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Optical Solitons for the Concatenation Model with Differential Group Delay Having Multiplicate White Noise

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Abstract. This work recovers optical soliton solutions to the concatenation model with differential group delay in the presence of white noise along both the components. Two integration algorithms have successfully recovered a full spectrum of solitons solutions to the model. It has been proved that the effect of white noise is visible only in the phase portion of the solitons along both components.

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1. Introduction

The concept of the concatenation model was first conceived during 2014 [1, 2]. This model is the conjunction of three well–known equations that are commonly visible in Nonlinear Optics. They are the nonlinear Schrödinger's equation, Lakshmanan–Porsezian–Daniel equation, and the Sasa–Satsuma equation. Subsequently, this concatenation model gained a lot of momentum during the past couple of years. A wide variety of features from this model have been addressed. They are the Painleve analysis, retrieval of soliton solutions and the conservation laws, addressing the model in magneto–optic waveguides and numerical study of the model by the Laplace–Adomian decomposition and recovering the quiescent optical solitons for nonlinear chromatic dispersion (CD) by Lie symmetry and other mathematical algorithms. This model was also considered with spatio–temporal dispersion in addition to the CD that was suggested to be used to control the Internet bottleneck effect. Later, the model was studied with differential group delay where the method of undetermined coefficients recovered the soliton solutions, and the complexiton solutions were also recovered for birefringent fibers [3–10].

The current paper revisits this concatenation model in birefringent fibers and is considered with the effect of white noise included. Two integration algorithms are applied to integrate the model. These are the enhanced Kudryashov and the new projective Riccati equations methods. These two schemes collectively recover the soliton solutions to the model. A full spectrum of optical solitons is thus recovered. In the soliton solutions, it has been observed that the effect of white noise appears only in the phase of the solitons along both components of birefringent fibers. This is consistent with the observation for the scalar version of the model with the white noise effect. The model is first introduced in the following section. Thereafter, the derivation of the soliton solutions to the model is carried out using the two integration algorithms mentioned. The details are exhibited in the rest of the paper.

2. Governing model

When we examine the concatenation model within birefringent fibers, taking into consideration the inclusion of spatio-temporal dispersion (STD) and the effects of multiplicative white noise in the Itô sense, the expression is as follows:

$$\begin{aligned} & iq_t + a_1 q_{xx} + b_1 q_{xt} + \left(c_1|q|^2 + d_1|r|^2\right)q \\ & + c_{11}\Big[\sigma_{11}q_{xxxx} + \left\{\alpha_1(q_x)^2 + \beta_1(r_x)^2\right\}q^* + \left(\gamma_1|q_x|^2 + \lambda_1|r_x|^2\right)q + \left(\delta_1|q|^2 + \zeta_1|r|^2\right)q_{xx} \\ & + (\mu_1 q^2 + \rho_1 r^2)q_{xx}^* + \left(f_1|q|^4 + g_1|q|^2|r|^2 + h_1|r|^4\right)q\Big] \\ & + ic_{21}\Big[\sigma_{71}q_{xxx} + \left(\eta_1|q|^2 + \theta_1|r|^2\right)q_x + (\varepsilon_1 q^2 + \tau_1 r^2)q_x^*\Big] + \sigma(q - ib_1 q_x)W_t(t) = 0, \end{aligned} \tag{1}$$

and

$$\begin{aligned} & ir_t + a_2 r_{xx} + b_2 r_{xt} + \left(c_2|r|^2 + d_2|q|^2\right)r \\ & + c_{12}\Big[\sigma_{12}r_{xxxx} + \left\{\alpha_2(r_x)^2 + \beta_2(q_x)^2\right\}r^* + \left(\gamma_2|r_x|^2 + \lambda_2|q_x|^2\right)r + \left(\delta_2|r|^2 + \zeta_2|q|^2\right)r_{xx} \\ & + (\mu_2 r^2 + \rho_2 q^2)r_{xx}^* + \left(f_2|r|^4 + g_2|r|^2|q|^2 + h_2|q|^4\right)r\Big] \\ & + ic_{22}\Big[\sigma_{72}r_{xxx} + \left(\eta_2|r|^2 + \theta_2|q|^2\right)r_x + (\varepsilon_2 r^2 + \tau_2 q^2)r_x^*\Big] + \sigma(r - ib_2 r_x)W_t(t) = 0. \end{aligned} \tag{2}$$

Eq's. (1) and (2) are the two components of the concatenation model for birefringent fibers, as studied earlier [7]. Additionally, b_1 and b_2 come from STD, while the standard Wiener process is represented by $W(t)$ with σ serving as the noise strength coefficient and $W_t(t)$ representing the white noise. Also, $q = q(x,t)$ and $r = r(x,t)$ represent complex-valued wave functions. The independent variables x and t represent spatial and temporal variables, respectively. Lastly, q^* and r^* denote the complex conjugate of q and r, respectively. The additional effects are included along the two components here are the effects of white noise and spatio-temporal dispersion, whose coefficients are σ and b_j ($j = 1,2$), respectively. In order to address the model, we start by:

$$q(x,t)=P_1(\xi)e^{i\phi(x,t)}, \tag{3}$$

$$r(x,t)=P_2(\xi)e^{i\phi(x,t)}. \tag{4}$$

Here we can describe the wave variable ξ as being

$$\xi=k(x-vt), \tag{5}$$

where k is wave width. Representing the amplitude components of the soliton solutions, $P_l(\xi)$, $l=1,2$, and describing the soliton speed as v, we also provide a definition for the phase component $\phi(x,t)$:

$$\phi(x,t)=-\kappa x+\omega t+\sigma W(t)-\sigma^2 t+\theta_0. \tag{6}$$

Within this context, κ takes on the role of denoting the wavenumber, ω is the frequency, σ signifies the noise coefficient, and θ_0 is indicative of the phase constant. Through the substitution of Eqs. (3) and (4) into equations (1) and (2) and the subsequent decomposition into their real and imaginary parts, we attain the following expressions

$$\begin{aligned}
&k^2\left(a_1-b_1v+3\kappa\left(\alpha_{71}c_{21}-2c_{11}\kappa\sigma_{11}\right)\right)P_1''+\begin{pmatrix}-a_1\kappa^2+b_1\kappa\left(\omega-\sigma^2\right)\\ -\alpha_{71}c_{21}\kappa^3+c_{11}\kappa^4\sigma_{11}+\sigma^2-\omega\end{pmatrix}P_1\\
&+\left(d_1-\kappa\left(c_{11}\kappa\left(\beta_1+\zeta_1-\lambda_1+\rho_1\right)+c_{21}\left(\tau_1-\theta_1\right)\right)\right)P_2^2P_1+c_{11}f_1P_1^5+c_{11}g_1P_2^2P_1^3+c_{11}h_1P_2^4P_1\\
&+c_{11}k^4\sigma_{11}P_1^{(4)}+c_{11}k^2\left(\alpha_1+\gamma_1\right)P_1\left(P_1'\right)^2+c_{11}k^2\left(\beta_1+\lambda_1\right)P_1\left(P_2'\right)^2+c_{11}k^2\left(\delta_1+\mu_1\right)P^2P_1''\\
&+c_{11}k^2\left(\zeta_1+\rho_1\right)P_2^2P_1''+\left(c_1-\kappa\left(c_{11}\kappa\left(\alpha_1-\gamma_1+\delta_1+\mu_1\right)+c_{21}\left(\varepsilon_1-\eta_1\right)\right)\right)P_1^3=0,
\end{aligned} \tag{7}$$

$$\begin{aligned}
&k^2\left(a_2-b_2v+3\kappa\left(\alpha_{72}c_{22}-2c_{12}\kappa\sigma_{12}\right)\right)P_2''+\begin{pmatrix}-a_2\kappa^2+b_2\kappa\left(\omega-\sigma^2\right)-\alpha_{72}c_{22}\kappa^3\\ +c_{12}\kappa^4\sigma_{12}+\sigma^2-\omega\end{pmatrix}P_2\\
&+\left(d_2-\kappa\left(c_{12}\kappa\left(\beta_2+\zeta_2-\lambda_2+\rho_2\right)+c_{22}\left(\tau_2-\theta_2\right)\right)\right)P_1^2P_2+c_{12}f_2P_2^5\\
&+c_{12}g_2P_1^2P_2^3+c_{12}h_2P_1{}^4P_2\\
&+c_{12}k^4\sigma_{12}P_2^{(4)}+c_{12}k^2\left(\alpha_2+\gamma_2\right)P_2\left(P_2'\right)^2\\
&+c_{12}k^2\left(\beta_2+\lambda_2\right)P_2\left(P_1'\right)^2+c_{12}k^2\left(\delta_2+\mu_2\right)P_2^2P_2''+c_{12}k^2\left(\zeta_2+\rho_2\right)P_1{}^2P_2''\\
&+\left(c_2-\kappa\left(c_{12}\kappa\left(\alpha_2-\gamma_2+\delta_2+\mu_2\right)+c_{22}\left(\varepsilon_2-\eta_2\right)\right)\right)P_2{}^3=0,
\end{aligned} \tag{8}$$

and the imaginary parts give

$$\begin{aligned}
&k\left(-2a_1\kappa+b_1\left(-\sigma^2+\kappa v+\omega\right)-3\alpha_{71}c_{21}\kappa^2+4c_{11}\kappa^3\sigma_{11}-v\right)P_1'\\
&+k^3\left(\alpha_{71}c_{21}-4c_{11}\kappa\sigma_{11}\right)P_1^{(3)}\\
&+k\left(c_{21}\left(\varepsilon_1+\eta_1\right)-2c_{11}\kappa\left(\alpha_1+\delta_1-\mu_1\right)\right)P_1{}^2P_1'-2\beta_1c_{11}\kappa kP_1P_2P_2'\\
&+kP_2{}^2P_1'\left(2c_{11}\kappa\left(\rho_1-\zeta_1\right)+c_{21}\left(\theta_1+\tau_1\right)\right)=0,
\end{aligned} \tag{9}$$

$$\begin{aligned}
&k\left(-2a_2\kappa+b_2\left(-\sigma^2+\kappa v+\omega\right)-3\alpha_{72}c_{22}\kappa^2+4c_{12}\kappa^3\sigma_{12}-v\right)P_2'\\
&+k^3\left(\alpha_{72}c_{22}-4c_{12}\kappa\sigma_{12}\right)P_2^{(3)}\\
&+k\left(c_{22}\left(\varepsilon_2+\eta_2\right)-2c_{12}\kappa\left(\alpha_2+\delta_2-\mu_2\right)\right)P_2{}^2P_2'-2\beta_2c_{12}\kappa kP_1P_2P_1'\\
&+k\left(2c_{12}\kappa\left(\rho_2-\zeta_2\right)+c_{22}\left(\theta_2+\tau_2\right)\right)P_1{}^2P_2'=0.
\end{aligned} \tag{10}$$

Using the balancing principle leads to $P_2=\varpi\, P_1$, then Eqs. (7) and (8) become

$$k^2\left(a_1-b_1v+3\kappa\left(\alpha_{71}c_{21}-2c_{11}\kappa\sigma_{11}\right)\right)P_1''+\begin{pmatrix}-a_1\kappa^2+b_1\kappa\left(\omega-\sigma^2\right)\\-\alpha_{71}c_{21}\kappa^3+c_{11}\kappa^4\sigma_{11}+\sigma^2-\omega\end{pmatrix}P_1$$
$$+\begin{pmatrix}-\kappa\left(c_{11}\kappa\left(\alpha_1-\gamma_1+\delta_1+\mu_1\right)+c_{21}\left(\varepsilon_1-\eta_1\right)\right)\\+\varpi^2\left(d_1-\kappa\left(c_{11}\kappa\left(\beta_1+\zeta_1-\lambda_1+\rho_1\right)+c_{21}\left(\tau_1-\theta_1\right)\right)\right)+c_1\end{pmatrix}P_1^3 \quad (11)$$
$$+c_{11}P_1^5\left(f_1+g_1v^2+h_1\varpi^4\right)+c_{11}k^4\sigma_{11}P_1^{(4)}+c_{11}k^2P_1\left(P_1'\right)^2\left(\alpha_1+\varpi^2\left(\beta_1+\lambda_1\right)+\gamma_1\right)$$
$$+c_{11}k^2\left(\delta_1+\varpi^2\left(\zeta_1+\rho_1\right)+\mu_1\right)P_1^2P_1''=0,$$

$$k^2\varpi\left(a_2-b_2v+3\kappa\left(\alpha_{72}c_{22}-2c_{12}\kappa\sigma_{12}\right)\right)P_1''+\varpi\begin{pmatrix}-a_2\kappa^2+b_2\kappa\left(\omega-\sigma^2\right)\\-\alpha_{72}c_{22}\kappa^3+c_{12}\kappa^4\sigma_{12}+\sigma^2-\omega\end{pmatrix}P_1$$
$$+\varpi P_1^3\begin{pmatrix}\varpi^2\left(c_2-\kappa\left(c_{12}\kappa\left(\alpha_2-\gamma_2+\delta_2+\mu_2\right)+c_{22}\left(\varepsilon_2-\eta_2\right)\right)\right)\\-\kappa\left(c_{12}\kappa\left(\beta_2+\zeta_2-\lambda_2+\rho_2\right)+c_{22}\left(\tau_2-\theta_2\right)\right)+d_2\end{pmatrix} \quad (12)$$
$$+c_{12}\varpi P_1^5\left(f_2v^4+g_2\varpi^2+h_2\right)+c_{12}k^4\sigma_{12}\varpi P_1^{(4)}+c_{12}k^2vP_1\left(P_1'\right)^2\left(\varpi\left(\alpha_2+\gamma_2\right)+\beta_2+\lambda_2\right)$$
$$+c_{12}k^2\varpi\left(\varpi^2\left(\delta_2+\mu_2\right)+\zeta_2+\rho_2\right)P_1^2P_1''=0,$$

and Eqs. (9) and (10) become

$$kP_1'\left(-2a_1\kappa+b_1\left(-\sigma^2+\kappa v+\omega\right)-3\alpha_{71}c_{21}\kappa^2+4c_{11}\kappa^3\sigma_{11}-v\right)$$
$$+k^3P_1^{(3)}\left(\alpha_{71}c_{21}-4c_{11}\kappa\sigma_{11}\right) \quad (13)$$
$$+k\left(c_{21}\left(\varepsilon_1+\eta_1+\varpi^2\left(\theta_1+\tau_1\right)\right)-2c_{11}\kappa\left(\alpha_1+\beta_1\varpi^2+\delta_1+\zeta_1\varpi^2-\mu_1-\rho_1\varpi^2\right)\right)P_1^2P_1'=0,$$

$$k\varpi P_1'\left(-2a_2\kappa+b_2\left(-\sigma^2+\kappa v+\omega\right)-3\alpha_{72}c_{22}\kappa^2+4c_{12}\kappa^3\sigma_{12}-v\right)$$
$$+k^3\varpi P_1^{(3)}\left(\alpha_{72}c_{22}-4c_{12}\kappa\sigma_{12}\right) \quad (14)$$
$$+k\varpi\left(c_{22}\left(\varepsilon_2\varpi^2+\eta_2\varpi^2+\theta_2+\tau_2\right)-2c_{12}\kappa\left(\alpha_2\varpi^2+\beta_2+\delta_2\varpi^2+\zeta_2-\mu_2\varpi^2-\rho_2\right)\right)P_1^2P_1'=0.$$

By comparing Eqs. (11) with (12) and reducing them to a single equation, we are able to derive the following parametric restrictions:

$$\varpi\begin{pmatrix}\varpi^2\left(c_2-\kappa\left(c_{12}\kappa\left(\alpha_2-\gamma_2+\delta_2+\mu_2\right)+c_{22}\left(\varepsilon_2-\eta_2\right)\right)\right)\\-\kappa\left(c_{12}\kappa\left(\beta_2+\zeta_2-\lambda_2+\rho_2\right)+c_{22}\left(\tau_2-\theta_2\right)\right)+d_2\end{pmatrix}=$$
$$\varpi^2\left(d_1-\kappa\left(c_{11}\kappa\left(\beta_1+\zeta_1-\lambda_1+\rho_1\right)+c_{21}\left(\tau_1-\theta_1\right)\right)\right) \quad (15)$$
$$-\kappa\left(c_{11}\kappa\left(\alpha_1-\gamma_1+\delta_1+\mu_1\right)+c_{21}\left(\varepsilon_1-\eta_1\right)\right)+c_1,$$

$$-a_1\kappa^2+b_1\kappa\left(\omega-\sigma^2\right)-\alpha_{71}c_{21}\kappa^3+c_{11}\kappa^4\sigma_{11}+\sigma^2-\omega$$
$$=\varpi\left(-a_2\kappa^2+b_2\kappa\left(\omega-\sigma^2\right)-\alpha_{72}c_{22}\kappa^3+c_{12}\kappa^4\sigma_{12}+\sigma^2-\omega\right), \quad (16)$$

$$c_{11}\left(f_1+g_1\varpi^2+h_1\varpi^4\right)=c_{12}\varpi\left(f_2\varpi^4+g_2\varpi^2+h_2\right), \quad (17)$$

$$c_{11}\left(\alpha_1+\varpi^2\left(\beta_1+\lambda_1\right)+\gamma_1\right)=c_{12}\varpi\left(\varpi^2\left(\alpha_2+\gamma_2\right)+\beta_2+\lambda_2\right), \quad (18)$$

$$a_1-b_1v+3\kappa\left(\alpha_{71}c_{21}-2c_{11}\kappa\sigma_{11}\right)$$
$$=\varpi\left(a_2-b_2v+3\kappa\left(\alpha_{72}c_{22}-2c_{12}\kappa\sigma_{12}\right)\right), \quad (19)$$

$$c_{11}\left(\delta_1+\varpi^2\left(\zeta_1+\rho_1\right)+\mu_1\right)$$
$$=c_{12}\left(\varpi^2\left(\delta_2+\mu_2\right)+\zeta_2+\rho_2\right), \quad (20)$$

$$c_{11}\sigma_{11}=c_{12}\sigma_{12}\varpi. \quad (21)$$

In the imaginary parts of Eqs. (13) and (14), we derive the speeds of the two components as follows

$$v=\frac{2\kappa a_l+\sigma^2 b_l-\omega b_l+3\kappa^2 c_{2l}\alpha_{7l}-4\kappa^3 c_{1l}\sigma_{1l}}{b_l\kappa-1},\tag{22}$$

where l=1,2 and the wavenumber is as follows

$$\kappa=\frac{c_{2l}\alpha_{7l}}{4c_{1l}\sigma_{1l}}.\tag{23}$$

Respectively, the third terms in Eqs. (13) and (14) give the constraints

$$\sigma_{11}=\frac{\alpha_{71}\left(\alpha_1+\beta_1\varpi^2+\delta_1+\zeta_1\varpi^2-\mu_1-\rho_1\varpi^2\right)}{2\left(\varepsilon_1+\eta_1+\varpi^2\left(\theta_1+\tau_1\right)\right)},\tag{24}$$

$$\sigma_{12}=\frac{\alpha_{72}\left(\alpha_2\varpi^2+\beta_2+\delta_2\varpi^2+\zeta_2-\mu_2\varpi^2-\rho_2\right)}{2\left(\varepsilon_2\varpi^2+\eta_2\varpi^2+\theta_2+\tau_2\right)}.\tag{25}$$

Eq. (11) can be set as

$$k^2P_1''''+w_6P_1^2P_1''+w_5P_1''+w_4P_1\left(P_1'\right)^2+w_3P_1^5+w_2P_1^3+w_1P_1=0,\tag{26}$$

with

$$\begin{cases} w_1=\dfrac{-a_1\kappa^2+b_1\kappa\left(\omega-\sigma^2\right)-\alpha_{71}c_{21}\kappa^3+c_{11}\kappa^4\sigma_{11}+\sigma^2-\omega}{c_{11}k^2\sigma_{11}},\\ w_2=\dfrac{-\kappa\begin{pmatrix}c_{11}\kappa\left(\alpha_1+\beta_1\varpi^2-\gamma_1+\delta_1+\zeta_1\varpi^2-\lambda_1\varpi^2+\mu_1+\rho_1\varpi^2\right)\\ +c_{21}\left(\varepsilon_1-\eta_1+\varpi^2\left(\tau_1-\theta_1\right)\right)\end{pmatrix}+c_1+d_1\varpi^2}{c_{11}k^2\sigma_{11}},\\ w_3=\dfrac{f_1+g_1\varpi^2+h_1\varpi^4}{k^2\sigma_{11}},\\ w_4=\dfrac{\alpha_1+\beta_1\varpi^2+\gamma_1+\lambda_1\varpi^2}{\sigma_{11}},\\ w_5=\dfrac{a_1-b_1v+3\kappa\left(\alpha_{71}c_{21}-2c_{11}\kappa\sigma_{11}\right)}{c_{11}\sigma_{11}},\\ w_6=\dfrac{\delta_1+\zeta_1\varpi^2+\mu_1+\rho_1\varpi^2}{\sigma_{11}},\end{cases}\tag{27}$$

where $\sigma_{11}\neq 0$.

3. Overview of the integration algorithms

A viable approach is to adopt a governing model

$$F\left(u,u_x,u_t,u_{xt},u_{xx},\ldots\right)=0,\tag{28}$$

where F is a polynomial in $u=u(x,t)$ and its partial derivatives. The wave profile is described by $u(x,t)$, with t and x used to specify the time and space variables, respectively. The use of the wave transformation

$$u(x,t)=U(\xi),\quad \xi=k(x-\upsilon t),\tag{29}$$

causes a reduction of Eq. (28) to

$$P\left(U,-k\upsilon U',kU',k^2U'',\ldots\right)=0,\tag{30}$$

where P is a polynomial in $U=U(\xi)$ and its derivatives (30).

3.1. The enhanced Kudryashov's scheme

This subsection presents a thorough overview of the basic procedures of the scheme.

Step–1: Herein, we detail the explicit solution for the reduced model Eq. (30):

$$U=\sigma_0+\sum_{i=1}^{N}\left\{\sigma_i R\ \xi)^i+\rho_i\left(\frac{R'(\xi)}{R(\xi)^i}\right)\right\}, \tag{31}$$

coupled with

$$R'(\xi)^2=R(\xi)^2(1-\chi R(\xi)^2). \tag{32}$$

The constants σ_0, χ, σ_i, and ρ_i (where $i=1,...,N$) will be provided, with N determined by the balancing procedure in Eq. (30).

Step–2: Soliton waves are described by Eq. (32):

$$R(\xi)=\frac{4c}{4c^2e^{\xi}+\chi e^{-\xi}}, \tag{33}$$

with c being a constant.

Step–3: Upon inserting Eq. (31) alongside Eq. (32) into Eq. (30), we can derive the requisite constants for the Eqs. (29) and (31). In order to incorporate the identified parametric restrictions, they can be substituted into Eq. (31) together with Eq. (33). Consequently, straddled solitons are obtained, which can be further classified as bright, dark, or singular solitons.

3.2. The new projective Riccati equation's approach

The following are the essential principles of the approach:

Step–1: Consider the formal solution of Eq. (30) to be

$$U(\xi)=a_0+\sum_{i=1}^{N}F^{i-1}(\xi)\left(a_iF(\xi)+b_iG(\xi)\right) \tag{34}$$

The ordinary differential equations for $F(\xi)$ and $G(\xi)$ are characterized by:

$$\begin{aligned}F'(\xi)&=-F(\xi)G(\xi),\\ G'(\xi)&=1-G^2(\xi)-\varepsilon F(\xi),\end{aligned} \tag{35}$$

with

$$G(\xi)^2=1-2\varepsilon F(\xi)+\zeta(\varepsilon)F(\xi)^2, \tag{36}$$

where $\zeta(\varepsilon)$ is the dependent variable with ε, which is given by the following cases 1-5. In Eq. (34), the incorporation of a non-zero constant ε and the consideration of a positive integer N result from the application of the balancing principle. The equation also consists of constants a_0, a_i and $b_i\ (i=0,1,...,N)$.

Step–2: Here is a list of the solutions of Eqs. (35):

Case-1: $\zeta(\varepsilon)=0$

$$F(\xi)=\frac{1}{2\varepsilon}\text{sech}^2\left[\frac{\xi}{2}\right],\ \text{ and }\ G(\xi)=\tanh\left[\frac{\xi}{2}\right], \tag{37}$$

or

$$F(\xi)=-\frac{1}{2\varepsilon}\text{csch}^2\left[\frac{\xi}{2}\right],\ \text{ and }\ G(\xi)=\coth\left[\frac{\xi}{2}\right]. \tag{38}$$

Case-2: $\zeta(\varepsilon)=\frac{24}{25}\varepsilon^2$

$$F(\xi)=\frac{1}{\varepsilon}\frac{5\,\mathrm{sech}[\xi]}{5\,\mathrm{sech}[\xi]\pm 1},\quad \text{and} \quad G(\xi)=\frac{\pm\tanh[\xi]}{5\,\mathrm{sech}[\xi]\pm 1}. \tag{39}$$

Case-3: $\zeta(\varepsilon)=\frac{5}{9}\varepsilon^2$

$$F(\xi)=\frac{1}{\varepsilon}\frac{3\,\mathrm{sech}[\xi]}{3\,\mathrm{sech}[\xi]\pm 2},\quad \text{and} \quad G(\xi)=\frac{2}{2\coth[\xi]\pm 3\,\mathrm{csch}[\xi]}. \tag{40}$$

Case-4: $\zeta(\varepsilon)=\varepsilon^2-1$

$$F(\xi)=\frac{4\,\mathrm{sech}[\xi]}{3\tanh[\xi]+4\varepsilon\,\mathrm{sech}[\xi]+5},\quad \text{and} \quad G(\xi)=\frac{5\tanh[\xi]+3}{3\tanh[\xi]+4\varepsilon\,\mathrm{sech}[\xi]+5}, \tag{41}$$

or

$$F(\xi)=\frac{\mathrm{sech}[\xi]}{\varepsilon\,\mathrm{sech}[\xi]+1},\quad \text{and} \quad G(\xi)=\frac{\tanh[\xi]}{\varepsilon\,\mathrm{sech}[\xi]+1}. \tag{42}$$

Case-5: $\zeta(\varepsilon)=\varepsilon^2+1$

$$F(\xi)=\frac{\mathrm{csch}[\xi]}{\varepsilon\,\mathrm{csch}[\xi]+1},\quad \text{and} \quad G(\xi)=\frac{\coth[\xi]}{\varepsilon\,\mathrm{csch}[\xi]+1}. \tag{43}$$

Step–3: Upon inserting Eqs. (34) alongside Eqs. (35) and (36) into Eq. (30), we can derive the requisite constants for the Eqs. (29) and (34). In order to incorporate the identified parametric restrictions, they can be substituted into Eq. (34) together with Eqs. (37)–(43). Consequently, bright, dark, or singular solitons are obtained.

4. Optical soliton solutions

4.1. The enhanced Kudryashov's technique

Balancing the terms P_1'''' and P_1^5 in Eq. (26) results in $N=1$. Following the enhanced Kudryashov technique, the solution is structured as indicated below

$$P_1(\xi)=\sigma_0+\sigma_1 R(\xi)+\rho_1\left(\frac{R'(\xi)}{R(\xi)}\right). \tag{44}$$

Using Eq. (44) and (32), and plugging them into Eq. (26), we arrive at a system of algebraic equations. When these equations are solved together, the results are as follows:

Result-1:

$$\begin{gathered}\sigma_0=0,\ \sigma_1=\pm\sqrt{-\frac{2\chi(10w_1+9w_5)}{w_2+w_4+w_6}},\ \rho_1=0,\ k=\sqrt{-(w_1+w_5)},\\ w_3=\frac{\left(3w_5(4w_2+w_4-2w_6)+2w_1(6w_2+w_4-4w_6)\right)(w_2+w_4+w_6)}{2(10w_1+9w_5)^2}.\end{gathered} \tag{45}$$

Hence, the model leads to the solutions:

$$q(x,t)=\left\{\frac{\pm 4c\sqrt{-\frac{2\chi(10w_1+9w_5)}{w_2+w_4+w_6}}}{4c^2 e^{\sqrt{-(w_1+w_5)}(x-vt)}+\chi e^{-\sqrt{-(w_1+w_5)}(x-vt)}}\right\} e^{i\left(-\left\{\frac{c_{21}\alpha_{71}}{4c_{11}\sigma_{11}}\right\}x+\omega t+\sigma W(t)-\sigma^2 t+\theta_0\right)}, \tag{46}$$

$$r(x,t)=\varpi\left\{\frac{\pm 4c\sqrt{-\dfrac{2\chi\left(10w_1+9w_5\right)}{w_2+w_4+w_6}}}{4c^2e^{\sqrt{-(w_1+w_5)}(x-vt)}+\chi e^{-\sqrt{-(w_1+w_5)}(x-vt)}}\right\} \times e^{i\left(-\left\{\frac{c_{22}\alpha_{72}}{4c_{12}\sigma_{12}}\right\}x+\omega t+\sigma W(t)-\sigma^2 t+\theta_0\right)}. \tag{47}$$

Selecting $\chi=\pm 4c^2$ in Eqs. (46) and (47) yields bright and singular solitons when $w_1+w_5<0$ and $\left(10w_1+9w_5\right)\left(w_2+w_4+w_6\right)<0$:

$$q(x,t)=\pm\sqrt{-\frac{2\left(10w_1+9w_5\right)}{w_2+w_4+w_6}}\operatorname{sech}\left[\sqrt{-\left(w_1+w_5\right)}\left(x-vt\right)\right] \times e^{i\left(-\left\{\frac{c_{21}\alpha_{71}}{4c_{11}\sigma_{11}}\right\}x+\omega t+\sigma W(t)-\sigma^2 t+\theta_0\right)}, \tag{48}$$

$$r(x,t)=\pm\varpi\sqrt{-\frac{2\left(10w_1+9w_5\right)}{w_2+w_4+w_6}}\operatorname{sech}\left[\sqrt{-\left(w_1+w_5\right)}\left(x-vt\right)\right] \times e^{i\left(-\left\{\frac{c_{22}\alpha_{72}}{4c_{12}\sigma_{12}}\right\}x+\omega t+\sigma W(t)-\sigma^2 t+\theta_0\right)}, \tag{49}$$

$$q(x,t)=\pm\sqrt{\frac{2\left(10w_1+9w_5\right)}{w_2+w_4+w_6}}\operatorname{csch}\left[\sqrt{-\left(w_1+w_5\right)}\left(x-vt\right)\right] \times e^{i\left(-\left\{\frac{c_{21}\alpha_{71}}{4c_{11}\sigma_{11}}\right\}x+\omega t+\sigma W(t)-\sigma^2 t+\theta_0\right)}, \tag{50}$$

$$r(x,t)=\pm\varpi\sqrt{\frac{2\left(10w_1+9w_5\right)}{w_2+w_4+w_6}}\operatorname{csch}\left[\sqrt{-\left(w_1+w_5\right)}\left(x-vt\right)\right] \times e^{i\left(-\left\{\frac{c_{22}\alpha_{72}}{4c_{12}\sigma_{12}}\right\}x+\omega t+\sigma W(t)-\sigma^2 t+\theta_0\right)}. \tag{51}$$

Figs. 1 through 12 present visual representations of numerical simulations illustrating various soliton solutions under specific parameter settings. The parameters used in these simulations are: $a_l=1$, $b_l=1$, $\omega=1$, $\sigma=1$, $k=1$, $\alpha_1=1$, $\beta_1=1$, $\gamma_1=1$, $\delta_1=1$, $\zeta_1=1$, $\lambda_1=1$, $\mu_1=1$, $\rho_1=1$, $\varepsilon_1=1$, $\eta_1=1$, $\tau_1=1$, $\theta_1=1$, $c_1=1$, $d_1=1$ and $\varpi=5.5$. In Figs. 1, 2, and 3, we present surface, contour, and 2D plots that showcase the bright soliton solutions (48) and (49). These plots provide a comprehensive visual understanding of the behavior of these solutions. Moving on to Figs. 4, 5, and 6, we shift our focus to the dark soliton solutions (55) and (56). These figures provide surface, contour, and 2D plots that depict the characteristics and properties of the dark solitons under the specified parameter values. Figs. 7, 8, and 9 introduce the combo bright–dark soliton solutions (82) and (83). These visual representations in surface, contour, and 2D plots give us valuable insights into the combined behavior of bright and dark solitons in the given parameter regime. Lastly, Figs. 10, 11, and 12 delve into the singular soliton solutions (57) and (58). Through surface, contour, and 2D plots, these figures offer a detailed exploration of the singular solitons, shedding light on their unique characteristics and features. Collectively, these figures and simulations provide a comprehensive and insightful visual representation of the different soliton solutions in the context of the specified parameters, aiding in the understanding and analyzing these complex mathematical phenomena.

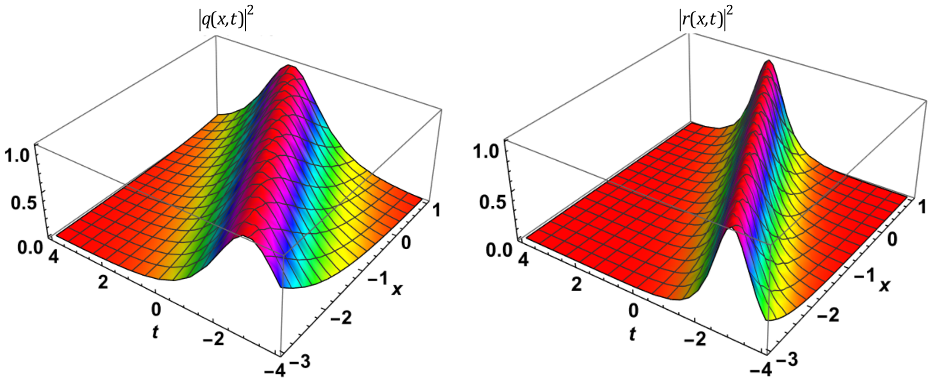

Fig. 1. Surface plots of a bright soliton.

Result-2:

$$\sigma_0 = \sigma_1 = 0,\ \rho_1 = \pm\sqrt{\frac{6w_5 - 5w_1}{2w_2 + w_4 - 4w_6}},$$
$$k = \frac{1}{2}\sqrt{\frac{2w_5\left(w_2 - w_4 - 2w_6\right) + w_1\left(2\left(w_4 + w_6\right) - w_2\right)}{2\left(2w_2 + w_4 - 4w_6\right)}}, \tag{52}$$
$$w_3 = \frac{\left(2w_2 + w_4 - 4w_6\right)\left(w_1\left(3w_2 - w_4 + 4w_6\right) - 6w_2w_5\right)}{\left(5w_1 - 6w_5\right)^2}.$$

Accordingly, the model results in the solutions:

$$q(x,t) = \pm\sqrt{\frac{6w_5 - 5w_1}{2w_2 + w_4 - 4w_6}}\left(\frac{\chi - 4c^2 e^{\sqrt{\frac{2w_5(w_2 - w_4 - 2w_6) + w_1(2(w_4 + w_6) - w_2)}{2(2w_2 + w_4 - 4w_6)}}(x - vt)}}{4c^2 e^{\sqrt{\frac{2w_5(w_2 - w_4 - 2w_6) + w_1(2(w_4 + w_6) - w_2)}{2(2w_2 + w_4 - 4w_6)}}(x - vt)} + \chi}\right) \tag{53}$$
$$\times e^{i\left(-\left\{\frac{c_{21}\alpha_{71}}{4c_{11}\sigma_{11}}\right\}x + \omega t + \sigma W(t) - \sigma^2 t + \theta_0\right)},$$

$$r(x,t) = \pm\varpi\sqrt{\frac{6w_5 - 5w_1}{2w_2 + w_4 - 4w_6}}\left(\frac{\chi - 4c^2 e^{\sqrt{\frac{2w_5(w_2 - w_4 - 2w_6) + w_1(2(w_4 + w_6) - w_2)}{2(2w_2 + w_4 - 4w_6)}}(x - vt)}}{4c^2 e^{\sqrt{\frac{2w_5(w_2 - w_4 - 2w_6) + w_1(2(w_4 + w_6) - w_2)}{2(2w_2 + w_4 - 4w_6)}}(x - vt)} + \chi}\right) \tag{54}$$
$$\times e^{i\left(-\left\{\frac{c_{22}\alpha_{72}}{4c_{12}\sigma_{12}}\right\}x + \omega t + \sigma W(t) - \sigma^2 t + \theta_0\right)}.$$

Selecting $\chi = \pm 4c^2$ recovers dark and singular solitons (see Eqs. (53) and (54)) when $\left(6w_5 - 5w_1\right)\left(2w_2 + w_4 - 4w_6\right) > 0$ and $\left(2w_5\left(w_2 - w_4 - 2w_6\right) + w_1\left(2\left(w_4 + w_6\right) - w_2\right)\right)\left(2w_2 + w_4 - 4w_6\right) > 0$:

$$q(x,t) = \mp\sqrt{\frac{6w_5 - 5w_1}{2w_2 + w_4 - 4w_6}}$$
$$\times\tanh\left[\frac{1}{2}\sqrt{\frac{2w_5\left(w_2 - w_4 - 2w_6\right) + w_1\left(2\left(w_4 + w_6\right) - w_2\right)}{2\left(2w_2 + w_4 - 4w_6\right)}}\left(x - vt\right)\right] \tag{55}$$
$$\times e^{i\left(-\left\{\frac{c_{21}\alpha_{71}}{4c_{11}\sigma_{11}}\right\}x + \omega t + \sigma W(t) - \sigma^2 t + \theta_0\right)},$$

$$r(x,t)=\mp\varpi\sqrt{\frac{6w_5-5w_1}{2w_2+w_4-4w_6}}\times\tanh\left[\frac{1}{2}\sqrt{\frac{2w_5(w_2-w_4-2w_6)+w_1(2(w_4+w_6)-w_2)}{2(2w_2+w_4-4w_6)}}(x-vt)\right]\times e^{i\left(-\left\{\frac{c_{22}\alpha_{72}}{4c_{12}\sigma_{12}}\right\}x+\omega t+\sigma W(t)-\sigma^2 t+\theta_0\right)}, \quad (56)$$

$$q(x,t)=\mp\sqrt{\frac{6w_5-5w_1}{2w_2+w_4-4w_6}}\times\coth\left[\frac{1}{2}\sqrt{\frac{2w_5(w_2-w_4-2w_6)+w_1(2(w_4+w_6)-w_2)}{2(2w_2+w_4-4w_6)}}(x-vt)\right]\times e^{i\left(-\left\{\frac{c_{21}\alpha_{71}}{4c_{11}\sigma_{11}}\right\}x+\omega t+\sigma W(t)-\sigma^2 t+\theta_0\right)}, \quad (57)$$

$$r(x,t)=\mp\varpi\sqrt{\frac{6w_5-5w_1}{2w_2+w_4-4w_6}}\times\coth\left[\frac{1}{2}\sqrt{\frac{2w_5(w_2-w_4-2w_6)+w_1(2(w_4+w_6)-w_2)}{2(2w_2+w_4-4w_6)}}(x-vt)\right]\times e^{i\left(-\left\{\frac{c_{22}\alpha_{72}}{4c_{12}\sigma_{12}}\right\}x+\omega t+\sigma W(t)-\sigma^2 t+\theta_0\right)}. \quad (58)$$

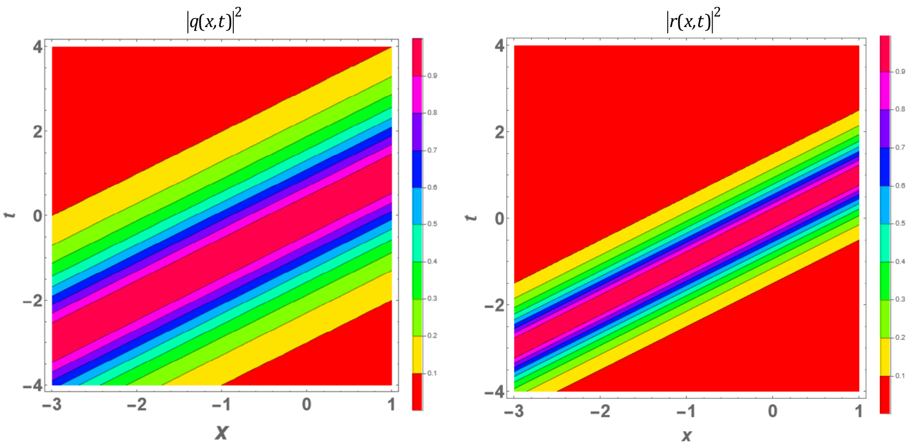

Fig. 2. Contour plots of a bright soliton.

Result-3:

$$\sigma_0=0,\ \sigma_1=-i\rho_1\sqrt{\chi},$$
$$\rho_1=\pm\sqrt{\frac{6w_5-20w_1}{8w_2+w_4-4w_6}},$$
$$k=\sqrt{\frac{4w_1(-2w_2+w_4+w_6)+w_5(4w_2-w_4-2w_6)}{8w_2+w_4-4w_6}}, \quad (59)$$
$$w_3=\frac{(8w_2+w_4-4w_6)(12w_1w_2-6w_5w_2-w_1w_4+4w_1w_6)}{4(10w_1-3w_5)^2}.$$

This leads to the solutions of the model:

$$q(x,t)=\pm\sqrt{\frac{6w_5-20w_1}{8w_2+w_4-4w_6}}$$
$$\times\left\{\frac{2\chi e^{-\sqrt{\frac{4w_1(-2w_2+w_4+w_6)+w_5(4w_2-w_4-2w_6)}{8w_2+w_4-4w_6}}(x-vt)}-i4c\sqrt{\chi}}{4c^2e^{\sqrt{\frac{4w_1(-2w_2+w_4+w_6)+w_5(4w_2-w_4-2w_6)}{8w_2+w_4-4w_6}}(x-vt)}+\chi e^{-\sqrt{\frac{4w_1(-2w_2+w_4+w_6)+w_5(4w_2-w_4-2w_6)}{8w_2+w_4-4w_6}}(x-vt)}}-1\right\} \tag{60}$$
$$\times e^{i\left(-\left\{\frac{c_{21}\alpha_{71}}{4c_{11}\sigma_{11}}\right\}x+\omega t+\sigma W(t)-\sigma^2t+\theta_0\right)}$$

$$r(x,t)=\pm\varpi\sqrt{\frac{6w_5-20w_1}{8w_2+w_4-4w_6}}$$
$$\times\left\{\frac{2\chi e^{-\sqrt{\frac{4w_1(-2w_2+w_4+w_6)+w_5(4w_2-w_4-2w_6)}{8w_2+w_4-4w_6}}(x-vt)}-i4c\sqrt{\chi}}{4c^2e^{\sqrt{\frac{4w_1(-2w_2+w_4+w_6)+w_5(4w_2-w_4-2w_6)}{8w_2+w_4-4w_6}}(x-vt)}+\chi e^{-\sqrt{\frac{4w_1(-2w_2+w_4+w_6)+w_5(4w_2-w_4-2w_6)}{8w_2+w_4-4w_6}}(x-vt)}}-1\right\} \tag{61}$$
$$\times e^{i\left(-\left\{\frac{c_{22}\alpha_{72}}{4c_{12}\sigma_{12}}\right\}x+\omega t+\sigma W(t)-\sigma^2t+\theta_0\right)}.$$

Complexitons and singular solitons (see Eqs. (60) and (61)) along with $\chi=\pm4c^2$ are:

$$q(x,t)=\pm\sqrt{\frac{6w_5-20w_1}{8w_2+w_4-4w_6}}$$
$$\times\left\{\tanh\left[\sqrt{\frac{4w_1(-2w_2+w_4+w_6)+w_5(4w_2-w_4-2w_6)}{8w_2+w_4-4w_6}}(x-vt)\right]\right.$$
$$\left.+i\,\mathrm{sech}\left[\sqrt{\frac{4w_1(-2w_2+w_4+w_6)+w_5(4w_2-w_4-2w_6)}{8w_2+w_4-4w_6}}(x-vt)\right]\right\} \tag{62}$$
$$\times e^{i\left(-\left\{\frac{c_{21}\alpha_{71}}{4c_{11}\sigma_{11}}\right\}x+\omega t+\sigma W(t)-\sigma^2t+\theta_0\right)},$$

$$r(x,t)=\pm\varpi\sqrt{\frac{6w_5-20w_1}{8w_2+w_4-4w_6}}$$
$$\times\left\{\tanh\left[\sqrt{\frac{4w_1(-2w_2+w_4+w_6)+w_5(4w_2-w_4-2w_6)}{8w_2+w_4-4w_6}}(x-vt)\right]\right.$$
$$\left.+i\,\mathrm{sech}\left[\sqrt{\frac{4w_1(-2w_2+w_4+w_6)+w_5(4w_2-w_4-2w_6)}{8w_2+w_4-4w_6}}(x-vt)\right]\right\} \tag{63}$$
$$\times e^{i\left(-\left\{\frac{c_{22}\alpha_{72}}{4c_{12}\sigma_{12}}\right\}x+\omega t+\sigma W(t)-\sigma^2t+\theta_0\right)},$$

$$q(x,t)=\pm\sqrt{\frac{6w_5-20w_1}{8w_2+w_4-4w_6}}$$
$$\times\coth\left[\frac{1}{2}\sqrt{\frac{4w_1(-2w_2+w_4+w_6)+w_5(4w_2-w_4-2w_6)}{8w_2+w_4-4w_6}}(x-vt)\right] \quad (64)$$
$$\times e^{i\left(-\left\{\frac{c_{21}\alpha_{71}}{4c_{11}\sigma_{11}}\right\}x+\omega t+\sigma W(t)-\sigma^2 t+\theta_0\right)},$$

$$r(x,t)=\pm\varpi\sqrt{\frac{6w_5-20w_1}{8w_2+w_4-4w_6}}$$
$$\times\coth\left[\frac{1}{2}\sqrt{\frac{4w_1(-2w_2+w_4+w_6)+w_5(4w_2-w_4-2w_6)}{8w_2+w_4-4w_6}}(x-vt)\right] \quad (65)$$
$$\times e^{i\left(-\left\{\frac{c_{22}\alpha_{72}}{4c_{12}\sigma_{12}}\right\}x+\omega t+\sigma W(t)-\sigma^2 t+\theta_0\right)}.$$

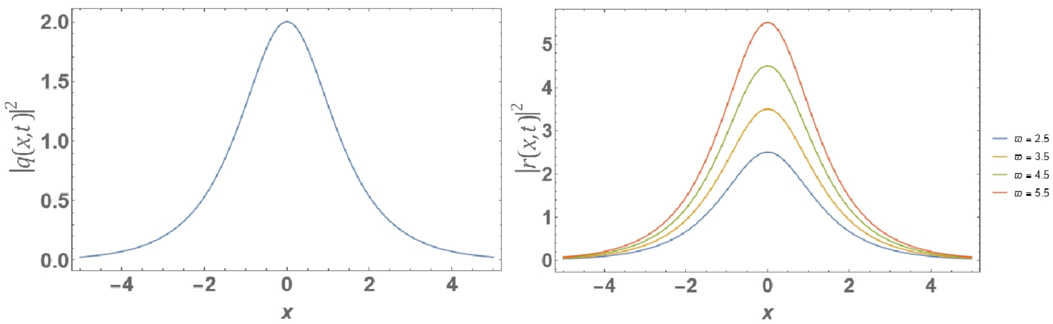

Fig. 3. 2D plots of a bright soliton.

4.2. The projective Riccati equations method

After balancing P_1'''' and P_1^5 in Eq. (14), we find $N=1$. In accordance with this, the solution is structured as below

$$P_1(\xi)=a_0+a_1F(\xi)+b_1G(\xi). \quad (66)$$

By combining Eqs. (66), (35), and Eq. (36), and plugging them into Eq. (26), a system of algebraic equations is formed. The solutions to this system, when obtained together, result in the following findings:

Case-1: $\zeta(\varepsilon)=0$

$$a_0=a_1=0,\ b_1=\pm\sqrt{\frac{6w_5-20w_1}{8w_2+w_4-4w_6}},\ k=\sqrt{\frac{4w_1(-2w_2+w_4+w_6)+w_5(4w_2-w_4-2w_6)}{8w_2+w_4-4w_6}},$$
$$w_3=\frac{(8w_2+w_4-4w_6)(w_1(12w_2-w_4+4w_6)-6w_2w_5)}{4(10w_1-3w_5)^2}. \quad (67)$$

As a consequence, the solutions of Eqs. (1) and (2) of the model are

$$q(x,t)=\pm\sqrt{\frac{6w_5-20w_1}{8w_2+w_4-4w_6}}$$
$$\times\tanh\left[\frac{1}{2}\sqrt{\frac{4w_1(-2w_2+w_4+w_6)+w_5(4w_2-w_4-2w_6)}{8w_2+w_4-4w_6}}\right] \quad (68)$$
$$\times e^{i\left(-\left\{\frac{c_{21}\alpha_{71}}{4c_{11}\sigma_{11}}\right\}x+\omega t+\sigma W(t)-\sigma^2 t+\theta_0\right)},$$

$$r(x,t)=\pm\varpi\sqrt{\frac{6w_5-20w_1}{8w_2+w_4-4w_6}}\times\tanh\left[\frac{1}{2}\sqrt{\frac{4w_1(-2w_2+w_4+w_6)+w_5(4w_2-w_4-2w_6)}{8w_2+w_4-4w_6}}\right]\times e^{i\left(-\left\{\frac{c_{22}\alpha_{72}}{4c_{12}\sigma_{12}}\right\}x+\omega t+\sigma W(t)-\sigma^2 t+\theta_0\right)}, \tag{69}$$

$$q(x,t)=\pm\sqrt{\frac{6w_5-20w_1}{8w_2+w_4-4w_6}}\times\coth\left[\frac{1}{2}\sqrt{\frac{4w_1(-2w_2+w_4+w_6)+w_5(4w_2-w_4-2w_6)}{8w_2+w_4-4w_6}}\right]\times e^{i\left(-\left\{\frac{c_{21}\alpha_{71}}{4c_{11}\sigma_{11}}\right\}x+\omega t+\sigma W(t)-\sigma^2 t+\theta_0\right)}, \tag{70}$$

$$r(x,t)=\pm\varpi\sqrt{\frac{6w_5-20w_1}{8w_2+w_4-4w_6}}\times\coth\left[\frac{1}{2}\sqrt{\frac{4w_1(-2w_2+w_4+w_6)+w_5(4w_2-w_4-2w_6)}{8w_2+w_4-4w_6}}\right]\times e^{i\left(-\left\{\frac{c_{22}\alpha_{72}}{4c_{12}\sigma_{12}}\right\}x+\omega t+\sigma W(t)-\sigma^2 t+\theta_0\right)}. \tag{71}$$

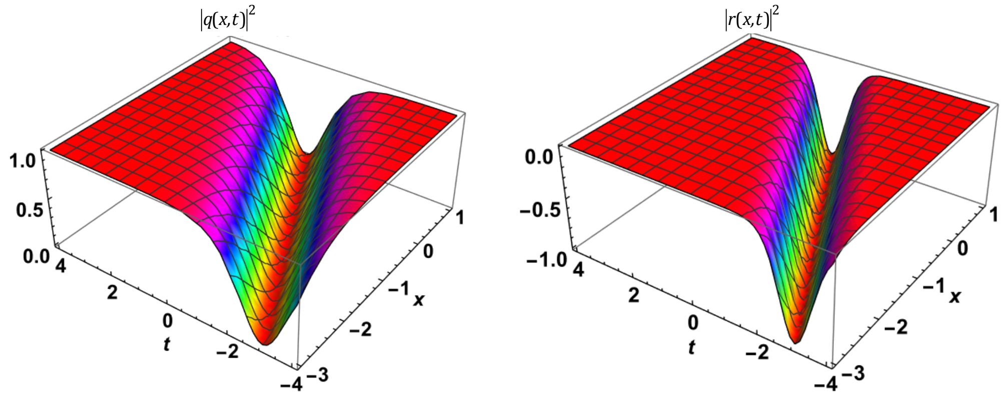

Fig. 4. Surface plots of a dark soliton.

Case-2: $\zeta(\varepsilon)=\frac{24}{25}\varepsilon^2$

Result-1:

$$a_0=0,\ a_1=\frac{12\varepsilon\sqrt{2w_5}}{5\sqrt{2w_4+3w_6}},\ b_1=0,\ k=\sqrt{-\frac{w_5}{5}},$$
$$w_1=-\frac{4w_5}{5}, w_2=\frac{1}{16}(6w_4+17w_6), \tag{72}$$
$$w_3=\frac{-2w_4^2-11w_6w_4-12w_6^2}{60w_5}.$$

In consequence, the model yields the solutions of Eqs. (1) and (2):

$$q(x,t)=\pm\left\{\frac{12\sqrt{2w_5}\ \mathrm{sech}\left[\sqrt{-\frac{w_5}{5}}(x-vt)\right]}{\sqrt{2w_4+3w_6}\left(5\mathrm{sech}\left[\sqrt{-\frac{w_5}{5}}(x-vt)\right]\pm1\right)}\right\} \tag{73}$$
$$\times e^{i\left(-\left\{\frac{c_{21}\alpha_{71}}{4c_{11}\sigma_{11}}\right\}x+\omega t+\sigma W(t)-\sigma^2 t+\theta_0\right)},$$

$$r(x,t)=\pm\varpi\left\{\frac{12\sqrt{2w_5}\ \mathrm{sech}\left[\sqrt{-\frac{w_5}{5}}(x-vt)\right]}{\sqrt{2w_4+3w_6}\left(5\mathrm{sech}\left[\sqrt{-\frac{w_5}{5}}(x-vt)\right]\pm1\right)}\right\} \tag{74}$$
$$\times e^{i\left(-\left\{\frac{c_{22}\alpha_{72}}{4c_{12}\sigma_{12}}\right\}x+\omega t+\sigma W(t)-\sigma^2 t+\theta_0\right)}.$$

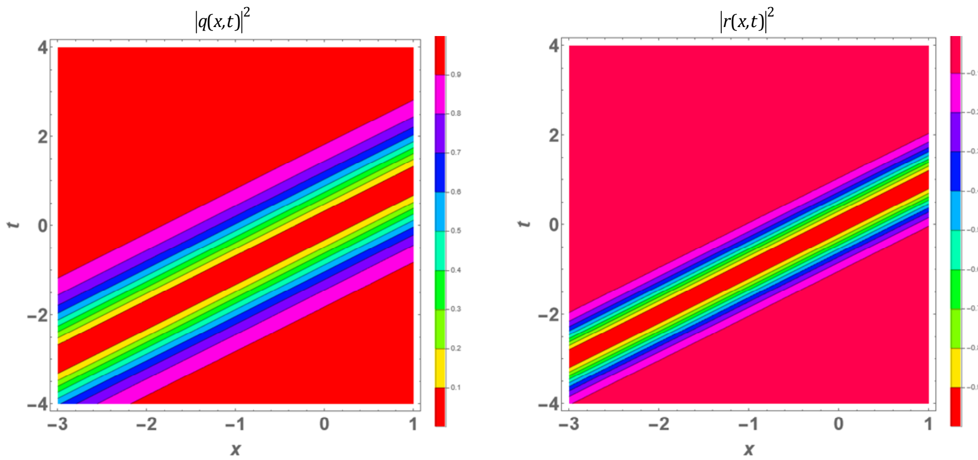

Fig. 5. Contour plots of a dark soliton.

Result-2:

$$a_0=0,\ a_1=0,\ b_1=\pm\frac{2\sqrt{6w_5}}{\sqrt{4w_4+7w_6}},$$
$$k=\sqrt{-\frac{2w_5(2w_4+3w_6)}{5(4w_4+7w_6)}},w_1=-\frac{w_5(32w_4+43w_6)}{10(4w_4+7w_6)}, \tag{75}$$
$$w_2=\frac{1}{48}(16w_4+47w_6),\ w_3=\frac{-4w_4^2-23w_6w_4-28w_6^2}{120w_5}.$$

Hence, the model leads to the solutions of Eqs. (1) and (2):

$$q(x,t)=\pm\left\{\frac{\pm2\sqrt{6w_5}\tanh\left[\sqrt{-\frac{2w_5(2w_4+3w_6)}{5(4w_4+7w_6)}}(x-vt)\right]}{\sqrt{4w_4+7w_6}\left(5\mathrm{sech}\left[\sqrt{-\frac{2w_5(2w_4+3w_6)}{5(4w_4+7w_6)}}(x-vt)\right]\pm1\right)}\right\} \tag{76}$$
$$\times e^{i\left(-\left\{\frac{c_{21}\alpha_{71}}{4c_{11}\sigma_{11}}\right\}x+\omega t+\sigma W(t)-\sigma^2 t+\theta_0\right)},$$

$$r(x,t)=\pm\varpi\left\{\frac{\left(\begin{array}{c}\pm 2\sqrt{6w_5}\\ \times\tanh\left[\sqrt{-\frac{2w_5(2w_4+3w_6)}{5(4w_4+7w_6)}}(x-vt)\right]\end{array}\right)}{\left(\begin{array}{c}\sqrt{4w_4+7w_6}\\ \times\left(5\operatorname{sech}\left[\sqrt{-\frac{2w_5(2w_4+3w_6)}{5(4w_4+7w_6)}}(x-vt)\right]\pm 1\right)\end{array}\right)}\right\}$$
$$\times e^{i\left(-\left\{\frac{c_{22}\alpha_{72}}{4c_{12}\sigma_{12}}\right\}x+\omega t+\sigma W(t)-\sigma^2 t+\theta_0\right)}. \tag{77}$$

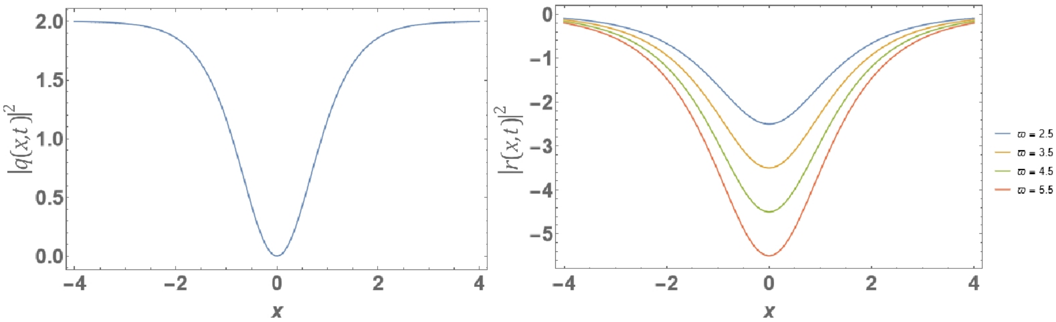

Fig. 6. 2D plots of a dark soliton.

<u>**Result-3**</u>**:**

$$a_0=0,\ b_1=\sqrt{\frac{6w_5-20w_1}{8w_2+w_4-4w_6}},$$
$$k=\sqrt{\frac{4w_1(-2w_2+w_4+w_6)+w_5(4w_2-w_4-2w_6)}{8w_2+w_4-4w_6}},$$
$$a_1=\frac{2\sqrt{6}\varepsilon}{5}b_1, \tag{78}$$
$$w_3=\frac{(8w_2+w_4-4w_6)\left(w_1(12w_2-w_4+4w_6)-6w_2w_5\right)}{4(10w_1-3w_5)^2}.$$

Accordingly, the model results in the solutions of Eqs. (1) and (2):

$$q(x,t)=\sqrt{\frac{6w_5-20w_1}{8w_2+w_4-4w_6}}$$
$$\times\frac{\left(\begin{array}{l}2\sqrt{6}\operatorname{sech}\left[\sqrt{\frac{4w_1(-2w_2+w_4+w_6)+w_5(4w_2-w_4-2w_6)}{8w_2+w_4-4w_6}}(x-vt)\right]\\ \pm\tanh\left[\sqrt{\frac{4w_1(-2w_2+w_4+w_6)+w_5(4w_2-w_4-2w_6)}{8w_2+w_4-4w_6}}(x-vt)\right]\end{array}\right)}{5\operatorname{sech}\left[\sqrt{\frac{4w_1(-2w_2+w_4+w_6)+w_5(4w_2-w_4-2w_6)}{8w_2+w_4-4w_6}}(x-vt)\right]\pm 1} \tag{79}$$
$$\times e^{i\left(-\left\{\frac{c_{21}\alpha_{71}}{4c_{11}\sigma_{11}}\right\}x+\omega t+\sigma W(t)-\sigma^2 t+\theta_0\right)},$$

$$r(x,t)=\varpi\sqrt{\frac{6w_5-20w_1}{8w_2+w_4-4w_6}}$$
$$\times\frac{\left(2\sqrt{6}\,\text{sech}\left[\sqrt{\frac{4w_1(-2w_2+w_4+w_6)+w_5(4w_2-w_4-2w_6)}{8w_2+w_4-4w_6}}(x-vt)\right] \pm\tanh\left[\sqrt{\frac{4w_1(-2w_2+w_4+w_6)+w_5(4w_2-w_4-2w_6)}{8w_2+w_4-4w_6}}(x-vt)\right]\right)}{\left(5\text{sech}\left[\sqrt{\frac{4w_1(-2w_2+w_4+w_6)+w_5(4w_2-w_4-2w_6)}{8w_2+w_4-4w_6}}(x-vt)\right]\pm 1\right)} \tag{80}$$
$$\times e^{i\left(-\left\{\frac{c_{22}\alpha_{72}}{4c_{12}\sigma_{12}}\right\}x+\omega t+\sigma W(t)-\sigma^2 t+\theta_0\right)}.$$

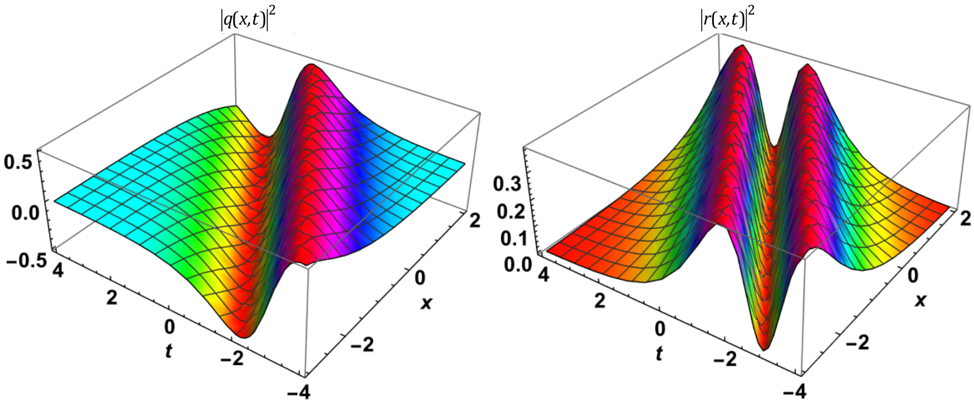

Fig. 7. Surface plots of a combo bright–dark soliton.

Case-3: $\zeta(\varepsilon)=\frac{5}{9}\varepsilon^2$

<u>Result-1</u>:

$$a_0=0,\ a_1=\frac{2\varepsilon\sqrt{5w_5}}{\sqrt{6w_4+9w_6}},\ b_1=0,\ k=\sqrt{-\frac{w_5}{5}},w_1=-\frac{4w_5}{5},$$
$$w_2=\frac{1}{15}(17w_4+33w_6),w_3=-\frac{(2w_4+3w_6)(w_4+4w_6)}{60w_5}. \tag{81}$$

In consequence, the model yields the solutions of Eqs. (1) and (2):

$$q(x,t)=\left\{\frac{6\sqrt{5w_5}\text{sech}\left[\sqrt{-\frac{w_5}{5}}(x-vt)\right]}{\sqrt{6w_4+9w_6}\left(3\text{sech}\left[\sqrt{-\frac{w_5}{5}}(x-vt)\right]\pm 2\right)}\right\} \tag{82}$$
$$\times e^{i\left(-\left\{\frac{c_{21}\alpha_{71}}{4c_{11}\sigma_{11}}\right\}x+\omega t+\sigma W(t)-\sigma^2 t+\theta_0\right)},$$

$$r(x,t)=\varpi\left\{\frac{6\sqrt{5w_5}\ \text{sech}\left[\sqrt{-\frac{w_5}{5}}(x-vt)\right]}{\sqrt{6w_4+9w_6}\left(3\text{sech}\left[\sqrt{-\frac{w_5}{5}}(x-vt)\right]\pm 2\right)}\right\} \tag{83}$$
$$\times e^{i\left(-\left\{\frac{c_{22}\alpha_{72}}{4c_{12}\sigma_{12}}\right\}x+\omega t+\sigma W(t)-\sigma^2 t+\theta_0\right)}.$$

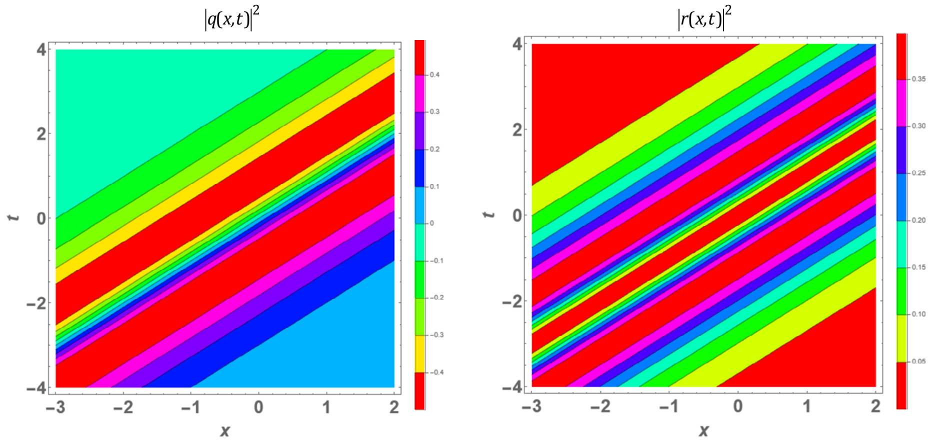

Fig. 8. Contour plots of a combo bright–dark soliton.

Result-2:

$$a_0 = a_1 = 0,\ b_1 = \frac{2\sqrt{15w_5}}{\sqrt{10w_4 + 63w_6}},\ k = \sqrt{-\frac{w_5(2w_4 + 3w_6)}{10w_4 + 63w_6}},\ w_1 = \frac{4w_5(3w_6 - 2w_4)}{10w_4 + 63w_6},$$
$$w_2 = \frac{1}{15}(5w_4 + 9w_6),\ w_3 = -\frac{(w_4 + 4w_6)(10w_4 + 63w_6)}{300w_5}. \tag{84}$$

Hence, the model leads to the solutions of Eqs. (1) and (2):

$$q(x,t) = \frac{4\sqrt{15w_5}}{\left(\sqrt{10w_4+63w_6}\left(\begin{array}{l}2\coth\left[\sqrt{-\frac{w_5(2w_4+3w_6)}{10w_4+63w_6}}(x-vt)\right]\\ \pm 3\operatorname{csch}\left[\sqrt{-\frac{w_5(2w_4+3w_6)}{10w_4+63w_6}}(x-vt)\right]\end{array}\right)\right)} \times e^{i\left(-\left\{\frac{c_{21}\alpha_{71}}{4c_{11}\sigma_{11}}\right\}x+\omega t+\sigma W(t)-\sigma^2 t+\theta_0\right)}, \tag{85}$$

$$r(x,t) = \varpi\frac{4\sqrt{15w_5}}{\left(\sqrt{10w_4+63w_6}\left(\begin{array}{l}2\coth\left[\sqrt{-\frac{w_5(2w_4+3w_6)}{10w_4+63w_6}}(x-vt)\right]\\ \pm 3\operatorname{csch}\left[\sqrt{-\frac{w_5(2w_4+3w_6)}{10w_4+63w_6}}(x-vt)\right]\end{array}\right)\right)} \times e^{i\left(-\left\{\frac{c_{22}\alpha_{72}}{4c_{12}\sigma_{12}}\right\}x+\omega t+\sigma W(t)-\sigma^2 t+\theta_0\right)}. \tag{86}$$

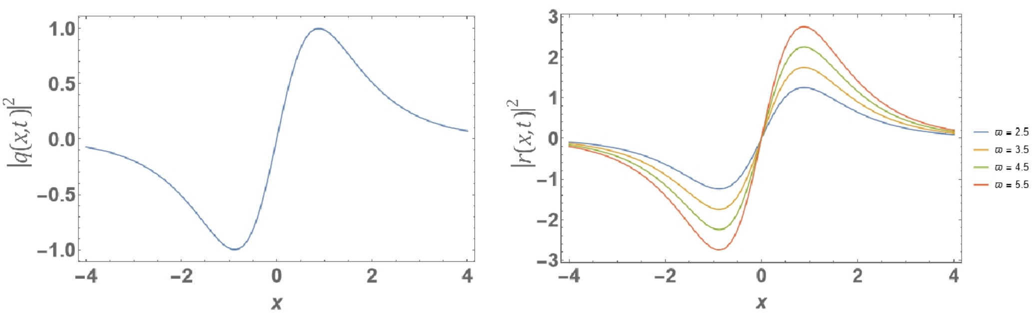

Fig. 9. 2D plots of a combo bright–dark soliton.

Result-3:

$$
\begin{aligned}
&a_0=0,\\
&a_1=\frac{\sqrt{5}\varepsilon}{3}b_1,\\
&b_1=\sqrt{\frac{6w_5-20w_1}{8w_2+w_4-4w_6}},\\
&k=\sqrt{\frac{4w_1(-2w_2+w_4+w_6)+w_5(4w_2-w_4-2w_6)}{8w_2+w_4-4w_6}},\\
&w_3=\frac{(8w_2+w_4-4w_6)(w_1(12w_2-w_4+4w_6)-6w_2w_5)}{4(10w_1-3w_5)^2}.
\end{aligned}
\tag{87}
$$

As a consequence, the solutions of Eqs. (1) and (2) of the model, reach

$$
\begin{aligned}
q(x,t)=&\sqrt{\frac{6w_5-20w_1}{8w_2+w_4-4w_6}}\\
&\times\left\{\frac{\sqrt{5}\operatorname{sech}\left[\sqrt{\frac{4w_1(-2w_2+w_4+w_6)+w_5(4w_2-w_4-2w_6)}{8w_2+w_4-4w_6}}(x-vt)\right]}{3\operatorname{sech}\left[\sqrt{\frac{4w_1(-2w_2+w_4+w_6)+w_5(4w_2-w_4-2w_6)}{8w_2+w_4-4w_6}}(x-vt)\right]\pm 2}\right.\\
&\left.+\frac{2}{2\coth\left[\sqrt{\frac{4w_1(-2w_2+w_4+w_6)+w_5(4w_2-w_4-2w_6)}{8w_2+w_4-4w_6}}(x-vt)\right]\pm 3\operatorname{csch}\left[\sqrt{\frac{4w_1(-2w_2+w_4+w_6)+w_5(4w_2-w_4-2w_6)}{8w_2+w_4-4w_6}}(x-vt)\right]}\right\},\\
&\times e^{i\left(-\left\{\frac{c_{21}\alpha_{71}}{4c_{11}\sigma_{11}}\right\}x+\omega t+\sigma W(t)-\sigma^2 t+\theta_0\right)},
\end{aligned}
\tag{88}
$$

$$
\begin{aligned}
r(x,t)=&\varpi\sqrt{\frac{6w_5-20w_1}{8w_2+w_4-4w_6}}\\
&\left\{\frac{\sqrt{5}\operatorname{sech}\left[\sqrt{\frac{4w_1(-2w_2+w_4+w_6)+w_5(4w_2-w_4-2w_6)}{8w_2+w_4-4w_6}}(x-vt)\right]}{3\operatorname{sech}\left[\sqrt{\frac{4w_1(-2w_2+w_4+w_6)+w_5(4w_2-w_4-2w_6)}{8w_2+w_4-4w_6}}(x-vt)\right]\pm 2}\right.\\
&\left.+\frac{2}{2\coth\left[\sqrt{\frac{4w_1(-2w_2+w_4+w_6)+w_5(4w_2-w_4-2w_6)}{8w_2+w_4-4w_6}}(x-vt)\right]\pm 3\operatorname{csch}\left[\sqrt{\frac{4w_1(-2w_2+w_4+w_6)+w_5(4w_2-w_4-2w_6)}{8w_2+w_4-4w_6}}(x-vt)\right]}\right\}\\
&\times e^{i\left(-\left\{\frac{c_{22}\alpha_{72}}{4c_{12}\sigma_{12}}\right\}x+\omega t+\sigma W(t)-\sigma^2 t+\theta_0\right)}.
\end{aligned}
\tag{89}
$$

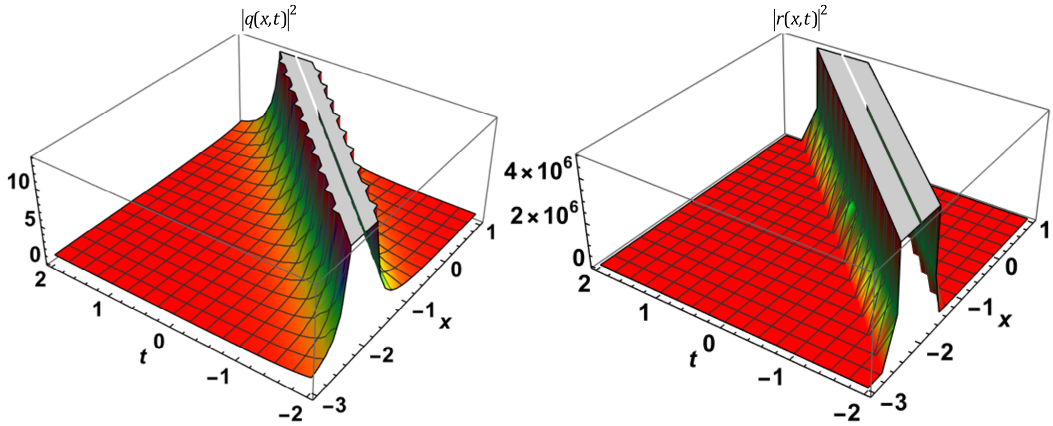

Fig. 10. Surface plots of a singular soliton.

Case-4: $\zeta(\varepsilon)=\varepsilon^2-1$

<u>Result-1</u>:

$$
\begin{aligned}
&a_0=0,\ a_1=\pm\sqrt{\frac{12(\varepsilon^2-1)w_5}{2w_4+3w_6}},\ b_1=0,\ k=\sqrt{-\frac{w_5}{5}},\\
&w_1=-\frac{1}{5}(4w_5), w_2=\frac{2(\varepsilon^2+2)w_4+3(2\varepsilon^2+1)w_6}{6(\varepsilon^2-1)},\\
&w_3=-\frac{(2w_4+3w_6)(w_4+4w_6)}{60w_5}.
\end{aligned}
\tag{90}
$$

Accordingly, the model results in the solutions of Eqs. (1) and (2):

$$
q(x,t)=\left\{\frac{4\sqrt{\frac{12(\varepsilon^2-1)w_5}{2w_4+3w_6}}\operatorname{sech}\left[\sqrt{-\frac{w_5}{5}}(x-vt)\right]}{4\varepsilon\operatorname{sech}\left[\sqrt{-\frac{w_5}{5}}(x-vt)\right]+3\tanh\left[\sqrt{-\frac{w_5}{5}}(x-vt)\right]+5}\right\} \times e^{i\left(-\left\{\frac{c_{21}\alpha_{71}}{4c_{11}\sigma_{11}}\right\}x+\omega t+\sigma W(t)-\sigma^2 t+\theta_0\right)}, \tag{91}
$$

$$
r(x,t)=\varpi\left\{\frac{4\sqrt{\frac{12(\varepsilon^2-1)w_5}{2w_4+3w_6}}\operatorname{sech}\left[\sqrt{-\frac{w_5}{5}}(x-vt)\right]}{4\varepsilon\operatorname{sech}\left[\sqrt{-\frac{w_5}{5}}(x-vt)\right]+3\tanh\left[\sqrt{-\frac{w_5}{5}}(x-vt)\right]+5}\right\} \times e^{i\left(-\left\{\frac{c_{22}\alpha_{72}}{4c_{12}\sigma_{12}}\right\}x+\omega t+\sigma W(t)-\sigma^2 t+\theta_0\right)}, \tag{92}
$$

or

$$
q(x,t)=\left\{\frac{\sqrt{\frac{12(\varepsilon^2-1)w_5}{2w_4+3w_6}}\operatorname{sech}\left[\sqrt{-\frac{w_5}{5}}(x-vt)\right]}{\varepsilon\operatorname{sech}\left[\sqrt{-\frac{w_5}{5}}(x-vt)\right]+1}\right\} \times e^{i\left(-\left\{\frac{c_{21}\alpha_{71}}{4c_{11}\sigma_{11}}\right\}x+\omega t+\sigma W(t)-\sigma^2 t+\theta_0\right)}, \tag{93}
$$

$$r(x,t)=\varpi\left\{\frac{\sqrt{\frac{12(\varepsilon^2-1)w_5}{2w_4+3w_6}}\operatorname{sech}\left[\sqrt{-\frac{w_5}{5}}(x-vt)\right]}{\varepsilon\operatorname{sech}\left[\sqrt{-\frac{w_5}{5}}(x-vt)\right]+1}\right\} \times e^{i\left(-\left\{\frac{c_{22}\alpha_{72}}{4c_{12}\sigma_{12}}\right\}x+\omega t+\sigma W(t)-\sigma^2 t+\theta_0\right)}. \quad (94)$$

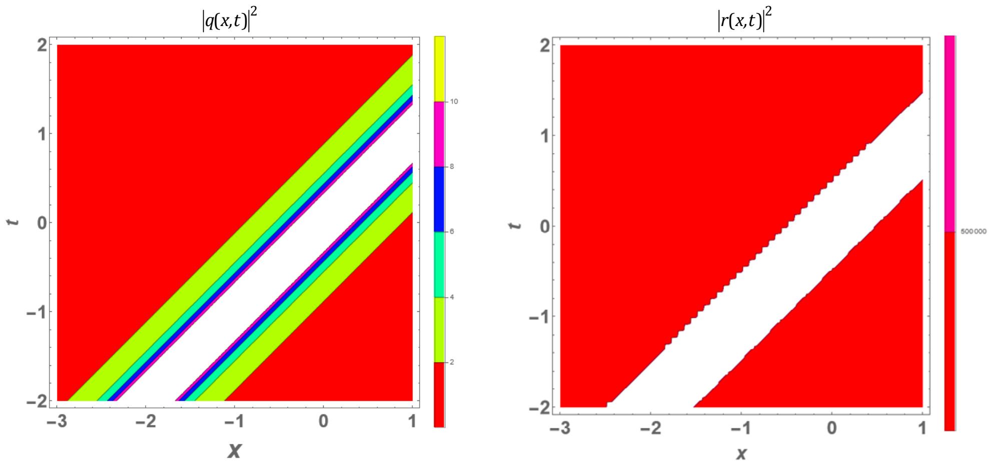

Fig. 11. Contour plots of a singular soliton.

Result-2:

$$\begin{aligned} &a_0=a_1=0,\ b_1=\sqrt{\frac{12(\varepsilon^2-1)w_5}{2(\varepsilon^2-1)w_4+3(\varepsilon^2+3)w_6}},\\ &k=\sqrt{-\frac{(\varepsilon^2-1)w_5(2w_4+3w_6)}{10(\varepsilon^2-1)w_4+15(\varepsilon^2+3)w_6}},\\ &w_1=-\frac{2w_5\left(4(\varepsilon^2-1)w_4+3(2\varepsilon^2-7)w_6\right)}{5\left(2(\varepsilon^2-1)w_4+3(\varepsilon^2+3)w_6\right)},\ w_2=\frac{(2\varepsilon^2-3)w_6}{2(\varepsilon^2-1)}+\frac{w_4}{3},\\ &w_3=-\frac{(w_4+4w_6)\left(2(\varepsilon^2-1)w_4+3(\varepsilon^2+3)w_6\right)}{60(\varepsilon^2-1)w_5}. \end{aligned} \quad (95)$$

In consequence, the model yields the solutions of Eqs. (1) and (2):

$$q(x,t)=\left\{\frac{5\sqrt{\frac{12(\varepsilon^2-1)w_5}{2(\varepsilon^2-1)w_4+3(\varepsilon^2+3)w_6}}\times\tanh\left[\sqrt{-\frac{(\varepsilon^2-1)w_5(2w_4+3w_6)}{10(\varepsilon^2-1)w_4+15(\varepsilon^2+3)w_6}}(x-vt)\right]+3}{3\tanh\left[\sqrt{-\frac{(\varepsilon^2-1)w_5(2w_4+3w_6)}{10(\varepsilon^2-1)w_4+15(\varepsilon^2+3)w_6}}(x-vt)\right]+4\varepsilon\operatorname{sech}\left[\sqrt{-\frac{(\varepsilon^2-1)w_5(2w_4+3w_6)}{10(\varepsilon^2-1)w_4+15(\varepsilon^2+3)w_6}}(x-vt)\right]+5}\right\} \times e^{i\left(-\left\{\frac{c_{21}\alpha_{71}}{4c_{11}\sigma_{11}}\right\}x+\omega t+\sigma W(t)-\sigma^2 t+\theta_0\right)}, \quad (96)$$

$$r(x,t)=\varpi\left\{\frac{5\sqrt{\frac{12(\varepsilon^2-1)w_5}{2(\varepsilon^2-1)w_4+3(\varepsilon^2+3)w_6}}\times\tanh\left[\sqrt{-\frac{(\varepsilon^2-1)w_5(2w_4+3w_6)}{10(\varepsilon^2-1)w_4+15(\varepsilon^2+3)w_6}}(x-vt)\right]+3}{3\tanh\left[\sqrt{-\frac{(\varepsilon^2-1)w_5(2w_4+3w_6)}{10(\varepsilon^2-1)w_4+15(\varepsilon^2+3)w_6}}(x-vt)\right]+4\varepsilon\,\mathrm{sech}\left[\sqrt{-\frac{(\varepsilon^2-1)w_5(2w_4+3w_6)}{10(\varepsilon^2-1)w_4+15(\varepsilon^2+3)w_6}}(x-vt)\right]+5}\right\}\times e^{i\left(-\left\{\frac{c_{22}\alpha_{72}}{4c_{12}\sigma_{12}}\right\}x+\omega t+\sigma W(t)-\sigma^2 t+\theta_0\right)},\tag{97}$$

or

$$q(x,t)=\left\{\frac{5\sqrt{\frac{12(\varepsilon^2-1)w_5}{2(\varepsilon^2-1)w_4+3(\varepsilon^2+3)w_6}}\times\tanh\left[\sqrt{-\frac{(\varepsilon^2-1)w_5(2w_4+3w_6)}{10(\varepsilon^2-1)w_4+15(\varepsilon^2+3)w_6}}(x-vt)\right]+3}{3\tanh\left[\sqrt{-\frac{(\varepsilon^2-1)w_5(2w_4+3w_6)}{10(\varepsilon^2-1)w_4+15(\varepsilon^2+3)w_6}}(x-vt)\right]+4\varepsilon\,\mathrm{sech}\left[\sqrt{-\frac{(\varepsilon^2-1)w_5(2w_4+3w_6)}{10(\varepsilon^2-1)w_4+15(\varepsilon^2+3)w_6}}(x-vt)\right]+5}\right\}\times e^{i\left(-\left\{\frac{c_{21}\alpha_{71}}{4c_{11}\sigma_{11}}\right\}x+\omega t+\sigma W(t)-\sigma^2 t+\theta_0\right)},\tag{98}$$

$$r(x,t)=\varpi\left\{\frac{5\sqrt{\frac{12(\varepsilon^2-1)w_5}{2(\varepsilon^2-1)w_4+3(\varepsilon^2+3)w_6}}\times\tanh\left[\sqrt{-\frac{(\varepsilon^2-1)w_5(2w_4+3w_6)}{10(\varepsilon^2-1)w_4+15(\varepsilon^2+3)w_6}}(x-vt)\right]+3}{3\tanh\left[\sqrt{-\frac{(\varepsilon^2-1)w_5(2w_4+3w_6)}{10(\varepsilon^2-1)w_4+15(\varepsilon^2+3)w_6}}(x-vt)\right]+4\varepsilon\,\mathrm{sech}\left[\sqrt{-\frac{(\varepsilon^2-1)w_5(2w_4+3w_6)}{10(\varepsilon^2-1)w_4+15(\varepsilon^2+3)w_6}}(x-vt)\right]+5}\right\}\times e^{i\left(-\left\{\frac{c_{22}\alpha_{72}}{4c_{12}\sigma_{12}}\right\}x+\omega t+\sigma W(t)-\sigma^2 t+\theta_0\right)}.\tag{99}$$

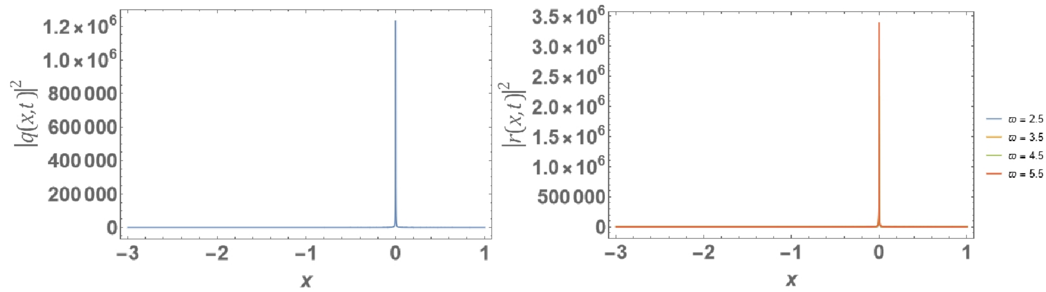

Fig. 12. 2D plots of a singular soliton.

Result-3:

$$\begin{aligned}
&a_0=0,\\
&a_1=b_1\sqrt{\varepsilon^2-1},\\
&b_1=\sqrt{\frac{6w_5-20w_1}{8w_2+w_4-4w_6}},\\
&k=\sqrt{\frac{4w_1(-2w_2+w_4+w_6)+w_5(4w_2-w_4-2w_6)}{8w_2+w_4-4w_6}},\\
&w_3=\frac{(8w_2+w_4-4w_6)(12w_1w_2-6w_5w_2-w_1w_4+4w_1w_6)}{4(10w_1-3w_5)^2}.
\end{aligned} \tag{100}$$

Hence, the model leads to the solutions of Eqs. (1) and (2):

$$\begin{aligned}
q(x,t)=&\sqrt{\frac{6w_5-20w_1}{8w_2+w_4-4w_6}}\\
&\times\left\{\frac{\left(\begin{array}{l}4\sqrt{\varepsilon^2-1}\operatorname{sech}\left[\sqrt{\frac{4w_1(-2w_2+w_4+w_6)+w_5(4w_2-w_4-2w_6)}{8w_2+w_4-4w_6}}(x-vt)\right]+\\ +5\tanh\left[\sqrt{\frac{4w_1(-2w_2+w_4+w_6)+w_5(4w_2-w_4-2w_6)}{8w_2+w_4-4w_6}}(x-vt)\right]+3\end{array}\right)}{\left(\begin{array}{l}4\varepsilon\operatorname{sech}\left[\sqrt{\frac{4w_1(-2w_2+w_4+w_6)+w_5(4w_2-w_4-2w_6)}{8w_2+w_4-4w_6}}(x-vt)\right]\\ +3\tanh\left[\sqrt{\frac{4w_1(-2w_2+w_4+w_6)+w_5(4w_2-w_4-2w_6)}{8w_2+w_4-4w_6}}(x-vt)\right]+5\end{array}\right)}\right\}\\
&\times e^{i\left(-\left\{\frac{c_{21}\alpha_{71}}{4c_{11}\sigma_{11}}\right\}x+\omega t+\sigma W(t)-\sigma^2t+\theta_0\right)},
\end{aligned} \tag{101}$$

$$\begin{aligned}
r(x,t)=&\varpi\sqrt{\frac{6w_5-20w_1}{8w_2+w_4-4w_6}}\\
&\times\left\{\frac{\left(\begin{array}{l}4\sqrt{\varepsilon^2-1}\operatorname{sech}\left[\sqrt{\frac{4w_1(-2w_2+w_4+w_6)+w_5(4w_2-w_4-2w_6)}{8w_2+w_4-4w_6}}(x-vt)\right]+\\ +5\tanh\left[\sqrt{\frac{4w_1(-2w_2+w_4+w_6)+w_5(4w_2-w_4-2w_6)}{8w_2+w_4-4w_6}}(x-vt)\right]+3\end{array}\right)}{\left(\begin{array}{l}4\varepsilon\operatorname{sech}\left[\sqrt{\frac{4w_1(-2w_2+w_4+w_6)+w_5(4w_2-w_4-2w_6)}{8w_2+w_4-4w_6}}(x-vt)\right]\\ +3\tanh\left[\sqrt{\frac{4w_1(-2w_2+w_4+w_6)+w_5(4w_2-w_4-2w_6)}{8w_2+w_4-4w_6}}(x-vt)\right]+5\end{array}\right)}\right\}\\
&\times e^{i\left(-\left\{\frac{c_{22}\alpha_{72}}{4c_{12}\sigma_{12}}\right\}x+\omega t+\sigma W(t)-\sigma^2t+\theta_0\right)},
\end{aligned} \tag{102}$$

or

$$q(x,t)=\sqrt{\frac{6w_5-20w_1}{8w_2+w_4-4w_6}}\times\left\{\frac{\left(\begin{array}{l}\sqrt{\varepsilon^2-1}\operatorname{sech}\left[\sqrt{\frac{4w_1(-2w_2+w_4+w_6)+w_5(4w_2-w_4-2w_6)}{8w_2+w_4-4w_6}}(x-vt)\right]\\+\tanh\left[\sqrt{\frac{4w_1(-2w_2+w_4+w_6)+w_5(4w_2-w_4-2w_6)}{8w_2+w_4-4w_6}}(x-vt)\right]\end{array}\right)}{\varepsilon\operatorname{sech}\left(\sqrt{\frac{4w_1(-2w_2+w_4+w_6)+w_5(4w_2-w_4-2w_6)}{8w_2+w_4-4w_6}}(x-vt)\right)+1}\right\}\times e^{i\left(-\left\{\frac{c_{21}\alpha_{71}}{4c_{11}\sigma_{11}}\right\}x+\omega t+\sigma W(t)-\sigma^2t+\theta_0\right)},\tag{103}$$

$$r(x,t)=\varpi\sqrt{\frac{6w_5-20w_1}{8w_2+w_4-4w_6}}\times\left\{\frac{\left(\begin{array}{l}\sqrt{\varepsilon^2-1}\operatorname{sech}\left[\sqrt{\frac{4w_1(-2w_2+w_4+w_6)+w_5(4w_2-w_4-2w_6)}{8w_2+w_4-4w_6}}(x-vt)\right]\\+\tanh\left[\sqrt{\frac{4w_1(-2w_2+w_4+w_6)+w_5(4w_2-w_4-2w_6)}{8w_2+w_4-4w_6}}(x-vt)\right]\end{array}\right)}{\varepsilon\operatorname{sech}\left(\sqrt{\frac{4w_1(-2w_2+w_4+w_6)+w_5(4w_2-w_4-2w_6)}{8w_2+w_4-4w_6}}(x-vt)\right)+1}\right\}\times e^{i\left(-\left\{\frac{c_{22}\alpha_{72}}{4c_{12}\sigma_{12}}\right\}x+\omega t+\sigma W(t)-\sigma^2t+\theta_0\right)}.\tag{104}$$

Case-5: $\zeta(\varepsilon)=\varepsilon^2+1$

<u>Result-1</u>:

$$a_0=0,\ a_1=\pm\sqrt{\frac{12(\varepsilon^2+1)w_5}{2w_4+3w_6}},\ b_1=0,\ k=\sqrt{-\frac{w_5}{5}},\ w_1=-\frac{4w_5}{5},$$
$$w_2=\frac{2(\varepsilon^2-2)w_4+3(2\varepsilon^2-1)w_6}{6(\varepsilon^2+1)},\ w_3=-\frac{(2w_4+3w_6)(w_4+4w_6)}{60w_5}.\tag{105}$$

Accordingly, the model results in the solutions of Eqs. (1) and (2):

$$q(x,t)=\left\{\frac{\sqrt{\frac{12(\varepsilon^2+1)w_5}{2w_4+3w_6}}\operatorname{csch}\left[\sqrt{-\frac{w_5}{5}}(x-vt)\right]}{\varepsilon\operatorname{csch}\left[\sqrt{-\frac{w_5}{5}}(x-vt)\right]+1}\right\}\times e^{i\left(-\left\{\frac{c_{21}\alpha_{71}}{4c_{11}\sigma_{11}}\right\}x+\omega t+\sigma W(t)-\sigma^2t+\theta_0\right)},\tag{106}$$

$$r(x,t)=\varpi\left\{\frac{\sqrt{\frac{12(\varepsilon^2+1)w_5}{2w_4+3w_6}}\operatorname{csch}\left[\sqrt{-\frac{w_5}{5}}(x-vt)\right]}{\varepsilon\operatorname{csch}\left[\sqrt{-\frac{w_5}{5}}(x-vt)\right]+1}\right\}\times e^{i\left(-\left\{\frac{c_{22}\alpha_{72}}{4c_{12}\sigma_{12}}\right\}x+\omega t+\sigma W(t)-\sigma^2t+\theta_0\right)}.\tag{107}$$

Result-2:

$$
\begin{aligned}
& a_0=a_1=0,\ b_1=\sqrt{\frac{12(\varepsilon^2+1)w_5}{2(\varepsilon^2+1)w_4+3(\varepsilon^2-3)w_6}}, \\
& k=\sqrt{-\frac{(\varepsilon^2+1)w_5(2w_4+3w_6)}{10(\varepsilon^2+1)w_4+15(\varepsilon^2-3)w_6}}, \\
& w_1=-\frac{2w_5\left(4(\varepsilon^2+1)w_4+3(2\varepsilon^2+7)w_6\right)}{5\left(2(\varepsilon^2+1)w_4+3(\varepsilon^2-3)w_6\right)}, \\
& w_2=\frac{(2\varepsilon^2-3)w_6}{2(\varepsilon^2+1)}+\frac{w_4}{3}, \\
& w_3=-\frac{(w_4+4w_6)\left(2(\varepsilon^2+1)w_4+3(\varepsilon^2-3)w_6\right)}{60(\varepsilon^2+1)w_5}.
\end{aligned}
\tag{108}
$$

In consequence, the model yields the solutions of Eqs. (1) and (2):

$$
q(x,t)=\left\{\frac{\sqrt{\frac{12(\varepsilon^2+1)w_5}{2(\varepsilon^2+1)w_4+3(\varepsilon^2-3)w_6}}\times\coth\left[\sqrt{-\frac{(\varepsilon^2+1)w_5(2w_4+3w_6)}{10(\varepsilon^2+1)w_4+15(\varepsilon^2-3)w_6}}(x-vt)\right]}{\varepsilon\,\mathrm{csch}\left[\sqrt{-\frac{(\varepsilon^2+1)w_5(2w_4+3w_6)}{10(\varepsilon^2+1)w_4+15(\varepsilon^2-3)w_6}}(x-vt)\right]+1}\right\}
\times e^{i\left(-\left\{\frac{c_{21}\alpha_{71}}{4c_{11}\sigma_{11}}\right\}x+\omega t+\sigma W(t)-\sigma^2 t+\theta_0\right)},
\tag{109}
$$

$$
r(x,t)=\varpi\left\{\frac{\sqrt{\frac{12(\varepsilon^2+1)w_5}{2(\varepsilon^2+1)w_4+3(\varepsilon^2-3)w_6}}\times\coth\left[\sqrt{-\frac{(\varepsilon^2+1)w_5(2w_4+3w_6)}{10(\varepsilon^2+1)w_4+15(\varepsilon^2-3)w_6}}(x-vt)\right]}{\varepsilon\,\mathrm{csch}\left[\sqrt{-\frac{(\varepsilon^2+1)w_5(2w_4+3w_6)}{10(\varepsilon^2+1)w_4+15(\varepsilon^2-3)w_6}}(x-vt)\right]+1}\right\}
\times e^{i\left(-\left\{\frac{c_{22}\alpha_{72}}{4c_{12}\sigma_{12}}\right\}x+\omega t+\sigma W(t)-\sigma^2 t+\theta_0\right)}.
\tag{110}
$$

Result-3:

$$
\begin{aligned}
& a_0=0, \\
& a_1=b_1\sqrt{\varepsilon^2+1}, \\
& b_1=\sqrt{\frac{6w_5-20w_1}{8w_2+w_4-4w_6}}, \\
& k=\sqrt{\frac{4w_1(-2w_2+w_4+w_6)+w_5(4w_2-w_4-2w_6)}{8w_2+w_4-4w_6}}, \\
& w_3=\frac{(8w_2+w_4-4w_6)(12w_1w_2-6w_5w_2-w_1w_4+4w_1w_6)}{4(10w_1-3w_5)^2}.
\end{aligned}
\tag{111}
$$

Hence, the model leads to the solutions of Eqs. (1) and (2):

$$q(x,t)=\sqrt{\frac{6w_5-20w_1}{8w_2+w_4-4w_6}}$$
$$\times\left\{\frac{\left(\begin{array}{l}\sqrt{\varepsilon^2+1}\operatorname{csch}\left[\sqrt{\frac{4w_1(-2w_2+w_4+w_6)+w_5(4w_2-w_4-2w_6)}{8w_2+w_4-4w_6}}(x-vt)\right]\\+\coth\left[\sqrt{\frac{4w_1(-2w_2+w_4+w_6)+w_5(4w_2-w_4-2w_6)}{8w_2+w_4-4w_6}}(x-vt)\right]\end{array}\right)}{\varepsilon\operatorname{csch}\left(\sqrt{\frac{4w_1(-2w_2+w_4+w_6)+w_5(4w_2-w_4-2w_6)}{8w_2+w_4-4w_6}}(x-vt)\right)+1}\right\} \tag{112}$$
$$\times e^{i\left(-\left\{\frac{c_{21}\alpha_{71}}{4c_{11}\sigma_{11}}\right\}x+\omega t+\sigma W(t)-\sigma^2 t+\theta_0\right)},$$

$$r(x,t)=\varpi\sqrt{\frac{6w_5-20w_1}{8w_2+w_4-4w_6}}$$
$$\times\left\{\frac{\left(\begin{array}{l}\sqrt{\varepsilon^2+1}\operatorname{csch}\left[\sqrt{\frac{4w_1(-2w_2+w_4+w_6)+w_5(4w_2-w_4-2w_6)}{8w_2+w_4-4w_6}}(x-vt)\right]\\+\coth\left[\sqrt{\frac{4w_1(-2w_2+w_4+w_6)+w_5(4w_2-w_4-2w_6)}{8w_2+w_4-4w_6}}(x-vt)\right]\end{array}\right)}{\varepsilon\operatorname{csch}\left(\sqrt{\frac{4w_1(-2w_2+w_4+w_6)+w_5(4w_2-w_4-2w_6)}{8w_2+w_4-4w_6}}(x-vt)\right)+1}\right\} \tag{113}$$
$$\times e^{i\left(-\left\{\frac{c_{22}\alpha_{72}}{4c_{12}\sigma_{12}}\right\}x+\omega t+\sigma W(t)-\sigma^2 t+\theta_0\right)}.$$

5. Conclusions

This paper addressed the soliton dynamics for the concatenation model in birefringent fibers that was with the effect of white noise along both components. The governing model is integrated with two integration schemes. They are the enhanced Kudryashov approach and the new projective Riccati equations approach. A full spectrum of soliton solutions collectively emerged from the schemes. It has been observed that the effect of white noise is with the phase portion along both components of the soliton solution. This is an interesting observation made for the first time in this paper.

The future of this paper holds very strong. Later, the model will be extended from differential group delay to dispersion flattened fibers that would portray a generalized version of the current results. Additional studies would also be conducted with the concatenation model, such as the retrieval of gap solitons, supercontinuum generation, and many others. These results will be recovered and reported across various journals after aligning them with the known results [11-32].

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Анотація. *У цій роботі отримуються розв'язки оптичних солітонів в моделі конкатенації з диференціальною груповою затримкою, за умови наявності білого шуму в обох компонентах. Повний спектр розв'язків солітонів для цієї моделі отримано за допомогою двох алгоритмів інтегрування. Було показано, що вплив білого шуму спостерігається лише у фазі солітонів в обох компонентах.*

Ключові слова: *вінерівський процес, метод Кудряшова, конкатенаційна модель, подвійне променезаломлення*