terms, while θ_j and μ_j, $(j=1,2)$ are the coefficients of the nonlinear dispersion terms. The system (1) and (2) is a manifested version of the standard models.

3. Mathematical analysis

With the goal in mind, we make the assumption that Eqs. (2) and (3) possess a formal solution:

$$\begin{aligned}q(x,t) &= U_1(\tau)e^{i\psi(x,t)},\\r(x,t) &= U_2(\tau)e^{i\psi(x,t)},\end{aligned} \tag{3}$$

here $U_j(\tau),(j=1,2)$ and $\psi(x,t)$ are real functions, such that

$$\tau = x - vt, \psi(x,t) = -\kappa x + \omega t + \theta_0, \tag{4}$$

and v, κ, ω and θ_0 are real constants. Here $U_j(\tau),(j=1,2)$ represent the pulse shapes, v is the velocity of the soliton, κ is the soliton wave number, ω is the soliton frequency and θ_0 is a phase constant. Substituting (3) and (4) into Eqs. (1) and (2), one gets the real parts as:

$$\begin{aligned}&a_6U_2^{(6)} + (a_4 - 5a_5\kappa - 15a_6\kappa^2)U_2^{(4)}\\&+(a_2 + 3a_3\kappa - 6a_4\kappa^2 - 10a_5\kappa^3 + 15a_6\kappa^4)U_2'' + 2l_1U_1^2U_1'' + 2l_1U_1U_1'^2\\&+2m_1U_1U_2U_2'' + 2m_1U_1U_2'^2 + (\alpha_1\kappa - \omega)U_1\\&+(\beta_1 + a_1\kappa - a_2\kappa^2 - a_3\kappa^3 + a_4\kappa^4 + a_5\kappa^5 - a_6\kappa^6)U_2\\&+\left[c_1 - \kappa(\gamma_1 + \mu_1)\right]U_1^3 + (d_1 + \sigma_1)U_1U_2^2 + e_1U_1^5 + f_1U_1^3U_2^2 + g_1U_1U_2^4 = 0,\end{aligned} \tag{5}$$

and

$$\begin{aligned}&b_6U_1^{(6)}+(b_4-5b_5\kappa-15b_6\kappa^2)U_1^{(4)}\\&+(b_2+3b_3\kappa-6b_4\kappa^2-10b_5\kappa^3+15b_6\kappa^4)U_1''+2l_2U_2^2U_2''+2l_2U_2U_2'^2\\&+2m_2U_2U_1U_1''+2m_2U_2U_1'^2+(\alpha_2\kappa-\omega)U_2\\&+(\beta_2+b_1\kappa-b_2\kappa^2-b_3\kappa^3+b_4\kappa^4+b_5\kappa^5-b_6\kappa^6)U_1\\&+\left[c_2-\kappa(\gamma_2+\mu_2)\right]U_2^3+(d_2+\sigma_2)U_2U_1^2+e_2U_2^5+f_2U_2^3U_1^2+g_2U_2U_1^4=0,\end{aligned}\tag{6}$$

where $U_2^{(4)}=\dfrac{d^4U_2}{d\tau^4}$ and $U_2^{(6)}=\dfrac{d^6U_2}{d\tau^6}$ while, the imaginary parts as:

$$\begin{aligned}&(a_5-6a_6\kappa)U_2^{(5)}+(a_3-4a_4\kappa-10a_5\kappa^2+20a_6\kappa^3)U_2'''-\left[3\gamma_1+2\theta_1+\mu_1\right]U_1^2U_1'\\&+(\alpha_1-v)U_1'+(a_1-2a_2\kappa-3a_3\kappa^2+4a_4\kappa^3+5a_5\kappa^4-6a_6\kappa^5)U_2'=0,\end{aligned}\tag{7}$$

and

$$\begin{aligned}&(b_5-6b_6\kappa)U_1^{(5)}+(b_3-4b_4\kappa-10b_5\kappa^2+20b_6\kappa^3)U_1'''-\left[3\gamma_2+2\theta_2+\mu_2\right]U_2^2U_2'\\&+(\alpha_2-v)U_2'+(b_1-2b_2\kappa-3b_3\kappa^2+4b_4\kappa^3+5b_5\kappa^4-6b_6\kappa^5)U_1'=0.\end{aligned}\tag{8}$$

Set

$$U_2(\tau)=AU_1(\tau),\tag{9}$$

where A is a constant, provided $A\neq 0$ or 1. Consequently, Eqs. (5)–(8) change to:

$$\begin{aligned}&a_6AU_1^{(6)}+(a_4-5a_5\kappa-15a_6\kappa^2)AU_1^{(4)}\\&+(a_2+3a_3\kappa-6a_4\kappa^2-10a_5\kappa^3+15a_6\kappa^4)AU_1''\\&+2(m_1A^2+l_1)U_1^2U_1''+2(m_1A^2+l_1)U_1U_1'^2\\&+\left[\alpha_1\kappa-\omega+(\beta_1+a_1\kappa-a_2\kappa^2-a_3\kappa^3+a_4\kappa^4+a_5\kappa^5-a_6\kappa^6)A\right]U_1\\&+\left[c_1-\kappa(\gamma_1+\mu_1)+(d_1+\sigma_1)A^2\right]U_1^3+(e_1+f_1A^2+g_1A^4)U_1^5=0,\end{aligned}\tag{10}$$

$$\begin{aligned}&b_6U_1^{(6)}+(b_4-5b_5\kappa-15b_6\kappa^2)U_1^{(4)}\\&+(b_2+3b_3\kappa-6b_4\kappa^2-10b_5\kappa^3+15b_6\kappa^4)U_1''+2A(m_2+l_2A^2)U_1^2U_1''\\&+2A(m_2+l_2A^2)U_1U_1'^2\\&+\left[(\alpha_2\kappa-\omega)A+\beta_2+b_1\kappa-b_2\kappa^2-b_3\kappa^3+b_4\kappa^4+b_5\kappa^5-b_6\kappa^6\right]U_1\\&+A\left[c_2A^2-\kappa(\gamma_2+\mu_2)A^2+d_2+\sigma_2\right]U^3+A(e_2A^4+f_2A^2+g_2)U_1^5=0,\end{aligned}\tag{11}$$

and

$$\begin{aligned}&(a_5-6a_6\kappa)AU_1^{(5)}+(a_3-4a_4\kappa-10a_5\kappa^2+20a_6\kappa^3)AU_1'''\\&-\left[3\gamma_1+2\theta_1+\mu_1\right]U_1^2U_1'\\&+\left[\alpha_1-v+(a_1-2a_2\kappa-3a_3\kappa^2+4a_4\kappa^3+5a_5\kappa^4-6a_6\kappa^5)A\right]U_1'=0,\end{aligned}\tag{12}$$

$$\begin{aligned}&(b_5-6b_6\kappa)U_1^{(5)}+(b_3-4b_4\kappa-10b_5\kappa^2+20b_6\kappa^3)U_1'''\\&-\left[3\gamma_2+2\theta_2+\mu_2\right]A^3U_1^2U_1'\\&+\left[(\alpha_2-v)A+b_1-2b_2\kappa-3b_3\kappa^2+4b_4\kappa^3+5b_5\kappa^4-6b_6\kappa^5\right]U_1'=0,\end{aligned}\tag{13}$$

respectively. From Eqs. (12) and (13), one gets:

$$\kappa=a_5/6a_6=b_5/6b_6,\tag{14}$$

$$\begin{aligned}&a_3-4a_4\kappa-10a_5\kappa^2+20a_6\kappa^3=0,\\&b_3-4b_4\kappa-10b_5\kappa^2+20b_6\kappa^3=0,\end{aligned}\tag{15}$$

$$3\gamma_j+2\theta_j+\mu_j=0,\ \text{for } j=1,2,\tag{16}$$

and

$$v=\alpha_1+(a_1-2a_2\kappa-3a_3\kappa^2+4a_4\kappa^3+5a_5\kappa^4-6a_6\kappa^5)A,$$
$$v=\frac{A\alpha_2+b_1-2b_2\kappa-3b_3\kappa^2+4b_4\kappa^3+5b_5\kappa^4-6b_6\kappa^5}{A}. \tag{17}$$

Eqs. (10) and (11) have the same form under the constraint conditions:

$$\begin{aligned}
&a_6A=b_6,\ (a_4-5a_5\kappa-15a_6\kappa^2)A=b_4-5b_5\kappa-15b_6\kappa^2,\\
&(a_2+3a_3\kappa-6a_4\kappa^2-10a_5\kappa^3+15a_6\kappa^4)A=b_2+3b_3\kappa-6b_4\kappa^2-10b_5\kappa^3+15b_6\kappa^4,\\
&l_1+m_1A^2=A\left(l_2A^2+m_2\right),\\
&\alpha_1\kappa-\omega+\left(\beta_1+a_1\kappa-a_2\kappa^2-a_3\kappa^3+a_4\kappa^4+a_5\kappa^5-a_6\kappa^6\right)A\\
&=\left(\alpha_2\kappa-\omega\right)A+\beta_2+b_1\kappa-b_2\kappa^2-b_3\kappa^3+b_4\kappa^4+b_5\kappa^5-b_6\kappa^6,\\
&c_1-\kappa\left(\gamma_1+\mu_1\right)+\left(d_1+\sigma_1\right)A^2=A\left[c_2A^2-\kappa\left(\gamma_2+\mu_2\right)A^2+d_2+\sigma_2\right],\\
&e_1+f_1A^2+g_1A^4=A\left(e_2A^4+f_2A^2+g_2\right).
\end{aligned} \tag{18}$$

From Eq. (18), one gets:

$$\omega=\frac{\begin{pmatrix}\beta_2-\beta_1A+\left(A\alpha_2-\alpha_1+b_1-a_1A\right)\kappa-\left(b_2-a_2A\right)\kappa^2\\ -\left(b_3-a_3A\right)\kappa^3+\left(b_4-a_4A\right)\kappa^4+\left(b_5-a_5A\right)\kappa^5-\left(b_6-a_6A\right)\kappa^6\end{pmatrix}}{A-1}. \tag{19}$$

Eq. (10) can be rewritten in the form:

$$U_1^{(6)}+\Theta_4U_1^{(4)}+\Theta_2U_1''+\Delta_0U_1^2U_1''+\Delta_1U_1U_1'^2+\Theta_1U_1+\Theta_3U_1^3+\Theta_5U_1^5=0, \tag{20}$$

where

$$\begin{aligned}
&\Theta_4=\frac{a_4-5a_5\kappa-15a_6\kappa^2}{a_6},\\
&\Theta_2=\frac{a_2+3a_3\kappa-6a_4\kappa^2-10a_5\kappa^3+15a_6\kappa^4}{a_6},\\
&\Delta_0=\frac{2\left(l_1+m_1A^2\right)}{a_6A},\ \Delta_1=\frac{2\left(m_1A^2+l_1\right)}{a_6A},\\
&\Theta_1=\frac{\alpha_1k-\omega+(\beta_1+a_1\kappa-a_2\kappa^2-a_3\kappa^3+a_4\kappa^4+a_5\kappa^5-a_6\kappa^6)A}{a_6A},\\
&\Theta_3=\frac{c_1-\kappa\left(\gamma_1+\mu_1\right)+\left(d_1+\sigma_1\right)A^2}{a_6A},\ \Theta_5=\frac{e_1+f_1A^2+g_1A^4}{a_6A}.
\end{aligned} \tag{21}$$

Next, balancing $U_1^{(6)}$ and U_1^5 in Eq. (20) deduces the balance number $N=3/2$. Thus, we take the transformation:

$$U_1\left(\tau\right)=Z^{\frac{3}{2}}\left(\tau\right), \tag{22}$$

where $Z\left(\tau\right)$ is a new positive function. Next, Eq. (20) changes to:

$$\begin{aligned}
&\frac{32}{105}Z^5Z^{(6)}+\frac{32}{35}Z^4Z'Z^{(5)}+\frac{16}{7}Z^4Z''Z''''-\frac{8}{7}Z^3Z'^2Z''''+\frac{32}{21}Z^4Z'''^2\\
&-\frac{32}{7}Z^3Z'Z''Z'''+\frac{16}{7}Z^2Z'^3Z'''-\frac{8}{7}Z^3Z''^3+\frac{36}{7}Z^2Z'^2Z''^2-\frac{30}{7}ZZ'^4Z''\\
&+Z'^6+\frac{32}{105}\Delta_0Z^8Z''+\frac{16}{105}\Delta_0Z^7Z'^2+\frac{16}{35}\Delta_1Z^7Z'^2+\frac{32}{105}\Theta_4Z^5Z''''\\
&+\frac{64}{105}\Theta_4Z^4Z'Z'''+\frac{16}{35}\Theta_4Z^4Z''^2-\frac{16}{35}\Theta_4Z^3Z'^2Z''+\frac{4}{35}\Theta_4Z^2Z'^4+\frac{32}{105}\Theta_2Z^5Z''\\
&+\frac{16}{105}\Theta_2Z^4Z'^2+\frac{64}{315}\Theta_1Z^6+\frac{64}{315}\Theta_3Z^9+\frac{64}{315}\Theta_5Z^{12}=0.
\end{aligned} \tag{23}$$

Here, $Z^{(6)} = \frac{d^6 Z}{d\tau^6}$, $Z^{(5)} = \frac{d^5 Z}{d\tau^5}$, $Z^{(4)} = \frac{d^4 Z}{d\tau^4}$, $Z''' = \frac{d^3 Z}{d\tau^3}$, $Z'' = \frac{d^2 Z}{d\tau^2}$, and $Z' = \frac{dZ}{d\tau}$. In the next section, we will solve Eq. (23) using the extended auxiliary equation approach:

4. Extended auxiliary equation approach

Using this method, deduces the formal solution of Eq. (23) as:

$$Z(\tau) = \sum_{s=0}^{N} \Omega_s R^s(\tau), \tag{24}$$

where $R(\tau)$ satisfies the first order auxiliary equation:

$$R'^2(\tau) = \sum_{z=0}^{4} h_z R^z(\tau). \tag{25}$$

Here $\Omega_s\,(s = 0,1,....,N)$ and $h_z\,(z = 0,1,2,3,4)$ are constants to be determined such that $\Omega_N \neq 0$ and $h_4 \neq 0$, where N is a positive integer. We determine the balance number N of Eq. (24) by using the homogeneous balance method as follows:

if $D(Z) = N$, $D(Z') = N+1$, $D(Z'') = N+2$ and hence

$$D\left[Z^m Z^{(n)}\right] = N(m+1)+n. \tag{26}$$

It is well known that Eq. (25) has the following types of solutions:

Type–1: If $h_0 = h_1 = h_3 = 0$, then Eq. (25) has the bright soliton solution:

$$R(\tau) = \pm\sqrt{-\frac{h_2}{h_4}}\,\mathrm{sech}\left(\sqrt{h_2}\,\tau\right), \tag{27}$$

provided $h_2 > 0$ and $h_4 < 0$, and the singular soliton solution:

$$R(\tau) = \pm\sqrt{\frac{h_2}{h_4}}\,\mathrm{csch}\left(\sqrt{h_2}\,\tau\right), \tag{28}$$

provided $h_2 > 0$ and $h_4 > 0$.

Type–2: If $h_1 = h_3 = 0$ and $h_0 = \frac{h_2^2}{4h_4}$, then Eq. (25) has the dark soliton solution:

$$R(\tau) = \pm\sqrt{-\frac{h_2}{2h_4}}\tanh\left(\sqrt{-\frac{h_2}{2}}\,\tau\right), \tag{29}$$

and the singular soliton solution:

$$R(\tau) = \pm\sqrt{-\frac{h_2}{2h_4}}\coth\left(\sqrt{-\frac{h_2}{2}}\,\tau\right), \tag{30}$$

provided $h_2 < 0$ and $h_4 > 0$.

Type–3: If $h_1 = h_3 = 0$, $h_0 = -\frac{m^2(1-m^2)h_2^2}{(2m^2-1)^2 h_4}$ and $0 < m < 1$, then Eq. (25) has the following Jacobi elliptic solution (JES):

$$R(\tau) = \pm\sqrt{-\frac{m^2 h_2}{(2m^2-1)h_4}}\,\mathrm{cn}\left(\sqrt{\frac{h_2}{2m^2-1}}\,\tau\right), \tag{31}$$

provided $(2m^2-1)h_2>0$ and $h_4<0$.

Type–4: If $h_1=h_3=0$, $h_0=\dfrac{(1-m^2)h_2^2}{(2-m^2)^2 h_4}$ and $0<m<1$, then Eq. (25) has the following JES:

$$R(\tau)=\pm\sqrt{-\frac{h_2}{(2-m^2)h_4}}\,\mathrm{dn}\left(\sqrt{\frac{h_2}{2-m^2}}\tau\right), \tag{32}$$

provided $h_2>0$ and $h_4<0$.

Type–5: If $h_1=h_3=0$, $h_0=\dfrac{m^2h_2^2}{(m^2+1)^2 h_4}$ and $0<m<1$, then Eq. (25) has the following JES:

$$R(\tau)=\pm\sqrt{-\frac{m^2h_2}{(1+m^2)h_4}}\,\mathrm{sn}\left(\sqrt{-\frac{h_2}{1+m^2}}\tau\right), \tag{33}$$

provided $h_2<0$ and $h_4>0$.

Type–6: If $h_0=h_1=0$, then Eq. (25) has the combo-bright-dark soliton solution:

$$R(\tau)=\frac{h_2\,\mathrm{sech}^2\left(\frac{1}{2}\sqrt{h_2}\tau\right)}{2\sqrt{h_2h_4}\tanh\left(\frac{1}{2}\sqrt{h_2}\tau\right)-h_3}, \tag{34}$$

and the combo-singular soliton solution:

$$R(\tau)=\frac{h_2\,\mathrm{csch}^2\left(\frac{1}{2}\sqrt{h_2}\tau\right)}{2\sqrt{h_2h_4}\coth\left(\frac{1}{2}\sqrt{h_2}\tau\right)+h_3}, \tag{35}$$

provided $h_2>0$ and $h_4>0$.

Type–7: If $h_0=h_1=0$ and $h_3=2\sqrt{h_2h_4}$, then Eq. (25) has the dark soliton solution:

$$R(\tau)=-\frac{1}{2}\sqrt{\frac{h_2}{h_4}}\left[1+\tanh\left(\frac{1}{2}\sqrt{h_2}\tau\right)\right], \tag{36}$$

and the singular soliton solution:

$$R(\tau)=-\frac{1}{2}\sqrt{\frac{h_2}{h_4}}\left[1+\coth\left(\frac{1}{2}\sqrt{h_2}\tau\right)\right], \tag{37}$$

provided $h_2>0$ and $h_4>0$.

4.1. Soliton solutions

In order to achieve this objective, we apply the balancing method (26) to balance $Z^5Z^{(6)}$ and Z^{12} in Eq. (23). By doing so, we determine that $N=1$, and the formal solution of Eq. (23) is provided as follows:

$$Z(\tau)=\Omega_0+\Omega_1R(\tau). \tag{38}$$

In this case, Ω_0 and Ω_1 represent constants, where Ω_1 is a non-zero value.

Set–1: In order to derive the solution, we can insert Eqs. (38) and (25) into Eq. (23) and make the assumption that $h_0=h_1=h_3=0$. By solving the resulting set of algebraic equations using Maple, we obtain the following outcome:

$$\Omega_0 = 0, \Omega_1 = \left(-\frac{135135}{64}\frac{h_4^3}{\Theta_5}\right)^{\frac{1}{6}},\ h_2 = -\frac{4}{179}\Theta_4, \tag{39}$$

and

$$\Theta_1 = \frac{53361}{5735339}\Theta_4^3,\ \Theta_2 = \frac{7459}{32041}\Theta_4^2,\ \Theta_3 = \frac{18}{895}\Delta_1\Theta_4,\ \Delta_0 = -\frac{3}{5}\Delta_1, \tag{40}$$

provided $h_4\Theta_5 < 0$. By substituting Eq. (39) into Eq. (38), along with Eqs. (27) and (28), we can obtain the bright soliton solution for Eqs. (1) and (2) as:

$$q(x,t) = \left[\left(-\frac{135135}{5735339}\frac{\Theta_4^3}{\Theta_5}\right)^{\frac{1}{6}}\operatorname{sech}\left(2\sqrt{-\frac{1}{179}\Theta_4}\,(x-vt)\right)\right]^{\frac{3}{2}} e^{i(-\kappa x+\omega t+\theta_0)}, \tag{41}$$

$$r(x,t) = A\left[\left(-\frac{135135}{5735339}\frac{\Theta_4^3}{\Theta_5}\right)^{\frac{1}{6}}\operatorname{sech}\left(2\sqrt{-\frac{1}{179}\Theta_4}\,(x-vt)\right)\right]^{\frac{3}{2}} e^{i(-\kappa x+\omega t+\theta_0)}, \tag{42}$$

provided $\Theta_4 < 0$ and $\Theta_5 > 0$, and the singular soliton solution as:

$$q(x,t) = \left[\left(\frac{135135}{5735339}\frac{\Theta_4^3}{\Theta_5}\right)^{\frac{1}{6}}\operatorname{csch}\left(2\sqrt{-\frac{1}{179}\Theta_4}\,(x-vt)\right)\right]^{\frac{3}{2}} e^{i(-\kappa x+\omega t+\theta_0)}, \tag{43}$$

$$r(x,t) = A\left[\left(\frac{135135}{5735339}\frac{\Theta_4^3}{\Theta_5}\right)^{\frac{1}{6}}\operatorname{csch}\left(2\sqrt{-\frac{1}{179}\Theta_4}\,(x-vt)\right)\right]^{\frac{3}{2}} e^{i(-\kappa x+\omega t+\theta_0)}, \tag{44}$$

provided $\Theta_4 < 0$ and $\Theta_5 < 0$. The conditions specified in Eq. (40) ensure that the solutions (41)-(44) satisfy the given Eqs. (1) and (2).

Set–2: In order to derive the solution, we can insert Eqs. (38) and (25) into Eq. (23) and make the assumption that $h_1 = h_3 = 0$ and $h_0 = \frac{h_2^2}{4h_4}$. By solving the resulting set of algebraic equations using Maple, we obtain the following outcome:

$$\Omega_0 = -\frac{\Delta_1}{10080}\left(-\frac{5005}{\Theta_5}\right)^{\frac{2}{3}},\ \Omega_1 = \left(-\frac{135135}{64}\frac{h_4^3}{\Theta_5}\right)^{\frac{1}{6}},\ h_2 = \frac{143}{1088640}\frac{\Delta_1^2}{\Theta_5}, \tag{45}$$

and

$$\begin{gathered}\Theta_1 = \frac{2924207}{936190547066880}\frac{\Delta_1^6}{\Theta_5^3},\ \Theta_2 = \frac{38669059}{4740548198400}\frac{\Delta_1^4}{\Theta_5^2},\\ \Theta_3 = \frac{4147}{19595520}\frac{\Delta_1^3}{\Theta_5},\ \Theta_3 = \frac{11869}{2177280}\frac{\Delta_1^3}{\Theta_5},\ \Delta_0 = -\frac{4}{3}\Delta_1,\end{gathered} \tag{46}$$

provided $h_4 > 0$ and $\Theta_5 < 0$. By substituting Eq. (45) into Eq. (38), along with Eqs. (29) and (30), we can obtain the dark soliton solution for Eq. (1) and (2) as:

$$q(x,t)=\left\{\frac{\Delta_1}{10080}\left(-\frac{5005}{\Theta_5}\right)^{\frac{2}{3}}\left[\tanh\left(\frac{\Delta_1}{15120}\sqrt{-\frac{15015}{\Theta_5}}(x-vt)\right)-1\right]\right\}^{\frac{3}{2}} \times e^{i(-\kappa x+\omega t+\theta_0)}, \tag{47}$$

$$r(x,t)=A\left\{\frac{\Delta_1}{10080}\left(-\frac{5005}{\Theta_5}\right)^{\frac{2}{3}}\left[\tanh\left(\frac{\Delta_1}{15120}\sqrt{-\frac{15015}{\Theta_5}}(x-vt)\right)-1\right]\right\}^{\frac{3}{2}} \times e^{i(-\kappa x+\omega t+\theta_0)}, \tag{48}$$

and the singular soliton solution as:

$$q(x,t)=\left\{\frac{\Delta_1}{10080}\left(-\frac{5005}{\Theta_5}\right)^{\frac{2}{3}}\left[\coth\left(\frac{\Delta_1}{15120}\sqrt{-\frac{15015}{\Theta_5}}(x-vt)\right)-1\right]\right\}^{\frac{3}{2}} \times e^{i(-\kappa x+\omega t+\theta_0)}, \tag{49}$$

$$r(x,t)=A\left\{\frac{\Delta_1}{10080}\left(-\frac{5005}{\Theta_5}\right)^{\frac{2}{3}}\left[\coth\left(\frac{\Delta_1}{15120}\sqrt{-\frac{15015}{\Theta_5}}(x-vt)\right)-1\right]\right\}^{\frac{3}{2}} \times e^{i(-\kappa x+\omega t+\theta_0)}, \tag{50}$$

provided $\Theta_5<0$. The conditions specified in Eq. (46) ensure that the solutions (47)-(50) satisfy the given Eqs. (1) and (2).

Set–3: In order to derive the solution, we can insert Eq. (38) and (25) into Eq. (23) and make the assumption that $h_0=h_1=0$. By solving the resulting set of algebraic equations using Maple, we obtain the following outcome:

$$\Omega_0=0,\ \Omega_1=\left(-\frac{135135}{64}\frac{h_4^3}{\Theta_5}\right)^{\frac{1}{6}},\ h_2=-\frac{4}{83}\Theta_5,\ h_3=h_4, \tag{51}$$

and

$$\begin{aligned}
\Delta_0&=\frac{192\Theta_4+4980h_4}{11869}\sqrt{-\frac{15015\Theta_5}{h_4}},\\
\Delta_1&=-\frac{320\Theta_4+4648h_4}{11869}\sqrt{-\frac{15015\Theta_5}{h_4}},\\
\Theta_1&=\frac{11025}{571787}\Theta_4^3,\\
\Theta_2&=\frac{1891}{6889}\Theta_4^2,\\
\Theta_3&=-\frac{4980\Theta_4h_4+1152\Theta_4^2-55112h_4^2}{985127}\sqrt{-\frac{15015\Theta_5}{h_4}},
\end{aligned} \tag{52}$$

provided $h_4\Theta_5<0$. By substituting Eq. (51) into Eq. (38), along with Eqs. (34) and (35), we can obtain the combo-bright-dark soliton solution for Eqs. (1) and (2) as:

$$q(x,t)=\left\{\left(-\frac{5005h_4^3}{\Theta_5}\right)^{\frac{1}{6}}\frac{2\sqrt{3}\Theta_4\mathrm{sech}^2\left(\frac{1}{83}\sqrt{-83\Theta_4}(x-vt)\right)}{4\sqrt{-83\Theta_4h_4}\tanh\left(\frac{1}{83}\sqrt{-83\Theta_4}(x-vt)\right)+83h_4}\right\}^{\frac{3}{2}} \times e^{i(-\kappa x+\omega t+\theta_0)}, \tag{53}$$

$$r(x,t)=A\left\{\left(-\frac{5005h_4^3}{\Theta_5}\right)^{\frac{1}{6}}\frac{2\sqrt{3}\Theta_4\mathrm{sech}^2\left(\frac{1}{83}\sqrt{-83\Theta_4}(x-vt)\right)}{4\sqrt{-83\Theta_4h_4}\tanh\left(\frac{1}{83}\sqrt{-83\Theta_4}(x-vt)\right)+83h_4}\right\}^{\frac{3}{2}} \times e^{i(-\kappa x+\omega t+\theta_0)}, \tag{54}$$

and the combo-singular soliton solution as:

$$q(x,t)=\left\{\left(-\frac{5005h_4^3}{\Theta_5}\right)^{\frac{1}{6}}\frac{2\sqrt{3}\Theta_4\mathrm{csch}^2\left(\frac{1}{83}\sqrt{-83\Theta_4}(x-vt)\right)}{4\sqrt{-83\Theta_4h_4}\coth\left(\frac{1}{83}\sqrt{-83\Theta_4}(x-vt)\right)-83h_4}\right\}^{\frac{3}{2}} e^{i(-\kappa x+\omega t+\theta_0)}, \tag{55}$$

$$r(x,t)=A\left\{\left(-\frac{5005h_4^3}{\Theta_5}\right)^{\frac{1}{6}}\frac{2\sqrt{3}\Theta_4\mathrm{csch}^2\left(\frac{1}{83}\sqrt{-83\Theta_4}(x-vt)\right)}{4\sqrt{-83\Theta_4h_4}\coth\left(\frac{1}{83}\sqrt{-83\Theta_4}(x-vt)\right)-83h_4}\right\}^{\frac{3}{2}} \times e^{i(-\kappa x+\omega t+\theta_0)}, \tag{56}$$

provided $\Theta_4<0$ and $h_4\Theta_5<0$. The conditions specified in Eq. (52) ensure that the solutions (53)-(56) satisfy the given Eqs. (1) and (2).

Set–4: In order to derive the solution, we can insert Eqs. (38) and (25) into Eq. (23) and make the assumption that $h_0=h_1=0$ and $h_3=2\sqrt{h_2h_4}$. By solving the resulting set of algebraic equations using Maple, we obtain the following outcome:

$$\Omega_0=0,\ \Omega_1=\left(-\frac{135135}{64}\frac{h_4^3}{\Theta_5}\right)^{\frac{1}{6}},\ h_2=\frac{4}{83}\Theta_4, \tag{57}$$

and

$$\Delta_0=-\frac{192\sqrt{12462450\Theta_5\Theta_4}}{11869},\ \Delta_1=-\frac{144\sqrt{12462450\Theta_5\Theta_4}}{11869},\ \Theta_1=\frac{11025}{571787}\Theta_4^3, \\ \Theta_2=\frac{1891}{6889}\Theta_4^2,\ \Theta_3=\frac{464\,\Theta_4\sqrt{12462450\Theta_5\Theta_4}}{985127}, \tag{58}$$

provided $h_4\Theta_5<0$ and $\Theta_5\Theta_4>0$. By substituting Eq. (57) into Eq. (38), along with Eqs. (36) and (37), we can obtain the dark soliton solution for Eqs. (1) and (2) as:

$$q(x,t)=\left\{\frac{1}{2}\sqrt{-\frac{3}{83}\Theta_4}\left(-\frac{5005}{\Theta_5}\right)^{\frac{1}{6}}\left[\tanh\left(\sqrt{-\frac{1}{83}\Theta_4}\,(x-vt)\right)-1\right]\right\}^{\frac{3}{2}} \times e^{i(-\kappa x+\omega t+\theta_0)}, \tag{59}$$

$$r(x,t)=A\left\{\frac{1}{2}\sqrt{-\frac{3}{83}\Theta_4}\left(-\frac{5005}{\Theta_5}\right)^{\frac{1}{6}}\left[\tanh\left(\sqrt{-\frac{1}{83}\Theta_4}\,(x-vt)\right)-1\right]\right\}^{\frac{3}{2}} \times e^{i(-\kappa x+\omega t+\theta_0)}, \tag{60}$$

and the singular soliton solution as:

$$q(x,t)=\left\{\frac{1}{2}\sqrt{-\frac{3}{83}\Theta_4}\left(-\frac{5005}{\Theta_5}\right)^{\frac{1}{6}}\left[\coth\left(\sqrt{-\frac{1}{83}\Theta_4}\,(x-vt)\right)-1\right]\right\}^{\frac{3}{2}} \times e^{i(-\kappa x+\omega t+\theta_0)}, \tag{61}$$

$$r(x,t)=A\left\{\frac{1}{2}\sqrt{-\frac{3}{83}\Theta_4}\left(-\frac{5005}{\Theta_5}\right)^{\frac{1}{6}}\left[\coth\left(\sqrt{-\frac{1}{83}\Theta_4}\,(x-vt)\right)-1\right]\right\}^{\frac{3}{2}} \times e^{i(-\kappa x+\omega t+\theta_0)}, \tag{62}$$

provided $\Theta_5<0$ and $\Theta_4<0$. The conditions specified in (58) ensure that the solutions (59)-(62) satisfy the given equations (1) and (2).

Remark: If we insert Eqs. (38) and (25) into Eq. (23) and use the assumptions on $h_j,(j=0,1,3)$ of Jacobi elliptic solutions (31)-(33) and solving the algebraic equations using the Maple, we get $m=1$, which gives the hyperbolic solutions considered before. Therefore, the original equations do have not Jacobi elliptic solutions..

5. Conclusions

The current paper retrieved highly dispersive gap optical solitons that emerged from Bragg gratings. The SPM structure of the fibers is in parabolic–nonlocal nonlinear form. The extended auxiliary equation integration algorithm has made this retrieval possible. The results are thus indeed promising to look further ahead with this work. One immediate thought would be to search for the conserved quantities with this model. Subsequently, the quasi-monochromatic solitons from the Bragg gratings structure can be recovered. The Laplace–Adomian decomposition scheme would lead to the numerical simulation of the bright and dark solitons, which would render a visual perspective of such solitons. The results would subsequently be disseminated elsewhere.

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Анотація. *У цій статті досліджуються щілинні солітони з параболічно-нелокальною формою самофазової модуляції. Солітонні розв'язки для цієї моделі виявляються за допомогою успішного застосування розширеного підходу з допоміжним рівнянням. Обмеження параметрів забезпечують існування таких щілинних солітонів.*

Ключові слова*: солітони, ґратки, розширене допоміжне рівняння, обмеження параметрів*