Optical solitons and complexitons for the concatenation model in birefringent fibers

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Received: 04.07.2023

Abstract. The current paper reveals optical soliton solutions to the concatenation model in birefringent fibers. Two integration approaches are adopted in the paper. They are the enhanced Kudryashov approach and the new projective Riccati equation's method. These reveal soliton and complexiton solutions that the method of undetermined coefficients failed to recover in earlier work. The parameter constraints are also displayed for the existence of the solitons and complexitons.

Keywords: solitons concatenation model, birefringence, parameter constraints, Kudryashov approach, Riccati equation's method

UDC: 535.32

1. Introduction

The concatenation model first came into light during 2014 and ever since its popularity has soared [1–8]. This model is a combination of three popular equations that are studied in the context of nonlinear fiber optics. They are the nonlinear Schrödinger's equation (NLSE), Lakshmanan–Porsezian–Daniel (LPD) equation, and the Sasa–Satsuma equation (SSE). The concatenation model has been gradually extensively studied from various perspectives. The rogue waves have been recovered [2]. Conservation laws have been identified with the usage of the multiplier approach [3]. Kudryashov's approaches have yielded straddled and other soliton solutions to the model [5]. The Painleve analysis has also been carried out [8].

Thereafter, moving further along, the model was addressed in birefringent fibers with differential group delay. In this context, a preliminary set of results has been identified. The method of undetermined coefficients has recovered the bright, dark, and singular soliton solutions [4]. However, as mentioned in the previous work [4], the method has its shortcomings. The approach failed to recover straddled soliton and also never recovered complexiton solutions. This paper, therefore, bridges that gap. The current work implements two integration algorithms to recover straddled soliton and complexiton solutions in birefringent fibers. These algorithms are the enhanced Kudryashov's method and the projective Riccati equation approach. The two approaches collectively reveal the same bright, dark, and singular 1–soliton solutions, as determined earlier by the method of undetermined coefficients, as well as straddled solitons and solitons of other functional forms and finally the complexiton solutions. These varieties of solutions are presented and displayed in the rest of the paper after a quick intro to the model and a succinct re-visitation of the integration schemes.

2. Governing model

For the scalar case, the concatenation model reads:

$$\begin{aligned} & iu_t + au_{xx} + b|u|^2 u + c_1\left[\sigma_1 u_{xxxx} + \sigma_2 (u_x)^2 u^* + \sigma_3 |u_x|^2 u + \sigma_4 |u|^2 u_{xx} + \sigma_5 u^2 u_{xx}^* + \sigma_6 |u|^4 u\right] \\ & + ic_2\left[\sigma_7 u_{xxx} + \sigma_8 |u|^2 u_x + \sigma_9 u^2 u_x^*\right] = 0 \end{aligned} . \quad (1)$$

In Eq. (1), $u(x,t)$ is the complex-valued wave profile that represents the electromagnetic wave, u_t gives the temporal dispersion, u_x is the spatial dispersion, u_{xx}, u_{xxx} and u_{xxxx} correspond to the higher-order dispersions, σ_i $(i = 1, 2, ..., 7)$ represent the coefficients of nonlinearity, while a and b are the coefficients of chromatic dispersion and Kerr law of nonlinearity, respectively. Here x and t represent the spatial and temporal variables, respectively. The first three terms correspond to the usual NLSE that is visible everywhere in fiber optics. The coefficient of c_1 is the LPD model, while the coefficient of c_2 is the SSE. Thus Eq. (1) is the conjunction of three standard models and is referred to as the concatenation model in the scalar case. When the pulse splits, the vector–coupled form of the concatenation model in birefringent fibers is given as:

$$\begin{aligned} & iq_t + a_1 q_{xx} + \left(b_1 |q|^2 + c_1 |r|^2\right) q \\ & + c_{11}\left[\sigma_{11} q_{xxxx} + \left\{\alpha_1 (q_x)^2 + \beta_1 (r_x)^2\right\} q^* + \left(\gamma_1 |q_x|^2 + \lambda_1 |r_x|^2\right) q + \left(\delta_1 |q|^2 + \zeta_1 |r|^2\right) q_{xx}\right. \\ & \left. + \left(\mu_1 q^2 + \rho_1 r^2\right) q_{xx}^* + \left(f_1 |q|^4 + g_1 |q|^2 |r|^2 + h_1 |r|^4\right) q\right] \\ & + ic_{21}\left[\sigma_{71} q_{xxx} + \left(\eta_1 |q|^2 + \theta_1 |r|^2\right) q_x + \left(\varepsilon_1 q^2 + \tau_1 r^2\right) q_x^*\right] = 0, \end{aligned} \quad (2)$$

and

$$\begin{aligned} & ir_t + a_2 r_{xx} + \left(b_2 |r|^2 + c_2 |q|^2\right) r \\ & + c_{12}\left[\sigma_{12} r_{xxxx} + \left\{\alpha_2 (r_x)^2 + \beta_2 (q_x)^2\right\} r^* + \left(\gamma_2 |r_x|^2 + \lambda_2 |q_x|^2\right) r + \left(\delta_2 |r|^2 + \zeta_2 |q|^2\right) r_{xx}\right. \\ & \left. + \left(\mu_2 r^2 + \rho_2 q^2\right) r_{xx}^* + \left(f_2 |r|^4 + g_2 |r|^2 |q|^2 + h_2 |q|^4\right) r\right] \\ & + ic_{22}\left[\sigma_{72} r_{xxx} + \left(\eta_2 |r|^2 + \theta_2 |q|^2\right) r_x + \left(\varepsilon_2 r^2 + \tau_2 q^2\right) r_x^*\right] = 0. \end{aligned} \quad (3)$$

Here, in Eqs. (2) and (3), the dependent variables $q(x,t)$ and $r(x,t)$ are the dependent variables along the two components in a birefringent fiber. For $j = 1, 2$, a_j represents the chromatic dispersion along the two components, while b_j accounts for self-phase modulation, and c_j

represents the cross-phase modulation. Then, σ_{1j} represents the fourth-order dispersions along the two components. Next up, α_j, β_j, γ_j, λ_j, δ_j, ζ_j, μ_j, ρ_j, f_j, g_j, and h_j are the respective split-ups of the coefficients σ_2 to σ_6 from the LPD model, along the two components for a birefringent fiber. σ_{7j} represents the coefficients of the fourth-order dispersion along the components, while in Equation (1), this effect is represented by the coefficient of σ_7. Finally, η_j, θ_j, ε_j, and τ_j are the components of soliton self-frequency shift along the two components of a birefringent fiber, which are designated by σ_8 and σ_9 in the SSE part.

In order to solve the coupled system (2) and (3), the following solution structure is assumed:

$$q(x,t) = U_1(\xi)e^{i\phi(x,t)}, \tag{4}$$

$$r(x,t) = U_2(\xi)e^{i\phi(x,t)}, \tag{5}$$

where, the wave variable ξ is given by

$$\xi = k(x - vt). \tag{6}$$

Here, $U_j(\xi)(j = 1,2)$ represents the amplitude component of the soliton solution with a width of k, which implies that there are two soliton amplitudes with a specified width and v is the speed of the soliton, while the phase component $\phi(x,t)$ is defined as

$$\phi(x,t) = -\kappa x + \omega t + \theta_0, \tag{7}$$

where κ is the wave number of the soliton, while ω represents the frequency, and θ_0 is the phase constant. Substituting Eqs. (4) and (5) into Eqs. (2) and (3) and then decomposing into real and imaginary parts, the real parts give

$$\begin{aligned}
&k^2\left(a_1 + 3\kappa\left(\sigma_{71}c_{21} - 2c_{11}\kappa\sigma_{11}\right)\right)U_1'' + \left(-a_1\kappa^2 - \sigma_{71}c_{21}\kappa^3 + c_{11}\kappa^4\sigma_{11} - \omega\right)U_1 \\
&+\left(b_1 - \kappa\left(c_{11}\kappa\left(\alpha_1 - \gamma_1 + \delta_1 + \mu_1\right) + c_{21}\left(\varepsilon_1 - \eta_1\right)\right)\right)U_1^3 + c_{11}f_1U_1^5 + c_{11}g_1U_2^2U_1^3 \\
&+c_{11}h_1U_2^4U_1 + c_{11}k^4\sigma_{11}U_1'''' + c_{11}k^2\left(\alpha_1 + \gamma_1\right)U_1U_{1'}^2 + c_{11}k^2\left(\beta_1 + \lambda_1\right)U_1U_{2'}^2 \\
&+c_{11}k^2\left(\delta_1 + \mu_1\right)U_1^2U_1'' + c_{11}k^2\left(\zeta_1 + \rho_1\right)U_2^2U_1'' \\
&+\left(c_1 - c_{11}\kappa^2\left(\beta_1 + \zeta_1 - \lambda_1 + \rho_1\right) - c_{21}\kappa\left(\tau_1 - \theta_1\right)\right)U_2^2U_1 = 0,
\end{aligned} \tag{8}$$

$$\begin{aligned}
&k^2\left(a_2 + 3\kappa\left(\sigma_{72}c_{22} - 2c_{12}\kappa\sigma_{12}\right)\right)U_2'' \\
&+\left(-a_2\kappa^2 - \sigma_{72}c_{22}\kappa^3 + c_{12}\kappa^4\sigma_{12} - \omega\right)U_2 \\
&+\left(b_2 - \kappa\left(c_{12}\kappa\left(\alpha_2 - \gamma_2 + \delta_2 + \mu_2\right) + c_{22}\left(\varepsilon_2 - \eta_2\right)\right)\right)U_2^3 \\
&+c_{12}f_2U_2^5 + c_{12}g_2U_1^2U_2^3 + c_{12}h_2U_1^4U_2 \\
&+c_{12}k^4\sigma_{12}U_2'''' + c_{12}k^2\left(\alpha_2 + \gamma_2\right)U_2U_{2'}^2 \\
&+c_{12}k^2\left(\beta_2 + \lambda_2\right)U_2U_{1'}^2 + c_{12}k^2\left(\delta_2 + \mu_2\right)U_2^2U_2'' \\
&+c_{12}k^2\left(\zeta_2 + \rho_2\right)U_1^2U_2'' \\
&+\left(c_2 - c_{12}\kappa^2\left(\beta_2 + \zeta_2 - \lambda_2 + \rho_2\right) - c_{22}\kappa\left(\tau_2 - \theta_2\right)\right)U_1^2U_2 = 0,
\end{aligned} \tag{9}$$

and the imaginary parts give

$$
\begin{aligned}
&-k\left(2a_1\kappa+3\sigma_{71}c_{21}\kappa^2-4c_{11}\kappa^3\sigma_{11}+v\right)U_{1'}\\
&+k\left(c_{21}\left(\varepsilon_1+\eta_1\right)-2c_{11}\kappa\left(\alpha_1+\delta_1-\mu_1\right)\right)U_1^2U_{1'}\\
&+k^3\left(\sigma_{71}c_{21}-4c_{11}\kappa\sigma_{11}\right)U_{1'''}-2\beta_1c_{11}\kappa k\ U_1U_2U_{2'}\\
&+k\left(2c_{11}\kappa\left(\rho_1-\zeta_1\right)+c_{21}\left(\theta_1+\tau_1\right)\right)U_2^2U_{1'}=0,
\end{aligned}
\tag{10}
$$

$$
\begin{aligned}
&-k\left(2a_2\kappa+3\sigma_{72}c_{22}\kappa^2-4c_{12}\kappa^3\sigma_{12}+v\right)U_{2'}\\
&+k\left(c_{22}\left(\varepsilon_2+\eta_2\right)-2c_{12}\kappa\left(\alpha_2+\delta_2-\mu_2\right)\right)U_2^2U_{2'}\\
&+k^3\left(\sigma_{72}c_{22}-4c_{12}\kappa\sigma_{12}\right)U_{2'''}-2\beta_2c_{12}\kappa k\ U_1U_2U_{1'}\\
&+k\left(2c_{12}\kappa\left(\rho_2-\zeta_2\right)+c_{22}\left(\theta_2+\tau_2\right)\right)U_1^2U_{2'}=0.
\end{aligned}
\tag{11}
$$

Here $U_{1'}=\dfrac{dU_1}{d\xi}$, $U_{1''}=\dfrac{d^2U_1}{d\xi^2}$, $U_{1'''}=\dfrac{d^3U_1}{d\xi^3}$, $U_{1'}^2=\left(\dfrac{dU_1}{d\xi}\right)^2$, $U_{2'}=\dfrac{dU_2}{d\xi}$, $U_{2''}=\dfrac{d^2U_2}{d\xi^2}$, $U_{2'''}=\dfrac{d^3U_2}{d\xi^3}$, and $U_{2'}^2=\left(\dfrac{dU_2}{d\xi}\right)^2$. Using the balancing principle leads to $U_2=\varrho\,U_1$, where ϱ is a constant, provided $\varrho\neq0$ or 1. Then Eqs. (8) and (9) become

$$
\begin{aligned}
&k^2\left(a_1+3\kappa\left(\sigma_{71}c_{21}-2c_{11}\kappa\sigma_{11}\right)\right)U_{1''}-\left(a_1\kappa^2+\sigma_{71}c_{21}\kappa^3-c_{11}\kappa^4\sigma_{11}+\omega\right)U_1\\
&+c_{11}\left(f_1+g_1\varrho^2+h_1\varrho^4\right)U_1^5\\
&+\Big[b_1-\kappa\Big(c_{11}\kappa\left(\alpha_1+\beta_1\varrho^2-\gamma_1+\delta_1+\zeta_1\varrho^2-\lambda_1\varrho^2+\mu_1+\rho_1\varrho^2\right)\\
&+c_{21}\left\{\varepsilon_1-\eta_1+\varrho^2\left(\tau_1-\theta_1\right)\right\}\Big)+c_1\varrho^2\Big]U_1^3\\
&+c_{11}k^4\sigma_{11}U_{1''''}+c_{11}k^2\left(\alpha_1+\varrho^2\left(\beta_1+\lambda_1\right)+\gamma_1\right)U_1U_{1'}^2\\
&+c_{11}k^2\left(\delta_1+\varrho^2\left(\zeta_1+\rho_1\right)+\mu_1\right)U_1^2U_{1''}=0,
\end{aligned}
\tag{12}
$$

$$
\begin{aligned}
&-k\left(2a_1\kappa+3\sigma_{71}c_{21}\kappa^2-4c_{11}\kappa^3\sigma_{11}+v\right)U_{1'}\\
&+k^3\left(\sigma_{71}c_{21}-4c_{11}\kappa\sigma_{11}\right)U_{1'''}+k\Big[c_{21}\left(\varepsilon_1+\eta_1+\varrho^2\left(\theta_1+\tau_1\right)\right)\\
&-2c_{11}\kappa\left(\alpha_1+\beta_1\varrho^2+\delta_1+\zeta_1\varrho^2-\mu_1-\rho_1\varrho^2\right)\Big]U_1^2U_{1'}=0,
\end{aligned}
\tag{13}
$$

and Eqs. (10) and (11) come out as:

$$
\begin{aligned}
&k^2\varrho\left(a_2+3\kappa\left(\alpha_{72}c_{22}-2c_{12}\kappa\sigma_{12}\right)\right)U_{1''}\\
&-\varrho\left(a_2\kappa^2+\alpha_{72}c_{22}\kappa^3-c_{12}\kappa^4\sigma_{12}+\omega\right)U_1\\
&+c_{12}\varrho\left(f_2\varrho^4+g_2\varrho^2+h_2\right)U_1^5\\
&+\varrho\Big(\varrho^2\left(b_2-\kappa\left[c_{12}\kappa\left(\alpha_2-\gamma_2+\delta_2+\mu_2\right)+c_{22}\left(\varepsilon_2-\eta_2\right)\right]\right)\\
&-c_{12}\kappa^2\left(\beta_2+\zeta_2-\lambda_2+\rho_2\right)+c_{22}\kappa\left(\theta_2-\tau_2\right)+c_2\Big)U_1^3\\
&+c_{12}k^4\sigma_{12}\varrho U_{1''''}+c_{12}k^2\varrho\left(\varrho^2\left(\alpha_2+\gamma_2\right)+\beta_2+\lambda_2\right)U_1U_{1'}^2\\
&+c_{12}k^2\varrho\left(\varrho^2\left(\delta_2+\mu_2\right)+\zeta_2+\rho_2\right)U_1^2U_{1''}=0,
\end{aligned}
\tag{14}
$$

$$
\begin{aligned}
&-k\varrho\left(2a_2\kappa+3\alpha_{72}c_{22}\kappa^2-4c_{12}\kappa^3\sigma_{12}+v\right)U_{1'}+k^3\varrho\left(\alpha_{72}c_{22}-4c_{12}\kappa\sigma_{12}\right)U_{1'''}\\
&+k\varrho\left(c_{22}\left(\varepsilon_2\varrho^2+\eta_2\varrho^2+\theta_2+\tau_2\right)-2c_{12}\kappa\left(\alpha_2\varrho^2+\beta_2+\delta_2\varrho^2+\zeta_2-\mu_2\varrho^2-\rho_2\right)\right)U_1^2U_{1'}=0.
\end{aligned}
\tag{15}
$$

By comparing Eqs. (12) with (14) and reducing them to a single equation, we are able to derive the following parametric restrictions:

$$
\begin{aligned}
&\varrho\left(\varrho^2\left(b_2-\kappa\left(c_{12}\kappa\left(\alpha_2-\gamma_2+\delta_2+\mu_2\right)+c_{22}\left(\varepsilon_2-\eta_2\right)\right)\right)\right.\\
&\left.-c_{12}\kappa^2\left(\beta_2+\zeta_2-\lambda_2+\rho_2\right)+c_{22}\kappa\left(\theta_2-\tau_2\right)+c_2\right)\\
&=b_1-\kappa\left(c_{11}\kappa\left(\alpha_1+\beta_1\varrho^2-\gamma_1+\delta_1+\zeta_1\varrho^2-\lambda_1\varrho^2+\mu_1+\rho_1\varrho^2\right)\right.\\
&\left.+c_{21}\left(\varepsilon_1-\eta_1+\varrho^2\left(\tau_1-\theta_1\right)\right)\right)+c_1\varrho^2,
\end{aligned}
\tag{16}
$$

$$
a_1\kappa^2+\sigma_{71}c_{21}\kappa^3-c_{11}\kappa^4\sigma_{11}+\omega=\varrho\left(a_2\kappa^2+\sigma_{72}c_{22}\kappa^3-c_{12}\kappa^4\sigma_{12}+\omega\right), \tag{17}
$$

$$
c_{11}\left(f_1+g_1\varrho^2+h_1\varrho^4\right)=c_{12}\varrho\left(f_2\varrho^4+g_2\varrho^2+h_2\right), \tag{18}
$$

$$
c_{11}\left(\alpha_1+\varrho^2\left(\beta_1+\lambda_1\right)+\gamma_1\right)=c_{12}\varrho\left(\varrho^2\left(\alpha_2+\gamma_2\right)+\beta_2+\lambda_2\right), \tag{19}
$$

$$
a_1+3\kappa\left(\sigma_{71}c_{21}-2c_{11}\kappa\sigma_{11}\right)=\varrho\left(a_2+3\kappa\left(\sigma_{72}c_{22}-2c_{12}\kappa\sigma_{12}\right)\right), \tag{20}
$$

$$
c_{11}\left(\delta_1+\varrho^2\left(\zeta_1+\rho_1\right)+\mu_1\right)=c_{12}\varrho\left(\varrho^2\left(\delta_2+\mu_2\right)+\zeta_2+\rho_2\right), \tag{21}
$$

$$
c_{11}\sigma_{11}=c_{12}\sigma_{12}\varrho. \tag{22}
$$

From the imaginary parts of Eqs. (13) and (15), we can equate the coefficients of the linearly independent functions with being zero then we get the speed of the two components as follows:

$$
v=-2\kappa a_1-3\kappa^2c_{21}\sigma_{71}+4\kappa^3c_{11}\sigma_{11}, \tag{23}
$$

$$
v=-2\kappa a_2-3\kappa^2c_{22}\sigma_{72}+4\kappa^3c_{12}\sigma_{12}, \tag{24}
$$

and the wave number as follows:

$$
\kappa=\frac{c_{22}\sigma_{72}}{4c_{12}\sigma_{12}}, \tag{25}
$$

$$
\kappa=\frac{c_{21}\sigma_{71}}{4c_{11}\sigma_{11}}. \tag{26}
$$

The third terms give the constraints:

$$
c_{21}\left(\varepsilon_1+\eta_1+\varrho^2\left(\theta_1+\tau_1\right)\right)=2c_{11}\kappa\left(\alpha_1+\beta_1\varrho^2+\delta_1+\zeta_1\varrho^2-\mu_1-\rho_1\varrho^2\right), \tag{27}
$$

$$
c_{22}\left(\varepsilon_2\varrho^2+\eta_2\varrho^2+\theta_2+\tau_2\right)=2c_{12}\kappa\left(\alpha_2\varrho^2+\beta_2+\delta_2\varrho^2+\zeta_2-\mu_2\varrho^2-\rho_2\right). \tag{28}
$$

By equating the velocities and the wave numbers of the two components, we get the following parametric restrictions:

$$
c_{22}=\frac{2\left(a_1-a_2\right)+\kappa\left(3\sigma_{71}c_{21}-4c_{11}\kappa\sigma_{11}+4c_{12}\kappa\sigma_{12}\right)}{3\sigma_{72}\kappa}, \tag{29}
$$

$$
c_{12}=\frac{c_{22}\sigma_{11}\sigma_{72}}{c_{21}\sigma_{12}\sigma_{71}}c_{11}. \tag{30}
$$

Eq. (12) can be set as

$$\vartheta_1 U_1 + \vartheta_2 U_1^3 + \vartheta_3 U_1^5 + \vartheta_4 U_1 U_{1'}^2 + \vartheta_5 U_{1''} + \vartheta_6 U_1^2 U_{1''} + k^2 U_{1''''} = 0, \tag{31}$$

with

$$\begin{cases} \vartheta_1 = -\dfrac{a_1\kappa^2 + \sigma_{71}c_{21}\kappa^3 - c_{11}\kappa^4\sigma_{11} + \omega}{c_{11}k^2\sigma_{11}}, \\ \vartheta_2 = \dfrac{b_1 - \kappa\begin{pmatrix} c_{11}\kappa\left(\alpha_1 + \beta_1\varrho^2 - \gamma_1 + \delta_1 + \zeta_1\varrho^2 - \lambda_1\varrho^2 + \mu_1 + \rho_1\varrho^2\right) \\ + c_{21}\left(\varepsilon_1 - \eta_1 + \varrho^2\left(\tau_1 - \theta_1\right)\right)\end{pmatrix} + c_1\varrho^2}{c_{11}k^2\sigma_{11}}, \\ \vartheta_3 = \dfrac{f_1 + g_1\varrho^2 + h_1\varrho^4}{k^2\sigma_{11}},\ \ \vartheta_4 = \dfrac{\alpha_1 + \varrho^2\left(\beta_1 + \lambda_1\right) + \gamma_1}{\sigma_{11}}, \\ \vartheta_5 = \dfrac{a_1 + 3\sigma_{71}c_{21}\kappa - 6c_{11}\kappa^2\sigma_{11}}{c_{11}\sigma_{11}},\ \ \vartheta_6 = \dfrac{\delta_1 + \varrho^2\left(\zeta_1 + \rho_1\right) + \mu_1}{\sigma_{11}}. \end{cases} \tag{32}$$

3. Integration algorithm: an overview

Consider a governing model:

$$G\left(u, u_x, u_t, u_{xt}, u_{xx}, \ldots\right) = 0, \tag{33}$$

where $u = u\left(x,t\right)$ denotes a wave profile, while t and x depict the time and space variables in sequence.

The relations

$$u\left(x,t\right) = U\left(\xi\right), \quad \xi = k\left(x - vt\right), \tag{34}$$

condenses Eq. (33) to

$$P\left(U, -kvU', kU', k^2U'', \ldots\right) = 0, \tag{35}$$

where k is the wave width, ξ is the wave variable, and v is the wave velocity.

3.1. Enhanced Kudryashov's method

The present subsection outlines the basic procedures of the new enhanced Kudryashov technique.

Step – 1: The reduced model (35) admits the explicit solution

$$U\left(\xi\right) = r_0 + \sum_{i=1}^{N} r_i R(\xi)^i + s_i\left(\frac{R'(\xi)}{R(\xi)^i}\right), \tag{36}$$

along with the auxiliary equation

$$R'(\xi)^2 = R(\xi)^2\left(1 - \chi R(\xi)^2\right). \tag{37}$$

Here r_0, χ, r_i, s_i $\left(i = 1, \ldots, N\right)$ are constants, where N come from the balancing technique in (35).

Step – 2: Eq. (37) gives the soliton wave

$$R\left(\xi\right) = \frac{4c}{4c^2e^{\xi} + \chi e^{-\xi}}, \tag{38}$$

where c is constant.

Step – 3: Putting Eq. (36) along with Eq. (37) into Eq. (35) gives us the constants in Eqs. (34) and (36). Finally, the acquired parametric restrictions can be applied by plugging them into Eq. (36)

along with Eq. (38). One arrives at straddled solitons, which can be reduced to bright, or dark or singular solitons.

3.2. Projective Riccati equation method

The central proceedings of the new projective Riccati equations method are as follows:

Step – 1: Eq. (35) has the formal solution

$$U(\xi)=\sigma_0+\sum_{i=1}^{N}F^{i-1}(\xi)\left(\sigma_i F(\xi)+\varrho_i G(\xi)\right), \tag{39}$$

where $F(\xi)$ and $G(\xi)$ satisfy the following ODEs:

$$F'(\xi)=-F(\xi)G(\xi),\ G'(\xi)=1-G^2(\xi)-\tau F(\xi), \tag{40}$$

where τ is constant. N is a positive integer that comes from the balancing principle in Eq. (35). σ_0,σ_i and $\varrho_i\ (i=0,1,...,N)$ are constants.

Step – 2: The solutions of Eq. (40) are listed as follows:

Family – 1:

$$\begin{cases} F(\xi)=\begin{cases}\dfrac{1}{2\tau}\operatorname{sech}^2\left[\dfrac{\xi}{2}\right]\\ -\dfrac{1}{2\tau}\operatorname{csch}^2\left[\dfrac{\xi}{2}\right]\end{cases}\\ G(\xi)=\begin{cases}\tanh\left[\dfrac{\xi}{2}\right]\\ \coth\left[\dfrac{\xi}{2}\right]\end{cases}\\ G(\xi)^2=1-2\tau F(\xi)\end{cases}. \tag{41}$$

Family – 2:

$$\begin{cases} F(\xi)=\dfrac{1}{\tau}\dfrac{5\operatorname{sech}[\xi]}{5\operatorname{sech}[\xi]\pm 1}\\ G(\xi)=\dfrac{\tanh[\xi]}{1\pm 5\operatorname{sech}[\xi]}\\ G(\xi)^2=1-2\tau F(\xi)+\dfrac{24}{25}\tau^2 F(\xi)^2\end{cases} \tag{42}$$

Family – 3:

$$\begin{cases} F(\xi)=\dfrac{1}{r}\dfrac{3\operatorname{sech}[\xi]}{3\operatorname{sech}[\xi]\pm 2}\\ G(\xi)=\dfrac{2}{2\coth[\xi]\pm 3\operatorname{csch}[\xi]}\\ G(\xi)^2=1-2\tau F(\xi)+\dfrac{5}{9}\tau^2 F(\xi)^2\end{cases}. \tag{43}$$

Family – 4:

$$\begin{cases} \epsilon = -1 \\ F(\xi) = \begin{cases} 4\,\mathrm{sech}[\xi]3\ \tanh[\xi] + 4\tau\,\mathrm{sech}[\xi] + 5 \\ \mathrm{sech}[\xi]\tau\,\mathrm{sech}[\xi] + 1 \end{cases}, \; G(\xi) = \begin{cases} 5\ \tanh[\xi] + 33\ \tanh[\xi] + 4\tau\,\mathrm{sech}[\xi] + 5 \\ \tanh[\xi]\tau\,\mathrm{sech}[\xi] + 1 \end{cases} \\ \epsilon = 1 \\ F(\xi) = \mathrm{csch}[\xi]\tau\,\mathrm{csch}[\xi] + 1, G(\xi) = \coth[\xi]\tau\,\mathrm{csch}[\xi] + 1 \\ G(\xi)^2 = 1 - 2\tau F(\xi) + \left(\tau^2 + \epsilon\right) F(\xi)^2 \end{cases} \tag{44}$$

Here ϵ is real-valued constant parameter.

4. Optical soliton solutions

The two integration schemes from the previous section will now be implemented to the model equations given by (2) and (3) to retrieve the optical soliton solutions. A wide variety of soliton solutions will be identified that are being reported for the first time in this work. The derivation details are exhibited and enumerated in the subsequent two subsections.

4.1. Enhanced Kudryashov's method

The solution to Eq. (31) can be derived by setting U_1''' equal to U_1^5 and solving for N, which yields $N = 1$, resulting in the following solution:

$$U_1(\xi) = r_0 + r_1 R(\xi) + s_1 \left(\frac{R'(\xi)}{R(\xi)} \right). \tag{45}$$

The substitution of Eqs. (45) and (37) into Eq. (31) leads to a set of algebraic equations that need to be solved, as shown below:

$$10r_0^3 s_1^2 \vartheta_3 + 3r_0 s_1^2 \vartheta_2 + 5r_0 s_1^4 \vartheta_3 + r_0^5 \vartheta_3 + r_0^3 \vartheta_2 + r_0 \vartheta_1 = 0, \tag{46}$$

$$20r_1 r_0^3 s_1 \vartheta_3 + 6r_1 r_0 s_1 \vartheta_2 + 20r_1 r_0 s_1^3 \vartheta_3 + 2r_1 r_0 s_1 \vartheta_6 = 0, \tag{47}$$

$$10r_0^2 s_1^3 \vartheta_3 + 3r_0^2 s_1 \vartheta_2 + 5r_0^4 s_1 \vartheta_3 + s_1^5 \vartheta_3 + s_1^3 \vartheta_2 + s_1 \vartheta_1 = 0, \tag{48}$$

$$\begin{aligned} &k^2 r_1 + 30r_1 r_0^2 s_1^2 \vartheta_3 + 3r_1 s_1^2 \vartheta_2 + 5r_1 s_1^4 \vartheta_3 \\ &+ r_1 s_1^2 \vartheta_6 + 5r_1 r_0^4 \vartheta_3 + 3r_1 r_0^2 \vartheta_2 + r_1 r_0^2 \vartheta_6 + r_1 \vartheta_1 + r_1 \vartheta_5 = 0, \end{aligned} \tag{49}$$

$$\begin{aligned} &-8k^2 s_1 \chi - 10r_0^2 s_1^3 \chi \vartheta_3 - 2r_0^2 s_1 \chi \vartheta_6 + 10r_1^2 s_1^3 \vartheta_3 + 3r_1^2 s_1 \vartheta_2 + 30r_0^2 r_1^2 s_1 \vartheta_3 + r_1^2 s_1 \vartheta_4 \\ &+ 2r_1^2 s_1 \vartheta_6 - 2s_1^5 \chi \vartheta_3 - s_1^3 \chi \vartheta_2 - 2s_1^3 \chi \vartheta_6 - 2s_1 \chi \vartheta_5 = 0, \end{aligned} \tag{50}$$

$$\begin{aligned} &-10r_0 s_1^4 \chi \vartheta_3 - 3r_0 s_1^2 \chi \vartheta_2 - 10r_0^3 s_1^2 \chi \vartheta_3 - 4r_0 s_1^2 \chi \vartheta_6 + 30r_0 r_1^2 s_1^2 \vartheta_3 \\ &+ 3r_0 r_1^2 \vartheta_2 + 10r_0^3 r_1^2 \vartheta_3 + r_0 r_1^2 \vartheta_4 + 2r_0 r_1^2 \vartheta_6 = 0, \end{aligned} \tag{51}$$

$$-20r_0 r_1 s_1^3 \chi \vartheta_3 - 2r_0 r_1 s_1 \chi \vartheta_4 - 8r_0 r_1 s_1 \chi \vartheta_6 + 20r_0 r_1^3 s_1 \vartheta_3 = 0, \tag{52}$$

$$\begin{aligned} &-20k^2 r_1 \chi - 10r_1 s_1^4 \chi \vartheta_3 - 3r_1 s_1^2 \chi \vartheta_2 - 30r_0^2 r_1 s_1^2 \chi \vartheta_3 \\ &-2r_1 s_1^2 \chi \vartheta_4 - 7r_1 s_1^2 \chi \vartheta_6 + 10r_1^3 s_1^2 \vartheta_3 - 2r_1 \chi \vartheta_5 \\ &-2r_0^2 r_1 \chi \vartheta_6 + r_1^3 \vartheta_2 + 10r_0^2 r_1^3 \vartheta_3 + r_1^3 \vartheta_4 + r_1^3 \vartheta_6 = 0, \end{aligned} \tag{53}$$

$$\begin{aligned} &24k^2 s_1 \chi^2 - 10r_1^2 s_1^3 \chi \vartheta_3 - 3r_1^2 s_1 \chi \vartheta_4 - 6r_1^2 s_1 \chi \vartheta_6 \\ &+ 5r_1^4 s_1 \vartheta_3 + s_1^5 \chi^2 \vartheta_3 + s_1^3 \chi^2 \vartheta_4 + 2s_1^3 \chi^2 \vartheta_6 = 0, \end{aligned} \tag{54}$$

$$5r_0s_1^4\chi^2\vartheta_3+r_0s_1^2\chi^2\vartheta_4+4r_0s_1^2\chi^2\vartheta_6-30r_0r_1^2s_1^2\chi\vartheta_3-r_0r_1^2\chi\vartheta_4-4r_0r_1^2\chi\vartheta_6+5r_0r_1^4\vartheta_3=0, \tag{55}$$

$$24k^2r_1\chi^2+5r_1s_1^4\chi^2\vartheta_3+3r_1s_1^2\chi^2\vartheta_4+6r_1s_1^2\chi^2\vartheta_6-10r_1^3s_1^2\chi\vartheta_3-r_1^3\chi\vartheta_4-2r_1^3\chi\vartheta_6+r_1^5\vartheta_3=0. \tag{56}$$

These equations, when solved together, result in the following outcomes:

<u>Result – 1</u>:

$$\begin{gathered} r_0=s_1=0,\ r_1=\pm\sqrt{-\frac{2\chi\left(10\vartheta_1+9\vartheta_5\right)}{\vartheta_2+\vartheta_4+\vartheta_6}},\ k=\sqrt{-\left(\vartheta_1+\vartheta_5\right)},\\ \vartheta_3=\frac{\left(3\vartheta_5\left(4\vartheta_2+\vartheta_4-2\vartheta_6\right)+2\vartheta_1\left(6\vartheta_2+\vartheta_4-4\vartheta_6\right)\right)\left(\vartheta_2+\vartheta_4+\vartheta_6\right)}{2\left(10\vartheta_1+9\vartheta_5\right)^2}. \end{gathered} \tag{57}$$

Using the parameters obtained from Eqs. (57) and (38), we can plug them into Eq. (45) to obtain:

$$\begin{aligned} q(x,t)=&\frac{\pm4c\sqrt{-\frac{2\chi\left(10\vartheta_1+9\vartheta_5\right)}{\vartheta_2+\vartheta_4+\vartheta_6}}}{4c^2\exp\left[\sqrt{-\left(\vartheta_1+\vartheta_5\right)}\left(x-vt\right)\right]+\chi\exp\left[-\sqrt{-\left(\vartheta_1+\vartheta_5\right)}\left(x-vt\right)\right]}\\ &\times\exp\left[i\left(-\frac{c_{21}\sigma_{71}}{4c_{11}\sigma_{11}}x+\omega t+\theta_0\right)\right], \end{aligned} \tag{58}$$

and

$$\begin{aligned} r(x,t)=\varrho&\frac{\pm4c\sqrt{-\frac{2\chi\left(10\vartheta_1+9\vartheta_5\right)}{\vartheta_2+\vartheta_4+\vartheta_6}}}{4c^2\exp\left[\sqrt{-\left(\vartheta_1+\vartheta_5\right)}\left(x-vt\right)\right]+\chi\exp\left[-\sqrt{-\left(\vartheta_1+\vartheta_5\right)}\left(x-vt\right)\right]}\\ &\times\exp\left[i\left(-\frac{c_{22}\sigma_{72}}{4c_{12}\sigma_{12}}x+\omega t+\theta_0\right)\right]. \end{aligned} \tag{59}$$

Bright solitons can be obtained by setting χ to $\pm4c^2$ that satisfy the conditions $\vartheta_1+\vartheta_5<0$ and $\left(10\vartheta_1+9\vartheta_5\right)\left(\vartheta_2+\vartheta_4+\vartheta_6\right)<0$, as described below

$$\begin{aligned} q(x,t)=&\pm\sqrt{-\frac{2\left(10\vartheta_1+9\vartheta_5\right)}{\vartheta_2+\vartheta_4+\vartheta_6}}\operatorname{sech}\left[\sqrt{-\left(\vartheta_1+\vartheta_5\right)}\left(x-vt\right)\right]\\ &\times\exp\left[i\left(-\frac{c_{21}\sigma_{71}}{4c_{11}\sigma_{11}}x+\omega t+\theta_0\right)\right], \end{aligned} \tag{60}$$

and

$$r(x,t)=\pm\varrho\sqrt{-\frac{2\left(10\vartheta_1+9\vartheta_5\right)}{\vartheta_2+\vartheta_4+\vartheta_6}}\operatorname{sech}\left[\sqrt{-\left(\vartheta_1+\vartheta_5\right)}\left(x-vt\right)\right]\exp\left[i\left(-\frac{c_{22}\sigma_{72}}{4c_{12}\sigma_{12}}x+\omega t+\theta_0\right)\right]. \tag{61}$$

The conditions $\vartheta_1+\vartheta_5<0$ and $\left(10\vartheta_1+9\vartheta_5\right)\left(\vartheta_2+\vartheta_4+\vartheta_6\right)>0$ lead to the existence of singular solitons:

$$q(x,t)=\pm\sqrt{\frac{2\left(10\vartheta_1+9\vartheta_5\right)}{\vartheta_2+\vartheta_4+\vartheta_6}}\operatorname{csch}\left[\sqrt{-\left(\vartheta_1+\vartheta_5\right)}\left(x-vt\right)\right]\exp\left[i\left(-\frac{c_{21}\sigma_{71}}{4c_{11}\sigma_{11}}x+\omega t+\theta_0\right)\right], \tag{62}$$

and

$$r(x,t)=\pm\varrho\sqrt{\frac{2\left(10\vartheta_1+9\vartheta_5\right)}{\vartheta_2+\vartheta_4+\vartheta_6}}\operatorname{csch}\left[\sqrt{-\left(\vartheta_1+\vartheta_5\right)}\left(x-vt\right)\right]\exp\left[i\left(-\frac{c_{22}\sigma_{72}}{4c_{12}\sigma_{12}}x+\omega t+\theta_0\right)\right]. \tag{63}$$

Result – 2:

$$r_0 = r_1 = 0,\ s_1 = \pm\sqrt{\frac{6\vartheta_5 - 5\vartheta_1}{2\vartheta_2 + \vartheta_4 - 4\vartheta_6}},\ k = \frac{1}{2}\sqrt{\frac{2\vartheta_5\left(\vartheta_2 - \vartheta_4 - 2\vartheta_6\right) + \vartheta_1\left(2\left(\vartheta_4 + \vartheta_6\right) - \vartheta_2\right)}{2\left(2\vartheta_2 + \vartheta_4 - 4\vartheta_6\right)}},$$
$$\vartheta_3 = \frac{\left(2\vartheta_2 + \vartheta_4 - 4\vartheta_6\right)\left(\vartheta_1\left(3\vartheta_2 - \vartheta_4 + 4\vartheta_6\right) - 6\vartheta_2\vartheta_5\right)}{\left(5\vartheta_1 - 6\vartheta_5\right)^2}. \tag{64}$$

Substituting the parameters derived from Eqs. (64) and (38) into Eq. (45) results in:

$$q(x,t) = \pm\sqrt{\frac{6\vartheta_5 - 5\vartheta_1}{2\vartheta_2 + \vartheta_4 - 4\vartheta_6}}\exp\left[i\left(-\frac{c_{21}\sigma_{71}}{4c_{11}\sigma_{11}}x + \omega t + \theta_0\right)\right]$$
$$\times\left(\frac{2\chi}{4c^2\exp\left[\sqrt{\frac{2\vartheta_5\left(\vartheta_2 - \vartheta_4 - 2\vartheta_6\right) + \vartheta_1\left(2\left(\vartheta_4 + \vartheta_6\right) - \vartheta_2\right)}{2\left(2\vartheta_2 + \vartheta_4 - 4\vartheta_6\right)}}\left(x - vt\right)\right] + \chi} - 1\right), \tag{65}$$

and

$$r(x,t) = \pm\varrho\sqrt{\frac{6\vartheta_5 - 5\vartheta_1}{2\vartheta_2 + \vartheta_4 - 4\vartheta_6}}\exp\left[i\left(-\frac{c_{22}\sigma_{72}}{4c_{12}\sigma_{12}}x + \omega t + \theta_0\right)\right]$$
$$\times\left(\frac{2\chi}{4c^2\exp\left[\sqrt{\frac{2\vartheta_5\left(\vartheta_2 - \vartheta_4 - 2\vartheta_6\right) + \vartheta_1\left(2\left(\vartheta_4 + \vartheta_6\right) - \vartheta_2\right)}{2\left(2\vartheta_2 + \vartheta_4 - 4\vartheta_6\right)}}\left(x - vt\right)\right] + \chi} - 1\right). \tag{66}$$

Dark and singular solitons can be found by plugging $\chi = \pm 4c^2$ into the equations, as indicated below:

$$q(x,t) = \pm\sqrt{\frac{6\vartheta_5 - 5\vartheta_1}{2\vartheta_2 + \vartheta_4 - 4\vartheta_6}}\times\exp\left[i\left(-\frac{c_{21}\sigma_{71}}{4c_{11}\sigma_{11}}x + \omega t + \theta_0\right)\right]$$
$$\times\tanh\left[\frac{1}{2}\sqrt{\frac{2\vartheta_5\left(\vartheta_2 - \vartheta_4 - 2\vartheta_6\right) + \vartheta_1\left(2\left(\vartheta_4 + \vartheta_6\right) - \vartheta_2\right)}{2\left(2\vartheta_2 + \vartheta_4 - 4\vartheta_6\right)}}\left(x - vt\right)\right], \tag{67}$$

$$r(x,t) = \pm\varrho\sqrt{\frac{6\vartheta_5 - 5\vartheta_1}{2\vartheta_2 + \vartheta_4 - 4\vartheta_6}}\exp\left[i\left(-\frac{c_{22}\sigma_{72}}{4c_{12}\sigma_{12}}x + \omega t + \theta_0\right)\right]$$
$$\times\tanh\left[\frac{1}{2}\sqrt{\frac{2\vartheta_5\left(\vartheta_2 - \vartheta_4 - 2\vartheta_6\right) + \vartheta_1\left(2\left(\vartheta_4 + \vartheta_6\right) - \vartheta_2\right)}{2\left(2\vartheta_2 + \vartheta_4 - 4\vartheta_6\right)}}\left(x - vt\right)\right], \tag{68}$$

$$q(x,t) = \pm\sqrt{\frac{6\vartheta_5 - 5\vartheta_1}{2\vartheta_2 + \vartheta_4 - 4\vartheta_6}}\exp\left[i\left(-\frac{c_{21}\sigma_{71}}{4c_{11}\sigma_{11}}x + \omega t + \theta_0\right)\right]$$
$$\times\coth\left[\frac{1}{2}\sqrt{\frac{2\vartheta_5\left(\vartheta_2 - \vartheta_4 - 2\vartheta_6\right) + \vartheta_1\left(2\left(\vartheta_4 + \vartheta_6\right) - \vartheta_2\right)}{2\left(2\vartheta_2 + \vartheta_4 - 4\vartheta_6\right)}}\left(x - vt\right)\right], \tag{69}$$

and

$$r(x,t)=\pm\varrho\sqrt{\frac{6\vartheta_5-5\vartheta_1}{2\vartheta_2+\vartheta_4-4\vartheta_6}}\exp\left[i\left(-\frac{c_{22}\sigma_{72}}{4c_{12}\sigma_{12}}x+\omega t+\theta_0\right)\right]$$
$$\times\coth\left[\frac{1}{2}\sqrt{\frac{2\vartheta_5\left(\vartheta_2-\vartheta_4-2\vartheta_6\right)+\vartheta_1\left(2\left(\vartheta_4+\vartheta_6\right)-\vartheta_2\right)}{2\left(2\vartheta_2+\vartheta_4-4\vartheta_6\right)}}\left(x-vt\right)\right]. \tag{70}$$

The conditions for the validity of these solitons are as follows: $2\vartheta_5\left(\vartheta_2-\vartheta_4-2\vartheta_6\right)+\vartheta_1\left(2\left(\vartheta_4+\vartheta_6\right)-\vartheta_2\right)>0$, $2\left(2\vartheta_2+\vartheta_4-4\vartheta_6\right)>0$, and $6\vartheta_5-5\vartheta_1>0$ must hold.

Result – 3:

$$\begin{aligned}
&r_1=\pm\sqrt{\frac{\chi\left(20\vartheta_1-6\vartheta_5\right)}{8\vartheta_2+\vartheta_4-4\vartheta_6}},\quad s_1=\pm\sqrt{\frac{6\vartheta_5-20\vartheta_1}{8\vartheta_2+\vartheta_4-4\vartheta_6}},\\
&k=\sqrt{\frac{4\vartheta_1\left(-2\vartheta_2+\vartheta_4+\vartheta_6\right)+\vartheta_5\left(4\vartheta_2-\vartheta_4-2\vartheta_6\right)}{8\vartheta_2+\vartheta_4-4\vartheta_6}},\\
&r_0=0,\\
&\vartheta_3=\frac{\left(8\vartheta_2+\vartheta_4-4\vartheta_6\right)\left(\vartheta_1\left(12\vartheta_2-\vartheta_4+4\vartheta_6\right)-6\vartheta_2\vartheta_5\right)}{4\left(10\vartheta_1-3\vartheta_5\right)^2}.
\end{aligned} \tag{71}$$

Upon substitution of the parameters obtained from Eqs. (71) and (38) into Eq. (45), we arrive at:

$$q(x,t)=\left\{\begin{aligned}
&\pm4c\sqrt{\frac{\chi\left(20\vartheta_1-6\vartheta_5\right)}{8\vartheta_2+\vartheta_4-4\vartheta_6}}\frac{\exp\left[\sqrt{\dfrac{\left(\begin{aligned}&4\vartheta_1\left(-2\vartheta_2+\vartheta_4+\vartheta_6\right)\\&+\vartheta_5\left(4\vartheta_2-\vartheta_4-2\vartheta_6\right)\end{aligned}\right)}{8\vartheta_2+\vartheta_4-4\vartheta_6}}(x-vt)\right]}{4c^2\exp\left[2\sqrt{\dfrac{\left(\begin{aligned}&4\vartheta_1\left(-2\vartheta_2+\vartheta_4+\vartheta_6\right)\\&+\vartheta_5\left(4\vartheta_2-\vartheta_4-2\vartheta_6\right)\end{aligned}\right)}{8\vartheta_2+\vartheta_4-4\vartheta_6}}(x-vt)\right]+\chi}\\
&\pm\sqrt{\frac{6\vartheta_5-20\vartheta_1}{8\vartheta_2+\vartheta_4-4\vartheta_6}}\left(\frac{2\chi}{4c^2\exp\left[2\sqrt{\dfrac{\left(\begin{aligned}&4\vartheta_1\left(-2\vartheta_2+\vartheta_4+\vartheta_6\right)\\&+\vartheta_5\left(4\vartheta_2-\vartheta_4-2\vartheta_6\right)\end{aligned}\right)}{8\vartheta_2+\vartheta_4-4\vartheta_6}}(x-vt)\right]+\chi}-1\right)
\end{aligned}\right\},$$
$$\times\exp\left[i\left(-\frac{c_{21}\sigma_{71}}{4c_{11}\sigma_{11}}x+\omega t+\theta_0\right)\right] \tag{72}$$

and

$$r(x,t)=\varrho\left\{\frac{\pm 4c\sqrt{\dfrac{\chi\left(20\vartheta_1-6\vartheta_5\right)}{8\vartheta_2+\vartheta_4-4\vartheta_6}}\exp\left[\sqrt{\dfrac{\begin{pmatrix}4\vartheta_1\left(-2\vartheta_2+\vartheta_4+\vartheta_6\right)\\+\vartheta_5\left(4\vartheta_2-\vartheta_4-2\vartheta_6\right)\end{pmatrix}}{8\vartheta_2+\vartheta_4-4\vartheta_6}}(x-vt)\right]}{4c^2\exp\left[2\sqrt{\dfrac{\begin{pmatrix}4\vartheta_1\left(-2\vartheta_2+\vartheta_4+\vartheta_6\right)\\+\vartheta_5\left(4\vartheta_2-\vartheta_4-2\vartheta_6\right)\end{pmatrix}}{8\vartheta_2+\vartheta_4-4\vartheta_6}}(x-vt)\right]+\chi}\pm\sqrt{\frac{6\vartheta_5-20\vartheta_1}{8\vartheta_2+\vartheta_4-4\vartheta_6}}\left(\frac{2\chi}{4c^2\exp\left[2\sqrt{\dfrac{\begin{pmatrix}4\vartheta_1\left(-2\vartheta_2+\vartheta_4+\vartheta_6\right)\\+\vartheta_5\left(4\vartheta_2-\vartheta_4-2\vartheta_6\right)\end{pmatrix}}{8\vartheta_2+\vartheta_4-4\vartheta_6}}(x-vt)\right]+\chi}-1\right)\right\}$$
$$\times\exp\left[i\left(-\frac{c_{22}\sigma_{72}}{4c_{12}\sigma_{12}}x+\omega t+\theta_0\right)\right]. \tag{73}$$

By setting χ to $\pm 4c^2$, complexiton solutions can be obtained that admit the conditions $6\vartheta_5-20\vartheta_1>0$, $8\vartheta_2+\vartheta_4-4\vartheta_6>0$, and $4\vartheta_1\left(-2\vartheta_2+\vartheta_4+\vartheta_6\right)+\vartheta_5\left(4\vartheta_2-\vartheta_4-2\vartheta_6\right)>0$, as described below

$$q(x,t)=\pm\sqrt{\frac{6\vartheta_5-20\vartheta_1}{8\vartheta_2+\vartheta_4-4\vartheta_6}}\times\left\{\begin{array}{l}\tanh\left[\sqrt{\dfrac{4\vartheta_1\left(-2\vartheta_2+\vartheta_4+\vartheta_6\right)+\vartheta_5\left(4\vartheta_2-\vartheta_4-2\vartheta_6\right)}{8\vartheta_2+\vartheta_4-4\vartheta_6}}(x-vt)\right]\\ \pm i\,\mathrm{sech}\left[\sqrt{\dfrac{4\vartheta_1\left(-2\vartheta_2+\vartheta_4+\vartheta_6\right)+\vartheta_5\left(4\vartheta_2-\vartheta_4-2\vartheta_6\right)}{8\vartheta_2+\vartheta_4-4\vartheta_6}}(x-vt)\right]\end{array}\right\}\times e^{i\left(-\frac{c_{21}\sigma_{71}}{4c_{11}\sigma_{11}}x+\omega t+\theta_0\right)}, \tag{74}$$

and

$$r(x,t)=\pm\varrho\sqrt{\frac{6\vartheta_5-20\vartheta_1}{8\vartheta_2+\vartheta_4-4\vartheta_6}}\times\left\{\begin{array}{l}\tanh\left[\sqrt{\dfrac{4\vartheta_1\left(-2\vartheta_2+\vartheta_4+\vartheta_6\right)+\vartheta_5\left(4\vartheta_2-\vartheta_4-2\vartheta_6\right)}{8\vartheta_2+\vartheta_4-4\vartheta_6}}(x-vt)\right]\\ \pm i\,\mathrm{sech}\left[\sqrt{\dfrac{4\vartheta_1\left(-2\vartheta_2+\vartheta_4+\vartheta_6\right)+\vartheta_5\left(4\vartheta_2-\vartheta_4-2\vartheta_6\right)}{8\vartheta_2+\vartheta_4-4\vartheta_6}}(x-vt)\right]\end{array}\right\}e^{i\left(-\frac{c_{22}\sigma_{72}}{4c_{12}\sigma_{12}}x+\omega t+\theta_0\right)}. \tag{75}$$

Additionally, such solutions can be reduced to dark and singular solitons:

$$q(x,t)=\pm\sqrt{\frac{6\vartheta_5-20\vartheta_1}{8\vartheta_2+\vartheta_4-4\vartheta_6}}\tanh\left[\frac{1}{2}\sqrt{\frac{\begin{pmatrix}4\vartheta_1\left(-2\vartheta_2+\vartheta_4+\vartheta_6\right)\\+\vartheta_5\left(4\vartheta_2-\vartheta_4-2\vartheta_6\right)\end{pmatrix}}{8\vartheta_2+\vartheta_4-4\vartheta_6}}(x-vt)\right]$$
$$\times\exp\left[i\left(-\frac{c_{21}\sigma_{71}}{4c_{11}\sigma_{11}}x+\omega t+\theta_0\right)\right], \tag{76}$$

$$r(x,t)=\pm\varrho\sqrt{\frac{6\vartheta_5-20\vartheta_1}{8\vartheta_2+\vartheta_4-4\vartheta_6}}\tanh\left[\frac{1}{2}\sqrt{\frac{\begin{pmatrix}4\vartheta_1\left(-2\vartheta_2+\vartheta_4+\vartheta_6\right)\\+\vartheta_5\left(4\vartheta_2-\vartheta_4-2\vartheta_6\right)\end{pmatrix}}{8\vartheta_2+\vartheta_4-4\vartheta_6}}(x-vt)\right]$$
$$\times\exp\left[i\left(-\frac{c_{22}\sigma_{72}}{4c_{12}\sigma_{12}}x+\omega t+\theta_0\right)\right], \tag{77}$$

$$q(x,t)=\pm\sqrt{\frac{6\vartheta_5-20\vartheta_1}{8\vartheta_2+\vartheta_4-4\vartheta_6}}\coth\left[\frac{1}{2}\sqrt{\frac{\begin{pmatrix}4\vartheta_1\left(-2\vartheta_2+\vartheta_4+\vartheta_6\right)\\+\vartheta_5\left(4\vartheta_2-\vartheta_4-2\vartheta_6\right)\end{pmatrix}}{8\vartheta_2+\vartheta_4-4\vartheta_6}}(x-vt)\right]$$
$$\times\exp\left[i\left(-\frac{c_{21}\sigma_{71}}{4c_{11}\sigma_{11}}x+\omega t+\theta_0\right)\right], \tag{78}$$

and

$$r(x,t)=\pm\varrho\sqrt{\frac{6\vartheta_5-20\vartheta_1}{8\vartheta_2+\vartheta_4-4\vartheta_6}}\coth\left[\frac{1}{2}\sqrt{\frac{\begin{pmatrix}4\vartheta_1\left(-2\vartheta_2+\vartheta_4+\vartheta_6\right)\\+\vartheta_5\left(4\vartheta_2-\vartheta_4-2\vartheta_6\right)\end{pmatrix}}{8\vartheta_2+\vartheta_4-4\vartheta_6}}(x-vt)\right]$$
$$\times\exp\left[i\left(-\frac{c_{22}\sigma_{72}}{4c_{12}\sigma_{12}}x+\omega t+\theta_0\right)\right]. \tag{79}$$

4.2. Projective Riccati equation method

We can obtain the solution by balancing U_1'''' and U_1^5 in Eq. (31), which leads to $N=1$, resulting in the form

$$U_1(\xi)=\sigma_0+\sigma_1F(\xi)+\varrho_1G(\xi). \tag{80}$$

Substituting the obtained values of Eqs. (80) and (40) as well as $G(\xi)^2=1-2\tau F(\xi)+R(\tau)F(\xi)^2$ into Eq. (31), we derive a system of algebraic equations:

$$\begin{aligned}&24k^2\sigma_1R(\tau)^2+\sigma_1^3\vartheta_4R(\tau)+2\sigma_1^3\vartheta_6R(\tau)\\&+10\sigma_1^3\vartheta_3\varrho_1^2R(\tau)+5\sigma_1\vartheta_3\varrho_1^4R(\tau)^2+3\sigma_1\vartheta_4\varrho_1^2R(\tau)^2\\&+6\sigma_1\vartheta_6\varrho_1^2R(\tau)^2+\sigma_1^5\vartheta_3=0,\end{aligned} \tag{81}$$

$$\begin{aligned}&24k^2\varrho_1 R(\tau)^2+10\sigma_1^2\vartheta_3\varrho_1^3 R(\tau)+3\sigma_1^2\vartheta_4\varrho_1 R(\tau)+6\sigma_1^2\vartheta_6\varrho_1 R(\tau)\\&+\vartheta_3\varrho_1^5 R(\tau)^2+\vartheta_4\varrho_1^3 R(\tau)^2+2\vartheta_6\varrho_1^3 R(\tau)^2+5\sigma_1^4\vartheta_3\varrho_1=0,\end{aligned}\tag{82}$$

$$\begin{aligned}&-60k^2\tau\sigma_1 R(\tau)+\sigma_0\sigma_1^2\vartheta_4 R(\tau)+4\sigma_0\sigma_1^2\vartheta_6 R(\tau)+5\sigma_0\vartheta_3\varrho_1^4 R(\tau)^2\\&-20\tau\sigma_1\vartheta_3\varrho_1^4 R(\tau)+30\sigma_0\sigma_1^2\vartheta_3\varrho_1^2 R(\tau)+\sigma_0\vartheta_4\varrho_1^2 R(\tau)^2\\&+4\sigma_0\vartheta_6\varrho_1^2 R(\tau)^2-8\tau\sigma_1\vartheta_4\varrho_1^2 R(\tau)-17\tau\sigma_1\vartheta_6\varrho_1^2 R(\tau)-2\tau\sigma_1^3\vartheta_4\\&-3\tau\sigma_1^3\vartheta_6-20\tau\sigma_1^3\vartheta_3\varrho_1^2+5\sigma_0\sigma_1^4\vartheta_3=0,\end{aligned}\tag{83}$$

$$\begin{aligned}&-36k^2\tau\varrho_1 R(\tau)+20\sigma_0\sigma_1\vartheta_3\varrho_1^3 R(\tau)+2\sigma_0\sigma_1\vartheta_4\varrho_1 R(\tau)+8\sigma_0\sigma_1\vartheta_6\varrho_1 R(\tau)\\&-4\tau\vartheta_3\varrho_1^5 R(\tau)-2\tau\vartheta_4\varrho_1^3 R(\tau)-5\tau\vartheta_6\varrho_1^3 R(\tau)-20\tau\sigma_1^2\vartheta_3\varrho_1^3-4\tau\sigma_1^2\vartheta_4\varrho_1\\&-7\tau\sigma_1^2\vartheta_6\varrho_1+20\sigma_0\sigma_1^3\vartheta_3\varrho_1=0,\end{aligned}\tag{84}$$

$$\begin{aligned}&30k^2\tau^2\sigma_1+20k^2\sigma_1 R(\tau)+20\tau^2\sigma_1\vartheta_3\varrho_1^4+5\tau^2\sigma_1\vartheta_4\varrho_1^2\\&+10\tau^2\sigma_1\vartheta_6\varrho_1^2+2\sigma_0^2\sigma_1\vartheta_6 R(\tau)+2\sigma_1\vartheta_5 R(\tau)-20\tau\sigma_0\vartheta_3\varrho_1^4 R(\tau)\\&+10\sigma_1\vartheta_3\varrho_1^4 R(\tau)-2\tau\sigma_0\vartheta_4\varrho_1^2 R(\tau)-10\tau\sigma_0\vartheta_6\varrho_1^2 R(\tau)\\&+30\sigma_0^2\sigma_1\vartheta_3\varrho_1^2 R(\tau)+3\sigma_1\vartheta_2\varrho_1^2 R(\tau)+2\sigma_1\vartheta_4\varrho_1^2 R(\tau)\\&+7\sigma_1\vartheta_6\varrho_1^2 R(\tau)-2\tau\sigma_0\sigma_1^2\vartheta_4-6\tau\sigma_0\sigma_1^2\vartheta_6-60\tau\sigma_0\sigma_1^2\vartheta_3\varrho_1^2\\&+10\sigma_0^2\sigma_1^3\vartheta_3+\sigma_1^3\vartheta_2+\sigma_1^3\vartheta_4+\sigma_1^3\vartheta_6+10\sigma_1^3\vartheta_3\varrho_1^2=0,\end{aligned}\tag{85}$$

$$\begin{aligned}&6k^2\tau^2\varrho_1+8k^2\varrho_1 R(\tau)+4\tau^2\vartheta_3\varrho_1^5+\tau^2\vartheta_4\varrho_1^3+2\tau^2\vartheta_6\varrho_1^3\\&+10\sigma_0^2\vartheta_3\varrho_1^3 R(\tau)+2\sigma_0^2\vartheta_6\varrho_1 R(\tau)+2\vartheta_3\varrho_1^5 R(\tau)+\vartheta_2\varrho_1^3 R(\tau)\\&+2\vartheta_6\varrho_1^3 R(\tau)+2\vartheta_5\varrho_1 R(\tau)-40\tau\sigma_0\sigma_1\vartheta_3\varrho_1^3-2\tau\sigma_0\sigma_1\vartheta_4\varrho_1\\&-8\tau\sigma_0\sigma_1\vartheta_6\varrho_1+10\sigma_1^2\vartheta_3\varrho_1^3+30\sigma_0^2\sigma_1^2\vartheta_3\varrho_1+3\sigma_1^2\vartheta_2\varrho_1\\&+\sigma_1^2\vartheta_4\varrho_1+2\sigma_1^2\vartheta_6\varrho_1=0,\end{aligned}\tag{86}$$

$$\begin{aligned}&-15k^2\tau\sigma_1+20\tau^2\sigma_0\vartheta_3\varrho_1^4+\tau^2\sigma_0\vartheta_4\varrho_1^2+4\tau^2\sigma_0\vartheta_6\varrho_1^2\\&+10\sigma_0\vartheta_3\varrho_1^4 R(\tau)+10\sigma_0^3\vartheta_3\varrho_1^2 R(\tau)+3\sigma_0\vartheta_2\varrho_1^2 R(\tau)\\&+4\sigma_0\vartheta_6\varrho_1^2 R(\tau)-3\tau\sigma_0^2\sigma_1\vartheta_6-3\tau\sigma_1\vartheta_5-20\tau\sigma_1\vartheta_3\varrho_1^4\\&-60\tau\sigma_0^2\sigma_1\vartheta_3\varrho_1^2-6\tau\sigma_1\vartheta_2\varrho_1^2-2\tau\sigma_1\vartheta_4\varrho_1^2-7\tau\sigma_1\vartheta_6\varrho_1^2\\&+10\sigma_0^3\sigma_1^2\vartheta_3+3\sigma_0\sigma_1^2\vartheta_2+\sigma_0\sigma_1^2\vartheta_4+2\sigma_0\sigma_1^2\vartheta_6+30\sigma_0\sigma_1^2\vartheta_3\varrho_1^2=0,\end{aligned}\tag{87}$$

$$\begin{aligned}&-k^2\tau\varrho_1-20\tau\sigma_0^2\vartheta_3\varrho_1^3-\tau\sigma_0^2\vartheta_6\varrho_1-4\tau\vartheta_3\varrho_1^5\\&-2\tau\vartheta_2\varrho_1^3-\tau\vartheta_6\varrho_1^3-\tau\vartheta_5\varrho_1+20\sigma_0\sigma_1\vartheta_3\varrho_1^3\\&+20\sigma_0^3\sigma_1\vartheta_3\varrho_1+6\sigma_0\sigma_1\vartheta_2\varrho_1+2\sigma_0\sigma_1\vartheta_6\varrho_1=0,\end{aligned}\tag{88}$$

$$\begin{aligned}&k^2\sigma_1-20\tau\sigma_0\vartheta_3\varrho_1^4-20\tau\sigma_0^3\vartheta_3\varrho_1^2-6\tau\sigma_0\vartheta_2\varrho_1^2-2\tau\sigma_0\vartheta_6\varrho_1^2\\&+5\sigma_0^4\sigma_1\vartheta_3+3\sigma_0^2\sigma_1\vartheta_2+\sigma_0^2\sigma_1\vartheta_6\\&+\sigma_1\vartheta_1+\sigma_1\vartheta_5+5\sigma_1\vartheta_3\varrho_1^4+30\sigma_0^2\sigma_1\vartheta_3\varrho_1^2+3\sigma_1\vartheta_2\varrho_1^2+\sigma_1\vartheta_6\varrho_1^2=0,\end{aligned}\tag{89}$$

$$10\sigma_0^2\vartheta_3\varrho_1^3+5\sigma_0^4\vartheta_3\varrho_1+3\sigma_0^2\vartheta_2\varrho_1+\vartheta_3\varrho_1^5+\vartheta_2\varrho_1^3+\vartheta_1\varrho_1=0,\tag{90}$$

$$\sigma_0^5\vartheta_3+\sigma_0^3\vartheta_2+\sigma_0\vartheta_1+10\sigma_0^3\vartheta_3\varrho_1^2+5\sigma_0\vartheta_3\varrho_1^4+3\sigma_0\vartheta_2\varrho_1^2=0.\tag{91}$$

The solution to these equations provides us with the following findings:

Family – 1: $R(\tau)=0$

$$\varrho_1=\pm\sqrt{\frac{6\vartheta_5-20\vartheta_1}{8\vartheta_2+\vartheta_4-4\vartheta_6}},\ k=\sqrt{\frac{4\vartheta_1(-2\vartheta_2+\vartheta_4+\vartheta_6)+\vartheta_5(4\vartheta_2-\vartheta_4-2\vartheta_6)}{8\vartheta_2+\vartheta_4-4\vartheta_6}},$$
$$\sigma_0=\sigma_1=0,\ \vartheta_3=\frac{(8\vartheta_2+\vartheta_4-4\vartheta_6)(\vartheta_1(12\vartheta_2-\vartheta_4+4\vartheta_6)-6\vartheta_2\vartheta_5)}{4(10\vartheta_1-3\vartheta_5)^2}. \quad (92)$$

We can obtain dark and singular solitons by substituting the parameters derived from Eqs. (92) and (41) into Eq. (80), as indicated below:

$$q(x,t)=\pm\sqrt{\frac{6\vartheta_5-20\vartheta_1}{8\vartheta_2+\vartheta_4-4\vartheta_6}}\tanh\left[\frac{1}{2}\sqrt{\frac{\begin{pmatrix}4\vartheta_1(-2\vartheta_2+\vartheta_4+\vartheta_6)\\+\vartheta_5(4\vartheta_2-\vartheta_4-2\vartheta_6)\end{pmatrix}}{8\vartheta_2+\vartheta_4-4\vartheta_6}}(x-vt)\right]e^{i\left(-\frac{c_{21}\sigma_{71}}{4c_{11}\sigma_{11}}x+\omega t+\theta_0\right)}, \quad (93)$$

$$r(x,t)=\pm\varrho\sqrt{\frac{6\vartheta_5-20\vartheta_1}{8\vartheta_2+\vartheta_4-4\vartheta_6}}\tanh\left[\frac{1}{2}\sqrt{\frac{\begin{pmatrix}4\vartheta_1(-2\vartheta_2+\vartheta_4+\vartheta_6)\\+\vartheta_5(4\vartheta_2-\vartheta_4-2\vartheta_6)\end{pmatrix}}{8\vartheta_2+\vartheta_4-4\vartheta_6}}(x-vt)\right]e^{i\left(-\frac{c_{22}\sigma_{72}}{4c_{12}\sigma_{12}}x+\omega t+\theta_0\right)}, \quad (94)$$

$$q(x,t)=\pm\sqrt{\frac{6\vartheta_5-20\vartheta_1}{8\vartheta_2+\vartheta_4-4\vartheta_6}}\coth\left[\frac{1}{2}\sqrt{\frac{\begin{pmatrix}4\vartheta_1(-2\vartheta_2+\vartheta_4+\vartheta_6)\\+\vartheta_5(4\vartheta_2-\vartheta_4-2\vartheta_6)\end{pmatrix}}{8\vartheta_2+\vartheta_4-4\vartheta_6}}(x-vt)\right]e^{i\left(-\frac{c_{21}\sigma_{71}}{4c_{11}\sigma_{11}}x+\omega t+\theta_0\right)}, \quad (95)$$

and

$$r(x,t)=\pm\varrho\sqrt{\frac{6\vartheta_5-20\vartheta_1}{8\vartheta_2+\vartheta_4-4\vartheta_6}}\coth\left[\frac{1}{2}\sqrt{\frac{\begin{pmatrix}4\vartheta_1(-2\vartheta_2+\vartheta_4+\vartheta_6)\\+\vartheta_5(4\vartheta_2-\vartheta_4-2\vartheta_6)\end{pmatrix}}{8\vartheta_2+\vartheta_4-4\vartheta_6}}(x-vt)\right]e^{i\left(-\frac{c_{22}\sigma_{72}}{4c_{12}\sigma_{12}}x+\omega t+\theta_0\right)}. \quad (96)$$

The resulting solitons satisfy the conditions $6\vartheta_5-20\vartheta_1>0$, $8\vartheta_2+\vartheta_4-4\vartheta_6>0$, and $4\vartheta_1(-2\vartheta_2+\vartheta_4+\vartheta_6)+\vartheta_5(4\vartheta_2-\vartheta_4-2\vartheta_6)>0$.

Family – 2: $R(\tau)=\frac{24}{25}\tau^2$

<u>**Result – 1**</u>:

$$\sigma_0=0,\ \sigma_1=\frac{4\tau}{5}\sqrt{\frac{9\vartheta_5-30\vartheta_1}{8\vartheta_2+\vartheta_4-4\vartheta_6}},\ \varrho_1=\sqrt{\frac{6\vartheta_5-20\vartheta_1}{8\vartheta_2+\vartheta_4-4\vartheta_6}},$$
$$k=\sqrt{\frac{4\vartheta_1(-2\vartheta_2+\vartheta_4+\vartheta_6)+\vartheta_5(4\vartheta_2-\vartheta_4-2\vartheta_6)}{8\vartheta_2+\vartheta_4-4\vartheta_6}}, \quad (97)$$
$$\vartheta_3=\frac{(8\vartheta_2+\vartheta_4-4\vartheta_6)(\vartheta_1(12\vartheta_2-\vartheta_4+4\vartheta_6)-6\vartheta_2\vartheta_5)}{4(10\vartheta_1-3\vartheta_5)^2}.$$

Upon plugging the parameters obtained from Eqs. (97) and (42) into Eq. (80), we obtain straddled singular–singular solitons that satisfy the conditions $6\vartheta_5 - 20\vartheta_1 > 0$, $8\vartheta_2 + \vartheta_4 - 4\vartheta_6 > 0$, and $4\vartheta_1\left(-2\vartheta_2 + \vartheta_4 + \vartheta_6\right) + \vartheta_5\left(4\vartheta_2 - \vartheta_4 - 2\vartheta_6\right) > 0$, as described below

$$q(x,t) = \sqrt{\frac{6\vartheta_5 - 20\vartheta_1}{8\vartheta_2 + \vartheta_4 - 4\vartheta_6}} \times \frac{1 \pm 2\sqrt{6}\,\mathrm{csch}\left[\sqrt{\frac{4\vartheta_1\left(-2\vartheta_2 + \vartheta_4 + \vartheta_6\right) + \vartheta_5\left(4\vartheta_2 - \vartheta_4 - 2\vartheta_6\right)}{8\vartheta_2 + \vartheta_4 - 4\vartheta_6}}(x - vt)\right]}{\left(\begin{array}{l} \coth\left[\sqrt{\frac{4\vartheta_1\left(-2\vartheta_2 + \vartheta_4 + \vartheta_6\right) + \vartheta_5\left(4\vartheta_2 - \vartheta_4 - 2\vartheta_6\right)}{8\vartheta_2 + \vartheta_4 - 4\vartheta_6}}(x - vt)\right] \\ \pm 5\,\mathrm{csch}\left[\sqrt{\frac{4\vartheta_1\left(-2\vartheta_2 + \vartheta_4 + \vartheta_6\right) + \vartheta_5\left(4\vartheta_2 - \vartheta_4 - 2\vartheta_6\right)}{8\vartheta_2 + \vartheta_4 - 4\vartheta_6}}(x - vt)\right] \end{array}\right)} \times e^{i\left(-\frac{c_{21}\sigma_{71}}{4c_{11}\sigma_{11}}x + \omega t + \theta_0\right)}, \tag{98}$$

and

$$r(x,t) = \varrho\sqrt{\frac{6\vartheta_5 - 20\vartheta_1}{8\vartheta_2 + \vartheta_4 - 4\vartheta_6}} \times \frac{1 \pm 2\sqrt{6}\,\mathrm{csch}\left[\sqrt{\frac{4\vartheta_1\left(-2\vartheta_2 + \vartheta_4 + \vartheta_6\right) + \vartheta_5\left(4\vartheta_2 - \vartheta_4 - 2\vartheta_6\right)}{8\vartheta_2 + \vartheta_4 - 4\vartheta_6}}(x - vt)\right]}{\left(\begin{array}{l} \coth\left[\sqrt{\frac{4\vartheta_1\left(-2\vartheta_2 + \vartheta_4 + \vartheta_6\right) + \vartheta_5\left(4\vartheta_2 - \vartheta_4 - 2\vartheta_6\right)}{8\vartheta_2 + \vartheta_4 - 4\vartheta_6}}(x - vt)\right] \\ \pm 5\,\mathrm{csch}\left[\sqrt{\frac{4\vartheta_1\left(-2\vartheta_2 + \vartheta_4 + \vartheta_6\right) + \vartheta_5\left(4\vartheta_2 - \vartheta_4 - 2\vartheta_6\right)}{8\vartheta_2 + \vartheta_4 - 4\vartheta_6}}(x - vt)\right] \end{array}\right)} \times e^{i\left(-\frac{c_{22}\sigma_{72}}{4c_{12}\sigma_{12}}x + \omega t + \theta_0\right)}. \tag{99}$$

Result – 2:

$$\sigma_0 = \varrho_1 = 0,\ \sigma_1 = \frac{12\sqrt{2}\tau}{5}\sqrt{\frac{\vartheta_5}{2\vartheta_4 + 3\vartheta_6}},\ k = \sqrt{-\frac{\vartheta_5}{5}},$$
$$\vartheta_3 = -\frac{2\vartheta_4^2 + 11\vartheta_6\vartheta_4 + 12\vartheta_6^2}{60\vartheta_5},\ \vartheta_2 = \frac{1}{16}\left(6\vartheta_4 + 17\vartheta_6\right),\ \vartheta_1 = -\frac{4\vartheta_5}{5}. \tag{100}$$

Straddled bright–bright solitons can be obtained from the parameters found in Eqs. (100) and (42) by inserting them into Eq. (80), which results in $\vartheta_5 < 0$ and $2\vartheta_4 + 3\vartheta_6 < 0$, as shown below:

$$q\left(x,t\right) = \left\{\frac{12\sqrt{2}\sqrt{\frac{\vartheta_5}{2\vartheta_4 + 3\vartheta_6}}\left(5\,\mathrm{sech}\left[\sqrt{-\frac{\vartheta_5}{5}}\left(x - vt\right)\right]\right)}{5\left(5\,\mathrm{sech}\left[\sqrt{-\frac{\vartheta_5}{5}}\left(x - vt\right)\right] \pm 1\right)}\right\} e^{i\left(-\frac{c_{21}\sigma_{71}}{4c_{11}\sigma_{11}}x + \omega t + \theta_0\right)}, \tag{101}$$

and

$$r(x,t)=\left\{\frac{12\sqrt{2}\varrho\sqrt{\frac{\vartheta_5}{2\vartheta_4+3\vartheta_6}}\left(5\operatorname{sech}\left[\sqrt{-\frac{\vartheta_5}{5}}(x-vt)\right]\right)}{5\left(5\operatorname{sech}\left[\sqrt{-\frac{\vartheta_5}{5}}(x-vt)\right]\pm1\right)}\right\}e^{i\left(-\frac{c_{22}\sigma_{72}}{4c_{12}\sigma_{12}}x+\omega t+\theta_0\right)}. \tag{102}$$

Result – 3:

$$\begin{gathered}\sigma_0=\sigma_1=0,\ \varrho_1=2\sqrt{6}\sqrt{\frac{\vartheta_5}{4\vartheta_4+7\vartheta_6}},\ k=\sqrt{\frac{2}{5}}\sqrt{-\frac{\vartheta_5(2\vartheta_4+3\vartheta_6)}{4\vartheta_4+7\vartheta_6}},\\ \vartheta_3=-\frac{4\vartheta_4^2+23\vartheta_6\vartheta_4+28\vartheta_6^2}{120\vartheta_5},\ \vartheta_2=\frac{\vartheta_4}{3}+\frac{47\vartheta_6}{48},\ \vartheta_1=-\frac{\vartheta_5(32\vartheta_4+43\vartheta_6)}{40\vartheta_4+70\vartheta_6}.\end{gathered} \tag{103}$$

When the values obtained from Eqs. (103) and (42) are inserted into Eq. (80), we observe the emergence of straddled dark–bright solitons exhibiting the restrictions $\vartheta_5>0$, $4\vartheta_4+7\vartheta_6>0$, and $2\vartheta_4+3\vartheta_6<0$, as described below:

$$q(x,t)=\left\{\frac{2\sqrt{6}\sqrt{\frac{\vartheta_5}{4\vartheta_4+7\vartheta_6}}\tanh\left[\sqrt{\frac{2}{5}}\sqrt{-\frac{\vartheta_5(2\vartheta_4+3\vartheta_6)}{4\vartheta_4+7\vartheta_6}}(x-vt)\right]}{1\pm5\operatorname{sech}\left[\sqrt{\frac{2}{5}}\sqrt{-\frac{\vartheta_5(2\vartheta_4+3\vartheta_6)}{4\vartheta_4+7\vartheta_6}}(x-vt)\right]}\right\}e^{i\left(-\frac{c_{21}\sigma_{71}}{4c_{11}\sigma_{11}}x+\omega t+\theta_0\right)}, \tag{104}$$

and

$$r(x,t)=\left\{\frac{2\sqrt{6}\varrho\sqrt{\frac{\vartheta_5}{4\vartheta_4+7\vartheta_6}}\tanh\left[\sqrt{\frac{2}{5}}\sqrt{-\frac{\vartheta_5(2\vartheta_4+3\vartheta_6)}{4\vartheta_4+7\vartheta_6}}(x-vt)\right]}{1\pm5\operatorname{sech}\left[\sqrt{\frac{2}{5}}\sqrt{-\frac{\vartheta_5(2\vartheta_4+3\vartheta_6)}{4\vartheta_4+7\vartheta_6}}(x-vt)\right]}\right\}e^{i\left(-\frac{c_{22}\sigma_{72}}{4c_{12}\sigma_{12}}x+\omega t+\theta_0\right)}. \tag{105}$$

***Family – 3**:* $R(\tau)=\frac{5}{9}\tau^2$

Result – 1:

$$\begin{aligned}&\sigma_0=0,\\ &\sigma_1=\frac{\sqrt{10}}{3}\tau\sqrt{\frac{3\vartheta_5-10\vartheta_1}{8\vartheta_2+\vartheta_4-4\vartheta_6}},\\ &\varrho_1=\sqrt{\frac{6\vartheta_5-20\vartheta_1}{8\vartheta_2+\vartheta_4-4\vartheta_6}},\\ &k=\sqrt{\frac{4\vartheta_1(-2\vartheta_2+\vartheta_4+\vartheta_6)+\vartheta_5(4\vartheta_2-\vartheta_4-2\vartheta_6)}{8\vartheta_2+\vartheta_4-4\vartheta_6}},\\ &\vartheta_3=\frac{(8\vartheta_2+\vartheta_4-4\vartheta_6)(\vartheta_1(12\vartheta_2-\vartheta_4+4\vartheta_6)-6\vartheta_2\vartheta_5)}{4(10\vartheta_1-3\vartheta_5)^2}.\end{aligned} \tag{106}$$

We can derive straddled singular–singular solitons by inserting the parameters from Eqs. (106) and (43) into Eq. (80), as presented below

$$q(x,t)=\sqrt{\frac{6\vartheta_5-20\vartheta_1}{8\vartheta_2+\vartheta_4-4\vartheta_6}}\times\exp\left[i\left(-\frac{c_{21}\sigma_{71}}{4c_{11}\sigma_{11}}x+\omega t+\theta_0\right)\right]$$
$$\times\frac{\sqrt{5}\operatorname{csch}\left[\sqrt{\frac{4\vartheta_1\left(-2\vartheta_2+\vartheta_4+\vartheta_6\right)+\vartheta_5\left(4\vartheta_2-\vartheta_4-2\vartheta_6\right)}{8\vartheta_2+\vartheta_4-4\vartheta_6}}(x-vt)\right]\pm 2}{\left(\begin{array}{c}3\operatorname{csch}\left[\sqrt{\frac{4\vartheta_1\left(-2\vartheta_2+\vartheta_4+\vartheta_6\right)+\vartheta_5\left(4\vartheta_2-\vartheta_4-2\vartheta_6\right)}{8\vartheta_2+\vartheta_4-4\vartheta_6}}(x-vt)\right]\\ \pm 2\coth\left[\sqrt{\frac{4\vartheta_1\left(-2\vartheta_2+\vartheta_4+\vartheta_6\right)+\vartheta_5\left(4\vartheta_2-\vartheta_4-2\vartheta_6\right)}{8\vartheta_2+\vartheta_4-4\vartheta_6}}(x-vt)\right]\end{array}\right)},\qquad(107)$$

and

$$r(x,t)=\varrho\sqrt{\frac{6\vartheta_5-20\vartheta_1}{8\vartheta_2+\vartheta_4-4\vartheta_6}}\times\exp\left[i\left(-\frac{c_{22}\sigma_{72}}{4c_{12}\sigma_{12}}x+\omega t+\theta_0\right)\right]$$
$$\times\frac{\sqrt{5}\operatorname{csch}\left(\sqrt{\frac{4\vartheta_1\left(-2\vartheta_2+\vartheta_4+\vartheta_6\right)+\vartheta_5\left(4\vartheta_2-\vartheta_4-2\vartheta_6\right)}{8\vartheta_2+\vartheta_4-4\vartheta_6}}(x-tv)\right)\pm 2}{\left(\begin{array}{c}3\operatorname{csch}\left(\sqrt{\frac{4\vartheta_1\left(-2\vartheta_2+\vartheta_4+\vartheta_6\right)+\vartheta_5\left(4\vartheta_2-\vartheta_4-2\vartheta_6\right)}{8\vartheta_2+\vartheta_4-4\vartheta_6}}(x-tv)\right)\\ \pm 2\coth\left[\sqrt{\frac{4\vartheta_1\left(-2\vartheta_2+\vartheta_4+\vartheta_6\right)+\vartheta_5\left(4\vartheta_2-\vartheta_4-2\vartheta_6\right)}{8\vartheta_2+\vartheta_4-4\vartheta_6}}(x-vt)\right]\end{array}\right)}.\qquad(108)$$

The resulting conditions are $6\vartheta_5-20\vartheta_1>0$, $8\vartheta_2+\vartheta_4-4\vartheta_6>0$, and $4\vartheta_1\left(-2\vartheta_2+\vartheta_4+\vartheta_6\right)+\vartheta_5\left(4\vartheta_2-\vartheta_4-2\vartheta_6\right)>0$.

<u>Result – 2</u>:

$$\begin{aligned}&\sigma_0=0,\ \sigma_1=2\sqrt{\frac{5}{3}}\tau\sqrt{\frac{\vartheta_5}{2\vartheta_4+3\vartheta_6}},\\&\varrho_1=0,\ k=\sqrt{-\frac{\vartheta_5}{5}},\\&\vartheta_1=-\frac{1}{5}\left(4\vartheta_5\right),\ \vartheta_2=\frac{1}{15}\left(17\vartheta_4+33\vartheta_6\right),\\&\vartheta_3=-\frac{2\vartheta_4^2+11\vartheta_6\vartheta_4+12\vartheta_6^2}{60\vartheta_5}.\end{aligned}\qquad(109)$$

The parameters obtained from Eqs. (109) and (43) can be substituted into Eq. (80) to yield straddled bright–bright solitons exhibiting the conditions $\vartheta_5<0$ and $2\vartheta_4+3\vartheta_6<0$, as given below

$$q\left(x,t\right)=\left\{\frac{2\sqrt{\frac{5}{3}}\sqrt{\frac{\vartheta_5}{2\vartheta_4+3\vartheta_6}}\left(3\operatorname{sech}\left[\sqrt{-\frac{\vartheta_5}{5}}\left(x-vt\right)\right]\right)}{3\operatorname{sech}\left[\sqrt{-\frac{\vartheta_5}{5}}\left(x-vt\right)\right]\pm 2}\right\}e^{i\left(-\frac{c_{21}\sigma_{71}}{4c_{11}\sigma_{11}}x+\omega t+\theta_0\right)},\qquad(110)$$

and

$$r(x,t)=\left\{\frac{2\sqrt{\frac{5}{3}}\varrho\sqrt{\frac{\vartheta_5}{2\vartheta_4+3\vartheta_6}}\left(3\operatorname{sech}\left[\sqrt{-\frac{\vartheta_5}{5}}(x-vt)\right]\right)}{3\operatorname{sech}\left[\sqrt{-\frac{\vartheta_5}{5}}(x-vt)\right]\pm 2}\right\}e^{i\left(-\frac{c_{22}\sigma_{72}}{4c_{12}\sigma_{12}}x+\omega t+\theta_0\right)}. \tag{111}$$

Result – 3:

$$\begin{gathered}\sigma_0=\sigma_1=0,\ \varrho_1=2\sqrt{15}\sqrt{\frac{\vartheta_5}{10\vartheta_4+63\vartheta_6}},\ k=\sqrt{-\frac{\vartheta_5(2\vartheta_4+3\vartheta_6)}{10\vartheta_4+63\vartheta_6}},\\ \vartheta_1=\frac{4\vartheta_5(3\vartheta_6-2\vartheta_4)}{10\vartheta_4+63\vartheta_6},\ \vartheta_2=\frac{\vartheta_4}{3}+\frac{3\vartheta_6}{5},\ \vartheta_3=-\frac{10\vartheta_4^2+103\vartheta_6\vartheta_4+252\vartheta_6^2}{300\vartheta_5}.\end{gathered} \tag{112}$$

After plugging the parameters found in Eqs. (112) and (43) into Eq. (80), we obtain straddled singular–singular solitons satisfying the conditions $\vartheta_5>0$, $4\vartheta_4+7\vartheta_6>0$, and $2\vartheta_4+3\vartheta_6<0$, as described below

$$q(x,t)=\frac{4\sqrt{15}\sqrt{\frac{\vartheta_5}{10\vartheta_4+63\vartheta_6}}}{2\coth\left[\sqrt{-\frac{\vartheta_5(2\vartheta_4+3\vartheta_6)}{10\vartheta_4+63\vartheta_6}}(x-vt)\right]\pm 3\operatorname{csch}\left[\sqrt{-\frac{\vartheta_5(2\vartheta_4+3\vartheta_6)}{10\vartheta_4+63\vartheta_6}}(x-vt)\right]}\times e^{i\left(-\frac{c_{21}\sigma_{71}}{4c_{11}\sigma_{11}}x+\omega t+\theta_0\right)} \tag{113}$$

and

$$r(x,t)=\frac{4\sqrt{15}\varrho\sqrt{\frac{\vartheta_5}{10\vartheta_4+63\vartheta_6}}}{2\coth\left[\sqrt{-\frac{\vartheta_5(2\vartheta_4+3\vartheta_6)}{10\vartheta_4+63\vartheta_6}}(x-vt)\right]\pm 3\operatorname{csch}\left[\sqrt{-\frac{\vartheta_5(2\vartheta_4+3\vartheta_6)}{10\vartheta_4+63\vartheta_6}}(x-vt)\right]}\times e^{i\left(-\frac{c_{22}\sigma_{72}}{4c_{12}\sigma_{12}}x+\omega t+\theta_0\right)}. \tag{114}$$

Family – 4: $R(\tau)=\tau^2+\epsilon$

Case – 1: $\epsilon=-1$

Result – 1:

$$\begin{gathered}\sigma_0=0,\sigma_1=\sqrt{\frac{(\tau^2-1)(6\vartheta_5-20\vartheta_1)}{8\vartheta_2+\vartheta_4-4\vartheta_6}},\\ \varrho_1=\sqrt{\frac{6\vartheta_5-20\vartheta_1}{8\vartheta_2+\vartheta_4-4\vartheta_6}},\\ k=\sqrt{\frac{4\vartheta_1(-2\vartheta_2+\vartheta_4+\vartheta_6)+\vartheta_5(4\vartheta_2-\vartheta_4-2\vartheta_6)}{8\vartheta_2+\vartheta_4-4\vartheta_6}},\\ \vartheta_3=\frac{(8\vartheta_2+\vartheta_4-4\vartheta_6)(\vartheta_1(12\vartheta_2-\vartheta_4+4\vartheta_6)-6\vartheta_2\vartheta_5)}{4(10\vartheta_1-3\vartheta_5)^2}.\end{gathered} \tag{115}$$

Upon plugging the parameters obtained in Eqs. (115) and (44) into Eq. (80), we observe the emergence of straddled bright–dark solitons that satisfy $6\vartheta_5 - 20\vartheta_1 > 0$, $8\vartheta_2 + \vartheta_4 - 4\vartheta_6 > 0$, and $4\vartheta_1\left(-2\vartheta_2 + \vartheta_4 + \vartheta_6\right) + \vartheta_5\left(4\vartheta_2 - \vartheta_4 - 2\vartheta_6\right) > 0$, as shown below

$$q(x,t) = \sqrt{\frac{6\vartheta_5 - 20\vartheta_1}{8\vartheta_2 + \vartheta_4 - 4\vartheta_6}} \times \frac{\left(\begin{array}{l} 4\sqrt{\tau^2 - 1}\,\mathrm{sech}\left[\sqrt{\frac{4\vartheta_1\left(-2\vartheta_2 + \vartheta_4 + \vartheta_6\right) + \vartheta_5\left(4\vartheta_2 - \vartheta_4 - 2\vartheta_6\right)}{8\vartheta_2 + \vartheta_4 - 4\vartheta_6}}(x - vt)\right] \\ +5\tanh\left[\sqrt{\frac{4\vartheta_1\left(-2\vartheta_2 + \vartheta_4 + \vartheta_6\right) + \vartheta_5\left(4\vartheta_2 - \vartheta_4 - 2\vartheta_6\right)}{8\vartheta_2 + \vartheta_4 - 4\vartheta_6}}(x - vt)\right] + 3 \end{array} \right)}{\left(\begin{array}{l} 4\tau\,\mathrm{sech}\left[\sqrt{\frac{4\vartheta_1\left(-2\vartheta_2 + \vartheta_4 + \vartheta_6\right) + \vartheta_5\left(4\vartheta_2 - \vartheta_4 - 2\vartheta_6\right)}{8\vartheta_2 + \vartheta_4 - 4\vartheta_6}}(x - vt)\right] \\ +3\tanh\left[\sqrt{\frac{4\vartheta_1\left(-2\vartheta_2 + \vartheta_4 + \vartheta_6\right) + \vartheta_5\left(4\vartheta_2 - \vartheta_4 - 2\vartheta_6\right)}{8\vartheta_2 + \vartheta_4 - 4\vartheta_6}}(x - vt)\right] + 5 \end{array} \right)} \times e^{i\left(-\frac{c_{21}\sigma_{71}}{4c_{11}\sigma_{11}}x + \omega t + \theta_0\right)}, \tag{116}$$

$$r(x,t) = \varrho\sqrt{\frac{6\vartheta_5 - 20\vartheta_1}{8\vartheta_2 + \vartheta_4 - 4\vartheta_6}} \times \frac{\left(\begin{array}{l} 4\sqrt{\tau^2 - 1}\,\mathrm{sech}\left[\sqrt{\frac{4\vartheta_1\left(-2\vartheta_2 + \vartheta_4 + \vartheta_6\right) + \vartheta_5\left(4\vartheta_2 - \vartheta_4 - 2\vartheta_6\right)}{8\vartheta_2 + \vartheta_4 - 4\vartheta_6}}(x - vt)\right] \\ +5\tanh\left[\sqrt{\frac{4\vartheta_1\left(-2\vartheta_2 + \vartheta_4 + \vartheta_6\right) + \vartheta_5\left(4\vartheta_2 - \vartheta_4 - 2\vartheta_6\right)}{8\vartheta_2 + \vartheta_4 - 4\vartheta_6}}(x - vt)\right] + 3 \end{array} \right)}{\left(\begin{array}{l} 4\tau\,\mathrm{sech}\left[\sqrt{\frac{4\vartheta_1\left(-2\vartheta_2 + \vartheta_4 + \vartheta_6\right) + \vartheta_5\left(4\vartheta_2 - \vartheta_4 - 2\vartheta_6\right)}{8\vartheta_2 + \vartheta_4 - 4\vartheta_6}}(x - vt)\right] \\ +3\tanh\left[\sqrt{\frac{4\vartheta_1\left(-2\vartheta_2 + \vartheta_4 + \vartheta_6\right) + \vartheta_5\left(4\vartheta_2 - \vartheta_4 - 2\vartheta_6\right)}{8\vartheta_2 + \vartheta_4 - 4\vartheta_6}}(x - vt)\right] + 5 \end{array} \right)} \times e^{i\left(-\frac{c_{22}\sigma_{72}}{4c_{12}\sigma_{12}}x + \omega t + \theta_0\right)}, \tag{117}$$

$$q(x,t)=\sqrt{\frac{6\vartheta_5-20\vartheta_1}{8\vartheta_2+\vartheta_4-4\vartheta_6}}\exp\left[i\left(-\frac{c_{21}\sigma_{71}}{4c_{11}\sigma_{11}}x+\omega t+\theta_0\right)\right]$$
$$\times\frac{\left(\sqrt{\tau^2-1}\operatorname{csch}\left[\sqrt{\frac{4\vartheta_1\left(-2\vartheta_2+\vartheta_4+\vartheta_6\right)+\vartheta_5\left(4\vartheta_2-\vartheta_4-2\vartheta_6\right)}{8\vartheta_2+\vartheta_4-4\vartheta_6}}(x-vt)\right]+1\right)}{\left(\begin{array}{l}\tau\operatorname{csch}\left[\sqrt{\frac{4\vartheta_1\left(-2\vartheta_2+\vartheta_4+\vartheta_6\right)+\vartheta_5\left(4\vartheta_2-\vartheta_4-2\vartheta_6\right)}{8\vartheta_2+\vartheta_4-4\vartheta_6}}(x-vt)\right]\\+\coth\left[\sqrt{\frac{4\vartheta_1\left(-2\vartheta_2+\vartheta_4+\vartheta_6\right)+\vartheta_5\left(4\vartheta_2-\vartheta_4-2\vartheta_6\right)}{8\vartheta_2+\vartheta_4-4\vartheta_6}}(x-vt)\right]\end{array}\right)},\tag{118}$$

and

$$r(x,t)=\varrho\sqrt{\frac{6\vartheta_5-20\vartheta_1}{8\vartheta_2+\vartheta_4-4\vartheta_6}}\exp\left[i\left(-\frac{c_{22}\sigma_{72}}{4c_{12}\sigma_{12}}x+\omega t+\theta_0\right)\right]$$
$$\times\frac{\left(\sqrt{\tau^2-1}\operatorname{csch}\left[\sqrt{\frac{4\vartheta_1\left(-2\vartheta_2+\vartheta_4+\vartheta_6\right)+\vartheta_5\left(4\vartheta_2-\vartheta_4-2\vartheta_6\right)}{8\vartheta_2+\vartheta_4-4\vartheta_6}}(x-vt)\right]+1\right)}{\left(\begin{array}{l}\tau\operatorname{csch}\left[\sqrt{\frac{4\vartheta_1\left(-2\vartheta_2+\vartheta_4+\vartheta_6\right)+\vartheta_5\left(4\vartheta_2-\vartheta_4-2\vartheta_6\right)}{8\vartheta_2+\vartheta_4-4\vartheta_6}}(x-vt)\right]\\+\coth\left[\sqrt{\frac{4\vartheta_1\left(-2\vartheta_2+\vartheta_4+\vartheta_6\right)+\vartheta_5\left(4\vartheta_2-\vartheta_4-2\vartheta_6\right)}{8\vartheta_2+\vartheta_4-4\vartheta_6}}(x-vt)\right]\end{array}\right)}.\tag{119}$$

<u>**Result – 2**</u>:

$$\begin{gathered}\sigma_1=2\sqrt{\frac{3\left(\tau^2-1\right)\vartheta_5}{2\vartheta_4+3\vartheta_6}},\ \sigma_0=\varrho_1=0,k=\sqrt{-\frac{\vartheta_5}{5}},\ \vartheta_1=-\frac{1}{5}\left(4\vartheta_5\right)\\ \vartheta_3=-\frac{2\vartheta_4^2+11\vartheta_6\vartheta_4+12\vartheta_6^2}{60\vartheta_5},\ \vartheta_2=\frac{2\left(\tau^2+2\right)\vartheta_4+3\left(2\tau^2+1\right)\vartheta_6}{6\left(\tau^2-1\right)}.\end{gathered}\tag{120}$$

Upon substitution of the parameters in Eqs. (120) with (44) into Eq. (80), we obtain straddled bright–dark and bright–bright solitons characterized by the inequalities $6\vartheta_5-20\vartheta_1>0$, $8\vartheta_2+\vartheta_4-4\vartheta_6>0$, and $4\vartheta_1\left(-2\vartheta_2+\vartheta_4+\vartheta_6\right)+\vartheta_5\left(4\vartheta_2-\vartheta_4-2\vartheta_6\right)>0$, as indicated below:

$$q(x,t)=\frac{8\sqrt{\frac{3\left(\tau^2-1\right)\vartheta_5}{2\vartheta_4+3\vartheta_6}}\left(\operatorname{sech}\left[\sqrt{-\frac{\vartheta_5}{5}}(x-vt)\right]\right)}{4\tau\operatorname{sech}\left[\sqrt{-\frac{\vartheta_5}{5}}(x-vt)\right]+3\tanh\left[\sqrt{-\frac{\vartheta_5}{5}}(x-vt)\right]+5}e^{i\left(-\frac{c_{21}\sigma_{71}}{4c_{11}\sigma_{11}}x+\omega t+\theta_0\right)},\tag{121}$$

$$r(x,t)=\frac{8\varrho\sqrt{\frac{3\left(\tau^2-1\right)\vartheta_5}{2\vartheta_4+3\vartheta_6}}\left(\operatorname{sech}\left[\sqrt{-\frac{\vartheta_5}{5}}(x-vt)\right]\right)}{4\tau\operatorname{sech}\left[\sqrt{-\frac{\vartheta_5}{5}}(x-vt)\right]+3\tanh\left[\sqrt{-\frac{\vartheta_5}{5}}(x-vt)\right]+5}e^{i\left(-\frac{c_{22}\sigma_{72}}{4c_{12}\sigma_{12}}x+\omega t+\theta_0\right)},\tag{122}$$

$$q(x,t)=\frac{2\sqrt{\frac{3\left(\tau^2-1\right)\vartheta_5}{2\vartheta_4+3\vartheta_6}}\operatorname{sech}\left[\sqrt{-\frac{\vartheta_5}{5}}\left(x-vt\right)\right]}{\tau\operatorname{sech}\left[\sqrt{-\frac{\vartheta_5}{5}}\left(x-vt\right)\right]+1}\exp\left[i\left(-\frac{c_{21}\sigma_{71}}{4c_{11}\sigma_{11}}x+\omega t+\theta_0\right)\right], \quad (123)$$

and

$$r(x,t)=\frac{2\varrho\sqrt{\frac{3\left(\tau^2-1\right)\vartheta_5}{2\vartheta_4+3\vartheta_6}}\operatorname{sech}\left[\sqrt{-\frac{\vartheta_5}{5}}\left(x-vt\right)\right]}{\tau\operatorname{sech}\left[\sqrt{-\frac{\vartheta_5}{5}}\left(x-vt\right)\right]+1}\exp\left[i\left(-\frac{c_{22}\sigma_{72}}{4c_{12}\sigma_{12}}x+\omega t+\theta_0\right)\right]. \quad (124)$$

Result – 3:

$$\begin{aligned}
&\sigma_0=\sigma_1=0,\\
&\varrho_1=2\sqrt{\frac{3\left(\tau^2-1\right)\vartheta_5}{2\left(\tau^2-1\right)\vartheta_4+3\left(\tau^2+3\right)\vartheta_6}},\quad k=\sqrt{-\frac{\left(\tau^2-1\right)\vartheta_5\left(2\vartheta_4+3\vartheta_6\right)}{10\left(\tau^2-1\right)\vartheta_4+15\left(\tau^2+3\right)\vartheta_6}},\\
&\vartheta_3=-\frac{\left(\vartheta_4+4\vartheta_6\right)\left(2\left(\tau^2-1\right)\vartheta_4+3\left(\tau^2+3\right)\vartheta_6\right)}{60\left(\tau^2-1\right)\vartheta_5}, \quad (125)\\
&\vartheta_2=\frac{\left(2\tau^2-3\right)\vartheta_6}{2\left(\tau^2-1\right)}+\frac{\vartheta_4}{3},\quad \vartheta_1=-\frac{2\vartheta_5\left(4\left(\tau^2-1\right)\vartheta_4+3\left(2\tau^2-7\right)\vartheta_6\right)}{5\left(2\left(\tau^2-1\right)\vartheta_4+3\left(\tau^2+3\right)\vartheta_6\right)}.
\end{aligned}$$

Inserting the findings in Eq. (125) together with Eq. (44) into Eq. (80) yields straddled dark–bright solitons that hold the constraints $\left(\tau^2-1\right)\vartheta_5>0$, $2\left(\tau^2-1\right)\vartheta_4+3\left(\tau^2+3\right)\vartheta_6>0$, and $2\vartheta_4+3\vartheta_6<0$, as presented below:

$$\begin{aligned}
q(x,t)=&\,2\sqrt{\frac{3\left(\tau^2-1\right)\vartheta_5}{2\left(\tau^2-1\right)\vartheta_4+3\left(\tau^2+3\right)\vartheta_6}}\\
&\times\frac{5\tanh\left[\sqrt{-\frac{\left(\tau^2-1\right)\vartheta_5\left(2\vartheta_4+3\vartheta_6\right)}{10\left(\tau^2-1\right)\vartheta_4+15\left(\tau^2+3\right)\vartheta_6}}(x-vt)\right]+3}{\left(\begin{array}{l}3\tanh\left[\sqrt{-\frac{\left(\tau^2-1\right)\vartheta_5\left(2\vartheta_4+3\vartheta_6\right)}{10\left(\tau^2-1\right)\vartheta_4+15\left(\tau^2+3\right)\vartheta_6}}(x-vt)\right]\\+4\tau\operatorname{sech}\left[\sqrt{-\frac{\left(\tau^2-1\right)\vartheta_5\left(2\vartheta_4+3\vartheta_6\right)}{10\left(\tau^2-1\right)\vartheta_4+15\left(\tau^2+3\right)\vartheta_6}}(x-vt)\right]+5\end{array}\right)} \quad (126)\\
&\times\exp\left[i\left(-\frac{c_{21}\sigma_{71}}{4c_{11}\sigma_{11}}x+\omega t+\theta_0\right)\right],
\end{aligned}$$

$$r(x,t)=2\varrho\sqrt{\frac{3\left(\tau^{2}-1\right)\vartheta_{5}}{2\left(\tau^{2}-1\right)\vartheta_{4}+3\left(\tau^{2}+3\right)\vartheta_{6}}}\exp\left[i\left(-\frac{c_{22}\sigma_{72}}{4c_{12}\sigma_{12}}x+\omega t+\theta_{0}\right)\right]$$
$$\times\frac{\left(5\tanh\left[\sqrt{-\frac{\left(\tau^{2}-1\right)\vartheta_{5}\left(2\vartheta_{4}+3\vartheta_{6}\right)}{10\left(\tau^{2}-1\right)\vartheta_{4}+15\left(\tau^{2}+3\right)\vartheta_{6}}}(x-vt)\right]+3\right)}{\left(\begin{array}{l}3\tanh\left[\sqrt{-\frac{\left(\tau^{2}-1\right)\vartheta_{5}\left(2\vartheta_{4}+3\vartheta_{6}\right)}{10\left(\tau^{2}-1\right)\vartheta_{4}+15\left(\tau^{2}+3\right)\vartheta_{6}}}(x-vt)\right]\\+4\tau\operatorname{sech}\left[\sqrt{-\frac{\left(\tau^{2}-1\right)\vartheta_{5}\left(2\vartheta_{4}+3\vartheta_{6}\right)}{10\left(\tau^{2}-1\right)\vartheta_{4}+15\left(\tau^{2}+3\right)\vartheta_{6}}}(x-vt)\right]+5\end{array}\right)},\tag{127}$$

$$q(x,t)=2\sqrt{\frac{3\left(\tau^{2}-1\right)\vartheta_{5}}{2\left(\tau^{2}-1\right)\vartheta_{4}+3\left(\tau^{2}+3\right)\vartheta_{6}}}\exp\left[i\left(-\frac{c_{21}\sigma_{71}}{4c_{11}\sigma_{11}}x+\omega t+\theta_{0}\right)\right]$$
$$\times\frac{\tanh\left[\sqrt{-\frac{\left(\tau^{2}-1\right)\vartheta_{5}\left(2\vartheta_{4}+3\vartheta_{6}\right)}{10\left(\tau^{2}-1\right)\vartheta_{4}+15\left(\tau^{2}+3\right)\vartheta_{6}}}(x-vt)\right]}{\tau\operatorname{sech}\left[\sqrt{-\frac{\left(r^{2}-1\right)\vartheta_{5}\left(2\vartheta_{4}+3\vartheta_{6}\right)}{10\left(\tau^{2}-1\right)\vartheta_{4}+15\left(\tau^{2}+3\right)\vartheta_{6}}}(x-vt)\right]+1},\tag{128}$$

and

$$r(x,t)=2\varrho\sqrt{\frac{3\left(\tau^{2}-1\right)\vartheta_{5}}{2\left(\tau^{2}-1\right)\vartheta_{4}+3\left(\tau^{2}+3\right)\vartheta_{6}}}\exp\left[i\left(-\frac{c_{22}\sigma_{72}}{4c_{12}\sigma_{12}}x+\omega t+\theta_{0}\right)\right]$$
$$\times\frac{\tanh\left[\sqrt{-\frac{\left(r^{2}-1\right)\vartheta_{5}\left(2\vartheta_{4}+3\vartheta_{6}\right)}{10\left(\tau^{2}-1\right)\vartheta_{4}+15\left(\tau^{2}+3\right)\vartheta_{6}}}(x-vt)\right]}{\tau\operatorname{sech}\left[\sqrt{-\frac{\left(\tau^{2}-1\right)\vartheta_{5}\left(2\vartheta_{4}+3\vartheta_{6}\right)}{10\left(\tau^{2}-1\right)\vartheta_{4}+15\left(\tau^{2}+3\right)\vartheta_{6}}}(x-vt)\right]+1}.\tag{129}$$

Case – 2: $\epsilon=1$

<u>**Result – 1**</u>:

$$\sigma_{0}=0,\ \sigma_{1}=\sqrt{-\frac{2\left(\tau^{2}+1\right)\left(10\vartheta_{1}-3\vartheta_{5}\right)}{8\vartheta_{2}+\vartheta_{4}-4\vartheta_{6}}},\ \varrho_{1}=\sqrt{\frac{6\vartheta_{5}-20\vartheta_{1}}{8\vartheta_{2}+\vartheta_{4}-4\vartheta_{6}}},$$
$$k=\sqrt{\frac{4\vartheta_{1}\left(-2\vartheta_{2}+\vartheta_{4}+\vartheta_{6}\right)+\vartheta_{5}\left(4\vartheta_{2}-\vartheta_{4}-2\vartheta_{6}\right)}{8\vartheta_{2}+\vartheta_{4}-4\vartheta_{6}}},\tag{130}$$
$$\vartheta_{3}=\frac{\left(8\vartheta_{2}+\vartheta_{4}-4\vartheta_{6}\right)\left(\vartheta_{1}\left(12\vartheta_{2}-\vartheta_{4}+4\vartheta_{6}\right)-6\vartheta_{2}\vartheta_{5}\right)}{4\left(10\vartheta_{1}-3\vartheta_{5}\right)^{2}}.$$

Upon inserting the outcomes from Eqs. (130) and (44) into Eq. (80), we arrive at straddled bright–dark solitons that satisfy $6\vartheta_5 - 20\vartheta_1 > 0$, $8\vartheta_2 + \vartheta_4 - 4\vartheta_6 > 0$, and $4\vartheta_1(-2\vartheta_2 + \vartheta_4 + \vartheta_6) + \vartheta_5(4\vartheta_2 - \vartheta_4 - 2\vartheta_6) > 0$, as shown below:

$$q(x,t) = \sqrt{\frac{6\vartheta_5 - 20\vartheta_1}{8\vartheta_2 + \vartheta_4 - 4\vartheta_6}} \exp\left[i\left(-\frac{c_{21}\sigma_{71}}{4c_{11}\sigma_{11}}x + \omega t + \theta_0\right)\right] \times \frac{\left(\sqrt{\tau^2+1}\operatorname{sech}\left[\sqrt{\frac{4\vartheta_1(-2\vartheta_2+\vartheta_4+\vartheta_6)+\vartheta_5(4\vartheta_2-\vartheta_4-2\vartheta_6)}{8\vartheta_2+\vartheta_4-4\vartheta_6}}(x-vt)\right]+1\right)}{\left(\begin{array}{l}\tau\operatorname{sech}\left[\sqrt{\frac{4\vartheta_1(-2\vartheta_2+\vartheta_4+\vartheta_6)+\vartheta_5(4\vartheta_2-\vartheta_4-2\vartheta_6)}{8\vartheta_2+\vartheta_4-4\vartheta_6}}(x-vt)\right]\\ +\tanh\left[\sqrt{\frac{4\vartheta_1(-2\vartheta_2+\vartheta_4+\vartheta_6)+\vartheta_5(4\vartheta_2-\vartheta_4-2\vartheta_6)}{8\vartheta_2+\vartheta_4-4\vartheta_6}}(x-vt)\right]\end{array}\right)}, \quad (131)$$

and

$$r(x,t) = \varrho\sqrt{\frac{6\vartheta_5 - 20\vartheta_1}{8\vartheta_2 + \vartheta_4 - 4\vartheta_6}} \exp\left[i\left(-\frac{c_{22}\sigma_{72}}{4c_{12}\sigma_{12}}x + \omega t + \theta_0\right)\right] \times \frac{\left(\sqrt{\tau^2+1}\operatorname{sech}\left[\sqrt{\frac{4\vartheta_1(-2\vartheta_2+\vartheta_4+\vartheta_6)+\vartheta_5(4\vartheta_2-\vartheta_4-2\vartheta_6)}{8\vartheta_2+\vartheta_4-4\vartheta_6}}(x-vt)\right]+1\right)}{\left(\begin{array}{l}\tau\operatorname{sech}\left[\sqrt{\frac{4\vartheta_1(-2\vartheta_2+\vartheta_4+\vartheta_6)+\vartheta_5(4\vartheta_2-\vartheta_4-2\vartheta_6)}{8\vartheta_2+\vartheta_4-4\vartheta_6}}(x-vt)\right]\\ +\tanh\left[\sqrt{\frac{4\vartheta_1(-2\vartheta_2+\vartheta_4+\vartheta_6)+\vartheta_5(4\vartheta_2-\vartheta_4-2\vartheta_6)}{8\vartheta_2+\vartheta_4-4\vartheta_6}}(x-vt)\right]\end{array}\right)}. \quad (132)$$

<u>Result – 2</u>:

$$\begin{aligned}
&\sigma_0 = \sigma_1 = 0,\ \varrho_1 = 2\sqrt{\frac{3(\tau^2+1)\vartheta_5}{2(\tau^2+1)\vartheta_4 + 3(\tau^2-3)\vartheta_6}},\\
&k = \sqrt{-\frac{(\tau^2+1)\vartheta_5(2\vartheta_4+3\vartheta_6)}{10(\tau^2+1)\vartheta_4 + 15(\tau^2-3)\vartheta_6}},\\
&\vartheta_3 = -\frac{(\vartheta_4+4\vartheta_6)\left(2(\tau^2+1)\vartheta_4 + 3(\tau^2-3)\vartheta_6\right)}{60(\tau^2+1)\vartheta_5},\\
&\vartheta_2 = \frac{(2\tau^2+3)\vartheta_6}{2(\tau^2+1)} + \frac{\vartheta_4}{3},\ \vartheta_1 = -\frac{2\vartheta_5\left(4(\tau^2+1)\vartheta_4 + 3(2\tau^2+7)\vartheta_6\right)}{5\left(2(\tau^2+1)\vartheta_4 + 3(\tau^2-3)\vartheta_6\right)}.
\end{aligned} \quad (133)$$

We obtain straddled singular–singular solitons with $\vartheta_5 > 0$, $2(\tau^2+1)\vartheta_4 + 3(\tau^2-3)\vartheta_6 > 0$, and $2\vartheta_4 + 3\vartheta_6 < 0$ by substituting the parameters obtained from Eqs. (133) and (44) into Eq. (80):

$$q(x,t)=\frac{2\sqrt{\frac{3\left(\tau^2+1\right)\vartheta_5}{2\left(\tau^2+1\right)\vartheta_4+3\left(\tau^2-3\right)\vartheta_6}}\coth\left[\sqrt{-\frac{\left(\tau^2+1\right)\vartheta_5\left(2\vartheta_4+3\vartheta_6\right)}{10\left(\tau^2+1\right)\vartheta_4+15\left(\tau^2-3\right)\vartheta_6}}(x-vt)\right]}{\tau\operatorname{csch}\left[\sqrt{-\frac{\left(\tau^2+1\right)\vartheta_5\left(2\vartheta_4+3\vartheta_6\right)}{10\left(\tau^2+1\right)\vartheta_4+15\left(\tau^2-3\right)\vartheta_6}}(x-vt)\right]+1} \times\exp\left[i\left(-\frac{c_{21}\sigma_{71}}{4c_{11}\sigma_{11}}x+\omega t+\theta_0\right)\right], \tag{134}$$

and

$$r(x,t)=\frac{2\varrho\sqrt{\frac{3\left(\tau^2+1\right)\vartheta_5}{2\left(\tau^2+1\right)\vartheta_4+3\left(\tau^2-3\right)\vartheta_6}}\coth\left[\sqrt{-\frac{\left(\tau^2+1\right)\vartheta_5\left(2\vartheta_4+3\vartheta_6\right)}{10\left(\tau^2+1\right)\vartheta_4+15\left(\tau^2-3\right)\vartheta_6}}(x-vt)\right]}{\tau\operatorname{csch}\left[\sqrt{-\frac{\left(\tau^2+1\right)\vartheta_5\left(2\vartheta_4+3\vartheta_6\right)}{10\left(\tau^2+1\right)\vartheta_4+15\left(\tau^2-3\right)\vartheta_6}}(x-vt)\right]+1} \times\exp\left[i\left(-\frac{c_{22}\sigma_{72}}{4c_{12}\sigma_{12}}x+\omega t+\theta_0\right)\right]. \tag{135}$$

<u>Result – 3</u>:

$$\begin{gathered}\sigma_0=\varrho_1=0,\quad \sigma_1=2\sqrt{\frac{3\left(\tau^2+1\right)\vartheta_5}{2\vartheta_4+3\vartheta_6}},\quad k=\sqrt{-\frac{\vartheta_5}{5}},\\ \vartheta_1=-\frac{4\vartheta_5}{5},\quad \vartheta_2=\frac{2\left(\tau^2-2\right)\vartheta_4+3\left(2\tau^2-1\right)\vartheta_6}{6\left(\tau^2+1\right)},\\ \vartheta_3=-\frac{2\vartheta_4^2+11\vartheta_6\vartheta_4+12\vartheta_6^2}{60\vartheta_5}.\end{gathered} \tag{136}$$

After plugging the parameters from Eqs. (136) and (44) into Eq. (80), we obtain straddled singular–singular solitons characterized by $\vartheta_5<0$ and $2\vartheta_4+3\vartheta_6<0$, as described below

$$q\left(x,t\right)=\left\{\frac{2\sqrt{\frac{3\left(\tau^2+1\right)\vartheta_5}{2\vartheta_4+3\vartheta_6}}\operatorname{csch}\left[\sqrt{-\frac{\vartheta_5}{5}}\left(x-vt\right)\right]}{\tau\operatorname{csch}\left[\sqrt{-\frac{\vartheta_5}{5}}\left(x-vt\right)\right]+1}\right\}\exp\left[i\left(-\frac{c_{21}\sigma_{71}}{4c_{11}\sigma_{11}}x+\omega t+\theta_0\right)\right], \tag{137}$$

and

$$r\left(x,t\right)=\varrho\left\{\frac{2\sqrt{\frac{3\left(\tau^2+1\right)\vartheta_5}{2\vartheta_4+3\vartheta_6}}\operatorname{csch}\left[\sqrt{-\frac{\vartheta_5}{5}}\left(x-vt\right)\right]}{\tau\operatorname{csch}\left[\sqrt{-\frac{\vartheta_5}{5}}\left(x-vt\right)\right]+1}\right\}\exp\left[i\left(-\frac{c_{22}\sigma_{72}}{4c_{12}\sigma_{12}}x+\omega t+\theta_0\right)\right]. \tag{138}$$

5. Conclusions

The current paper is an extensive study of optical solitons for the concatenation model in birefringent fibers. A couple of algorithms have revealed a wide range of soliton and complexiton solutions that are being reported for the first time. Earlier, the method of undetermined coefficients identified only bright, dark, and singular 1–soliton solutions to the model. The current paper has gone far and beyond to recover the additional form of soliton and complexiton solutions. Nevertheless, the approaches have their own limitations. The two schemes fail to recover soliton radiation to the model. This is one of the future avenues to look into. The conservation laws for solitons in birefringent fibers for the concatenation model are yet to be looked at. Next up, the model needs to be studied with perturbation terms that can be addressed using the method of semi–inverse variational principle, especially when the perturbation terms appear with maximum intensity. Subsequently, it would be necessary to have a look at solitons in magneto–optic waveguides, or gap solitons, as well as solitons in optical couplers for the concatenation model. Such studies are underway, and those results would be reported as soon as they are available and aligned with the pre–existing works [1—10].

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Анотація. *Ця стаття розкриває розв'язки оптичних солітонів для моделі конкатенації у двопроменезаломлюючих волокнах. У статті застосовано два підходи до інтегрування: покращений підхід Кудряшова та новий метод проективного рівняння Ріккаті. Ці підходи дозволяють знайти розв'язки солітонів і комплексітонів, які не вдалося відновити за допомогою методу невизначених коефіцієнтів у попередніх дослідженнях. Також наведені обмеження параметрів для існування солітонів та комплексітонів.*

Ключові слова: *модель конкатенації солітонів, подвійне променезаломлення, обмеження параметрів, підхід Кудряшова, метод рівняння Ріккаті.*