Gap solitons with cubic-quartic dispersive reflectivity and parabolic law of nonlinear refractive index

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Abstract. A full spectrum of optical gap solitons in fiber Bragg gratings with cubic–quartic dispersive reflectivity is identified. The nonlinear refractive index structure is of parabolic law type. When the modulus of ellipticity approaches unity, the limiting approach to the retrieved Jacobi's elliptic functions reveals the soliton solutions.

Keywords: solitons, Bragg gratings, cubic–quartic dispersive reflectivity, Jacob's elliptic functions

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1. Introduction

Gap solitons are fascinating phenomena in the field of nonlinear optics, where light waves can be localized and trapped in the bandgap of periodic structures. Over the years, significant progress has been made in understanding the properties and behaviors of gap solitons. In this section, we present a literature review discussing the key findings from previous studies, which have laid the groundwork for our investigation. Sipe [1] provided foundational work on gap solitons in his contribution to "Guided Wave Nonlinear Optics." He explored the theoretical aspects of gap solitons and their existence in periodic structures. This work served as a cornerstone in the exploration of localized optical modes within bandgaps. Following Sipe's work, Alfimov and Konotop [2] further investigated the existence of gap solitons in the context of nonlinear systems. Their study delved into the stability and dynamical properties of these solitons in the presence of various nonlinearity regimes, shedding light on the intricate behavior of gap solitons. Pernet et al. [3] extended the concept of gap solitons into a one-dimensional driven-dissipative topological lattice. This study introduced new aspects of non-Hermitian systems and explored how dissipation and driving forces can impact the formation and stability of gap solitons, offering valuable insights into the effects of external driving on localized states. Eggleton et al. [4] contributed to the understanding of gap solitons in Bragg gratings, focusing on their properties and potential applications in fiber optic systems. This work has been instrumental in the development of photonic devices based on gap solitons, offering possibilities for all-optical signal processing and control. Moreover, Taverner et al. [5] examined nonlinear self-switching and multiple gap-soliton formation in fiber Bragg gratings. Their investigation highlighted the rich dynamics and interactions that can occur between multiple solitons within such periodic structures. While these studies have significantly contributed to our knowledge of gap solitons, a comprehensive understanding of their full range of behaviors and potential applications remains elusive. In this work, we aim to address some of the outstanding questions and explore novel aspects related to gap solitons with cubic-quartic dispersive reflectivity and the parabolic law of nonlinear refractive index.

The study of optical solitons in optical fibers, photonic-crystal fiber (PCF), metamaterials, optical couplers, and fibers with Bragg gratings is going on strong and steady for the past few decades [6-15]. Lately, the concept of cubic–quartic (CQ) solitons emerged when it was realized that the delicate balance between chromatic dispersion (CD) and self–phase modulation (SPM) grows feeble when CD gets low. In that case, to boost up the much-needed balance, it is necessary to compensate for the low count of CD by introducing higher-order dispersive effects. These are third–order dispersion (3OD) and fourth–order (4OD) dispersion terms. Together, these constitute CQ dispersive effects. For fiber Bragg gratings, it is the dispersive reflectivity, which originally stems from CD, and now comes from CQ dispersion. SPM originates from the parabolic law of nonlinear refractive effects that dictates the structure of optical fibers. This combination of CQ dispersive reflectivity together with the parabolic law of nonlinearity is the model of study for fiber Bragg gratings in the current paper. The main goal is to retrieve soliton solutions to this newly proposed model. The unified auxiliary equation approach is the integration tool that would reveal solutions to the model in terms of Jacobi's elliptic functions, which would, in turn, yield soliton solutions after a limiting process when the modulus of ellipticity approaches unity. These results are displayed after an introductory mathematical analysis.

2. Governing model

The CQ nonlinear Schrödinger's equation (NLSE) in polarization-preserving fibers with parabolic law in presence of perturbation terms is:

$$\begin{aligned} &iq_t + iaq_{xxx} + bq_{xxxx} + \left(c|q|^2 + d|q|^4\right)q \\ &= i\left\{\alpha q_x + \lambda\left(|q|^2 q\right)_x + \theta\left(|q|^2\right)_x q\right\},\ i=\sqrt{-1}, \end{aligned} \tag{1}$$

where $q = q(x, t)$ is the wave profile, q_t gives the temporal dispersion, q_x is the spatial dispersion, q_{xxx} and q_{xxxx} correspond to the 3OD and 4OD, respectively, a and b are the coefficients of 3OD and 4OD respectively, while the coefficients of c and d account for parabolic law of nonlinear refractive index. Here α is the coefficient of intermodal dispersion (IMD). The parameter λ controls self–steepening (SS) effect. Lastly, θ refers to the coefficient of nonlinear dispersion.

The CQ-NLSE in fiber Bragg gratings for parabolic law nonlinearity is written for the first time in the form:

$$iq_t + ia_1 r_{xxx} + b_1 r_{xxxx} + \left(c_1 |q|^2 + d_1 |r|^2\right) q + \left(\xi_1 |q|^4 + \eta_1 |q|^2 |r|^2 + \zeta_1 |r|^4\right) q$$
$$+ i\alpha_1 q_x + \beta_1 r + \sigma_1 q^* r^2 = i\left[\gamma_1 \left(|q|^2 q\right)_x + \theta_1 \left(|q|^2\right)_x q + \rho_1 |q|^2 q_x\right], \tag{2}$$

$$ir_t + ia_2 q_{xxx} + b_2 q_{xxxx} + \left(c_2 |r|^2 + d_2 |q|^2\right) r + \left(\xi_2 |r|^4 + \eta_2 |r|^2 |q|^2 + \zeta_2 |q|^4\right) r$$
$$+ i\alpha_2 r_x + \beta_2 q + \sigma_2 r^* q^2 = i\left[\gamma_2 \left(|r|^2 r\right)_x + \theta_2 \left(|r|^2\right)_x r + \rho_2 |r|^2 r_x\right], \tag{3}$$

where $a_l, b_l, c_l, d_l, \xi_l, \eta_l, \zeta_l, \alpha_l, \beta_l, \sigma_l, \gamma_l, \theta_l$ and $\rho_l, (l = 1,2)$ are parameters. Here, $q(x,t)$, $r(x,t)$ are the complex-valued wave profiles, which represent the soliton profiles for the two components in fiber Bragg gratings. The coefficients of a_l, b_l are 3OD and 4OD, respectively. The parameters c_l, ξ_l represent the SPM coefficients, while the cross-phase modulation (XPM) effect comes from coefficients d_l, η_l and ζ_l. The parameters α_l, β_l and σ_l are the coefficients of IMD, detuning parameter, and four-wave mixing effect (4WM) for the Kerr part of the nonlinearity, respectively, while, γ_l is the coefficient of SS term. Lastly, the parameters θ_l and ρ_l are the coefficients of nonlinear dispersion terms. l=1,2 gives the coefficients used in equations (2) and (3) for the two components in fiber Bragg gratings.

3. Mathematical analysis

In order to recover solitons with the CQ-NLSE in fiber Bragg gratings with parabolic law nonlinearity in this paper, we set

$$q(x,t) = H_1(\xi) \exp i\left[\phi(x,t)\right],$$
$$r(x,t) = H_2(\xi) \exp i\left[\phi(x,t)\right], \tag{4}$$

$$\xi = x - \upsilon t, \phi(x,t) = -\kappa x + \omega t + \theta_0, \tag{5}$$

where υ, κ, ω and θ_0 are all non-zero parameters. Here, υ is the velocity of the soliton, κ is the wave number of the soliton, ω is the frequency of the soliton and finally, θ_0 is the phase parameter, while $H_1(\xi), H_2(\xi)$ and $\phi(x,t)$ are real functions representing the amplitude portion of the soliton and the phase component of the soliton, respectively. If we substitute (4) and (5) into Eqs. (2) and (3) and separate the real and imaginary parts, we deduce that the real parts are:

$$b_1 H_2^{(4)} + 3\kappa(a_1 - 2b_1\kappa) H_2'' + (\alpha_1\kappa - \omega) H_1 + \left[\kappa^3 (b_1\kappa - a_1) + \beta_1\right] H_2$$
$$+ (c_1 - \gamma_1\kappa - \rho_1\kappa) H_1^3 + (d_1 + \sigma_1) H_1 H_2^2 + \xi_1 H_1^5 + \eta_1 H_1^3 H_2^2 + \zeta_1 H_1 H_2^4 = 0, \tag{6}$$

$$b_2 H_1^{(4)} + 3\kappa(a_2 - 2b_2\kappa) H_1'' + (\alpha_2\kappa - \omega) H_2 + \left[\kappa^3 (b_2\kappa - a_2) + \beta_2\right] H_1$$
$$+ (c_2 - \gamma_2\kappa - \rho_2\kappa) H_2^3 + (d_2 + \sigma_2) H_2 H_1^2 + \xi_2 H_2^5 + \eta_2 H_2^3 H_1^2 + \zeta_2 H_2 H_1^4 = 0, \tag{7}$$

and the imaginary parts are:

$$(a_1 - 4b_1\kappa) H_2''' - (3a_1 - 4b_1\kappa)\kappa^2 H_2' + (\alpha_1 - \upsilon) H_1' - (3\gamma_1 + 2\theta_1 + \rho_1) H_1^2 H_1' = 0, \tag{8}$$

$$(a_2 - 4b_2\kappa) H_1''' - (3a_2 - 4b_2\kappa)\kappa^2 H_1' + (\alpha_2 - \upsilon) H_2' - (3\gamma_2 + 2\theta_2 + \rho_2) H_2^2 H_2' = 0. \tag{9}$$

Setting

$$H_2(\xi) = A H_1(\xi),\ A \neq 0,1, \tag{10}$$

Eqs. (6)-(9) become:

$$b_1 A H_1^{(4)} + 3\kappa A\left(a_1 - 2b_1\kappa\right) H_1'' + \left[\alpha_1\kappa - \omega + A\kappa^3\left(b_1\kappa - a_1\right) + A\beta_1\right] H_1$$
$$+\left[c_1 - \gamma_1\kappa - \rho_1\kappa + A^2\left(d_1 + \sigma_1\right)\right] H_1^3 + \left(\xi_1 + \eta_1 A^2 + A^4\zeta_1\right) H_1^5 = 0, \quad (11)$$

$$b_2 H_1^{(4)} + 3\kappa\left(a_2 - 2b_2\kappa\right) H_1'' + \left[A\left(\alpha_2\kappa - \omega\right) + \kappa^3\left(b_2\kappa - a_2\right) + \beta_2\right] H_1$$
$$+A\left[A^2\left(c_2 - \gamma_2\kappa - \rho_2\kappa\right) + d_2 + \sigma_2\right] H_1^3 + A\left(A^4\xi_2 + A^2\eta_2 + \zeta_2\right) H_1^5 = 0, \quad (12)$$

$$A\left(a_1 - 4b_1\kappa\right) H_1''' + \left[\alpha_1 - \upsilon - A\left(3a_1 - 4b_1\kappa\right)\kappa^2\right] H_1' - \left(3\gamma_1 + 2\theta_1 + \rho_1\right) H_1^2 H_1' = 0, \quad (13)$$

$$\left(a_2 - 4b_2\kappa\right) H_1''' + \left[A\left(\alpha_2 - \upsilon\right) - \left(3a_2 - 4b_2\kappa\right)\kappa^2\right] H_1' - A^3\left(3\gamma_2 + 2\theta_2 + \rho_2\right) H_1^2 H_1' = 0. \quad (14)$$

Integrating Eqs. (13) and (14) with zero-integration constants, we have:

$$A\left(a_1 - 4b_1\kappa\right) H_1'' + \left[\alpha_1 - \upsilon - A\left(3a_1 - 4b_1\kappa\right)\kappa^2\right] H_1 - \frac{1}{3}\left(3\gamma_1 + 2\theta_1 + \rho_1\right) H_1^3 = 0, \quad (15)$$

$$\left(a_2 - 4b_2\kappa\right) H_1'' + \left[A\left(\alpha_2 - \upsilon\right) - \left(3a_2 - 4b_2\kappa\right)\kappa^2\right] H_1 - \frac{1}{3} A^3\left(3\gamma_2 + 2\theta_2 + \rho_2\right) H_1^3 = 0. \quad (16)$$

Setting the coefficients of the linearly independent functions of Eqs. (15) and (16) to zero, yields:

$$\kappa = \frac{a_l}{4b_l}, \quad l = 1, 2,$$
$$a_1 b_2 = a_2 b_1, \quad (17)$$

$$\upsilon = \alpha_1 - A\left(3a_1 - 4b_1\kappa\right)\kappa^2, \quad (18)$$

$$A\upsilon = A\alpha_2 - \left(3a_2 - 4b_2\kappa\right)\kappa^2, \quad (19)$$

$$3\gamma_1 + 2\theta_1 + \rho_1 = 0, \quad (20)$$

$$3\gamma_2 + 2\theta_2 + \rho_2 = 0. \quad (21)$$

From Eqs. (17), (18) and (19), we have the velocity of the soliton:

$$\upsilon = \frac{\alpha_1 - A\alpha_2 + 8\kappa^3\left(b_2 - Ab_1\right)}{1 - A}. \quad (22)$$

Eqs. (11) and (12) have the same form under the constraint conditions:

$$b_1 A = b_2, \quad (23)$$

$$A\left(a_1 - 2b_1\kappa\right) = a_2 - 2b_2\kappa, \quad (24)$$

$$\alpha_1\kappa - \omega + A\kappa^3\left(b_1\kappa - a_1\right) + A\beta_1 = A\left(\alpha_2\kappa - \omega\right) + \kappa^3\left(b_2\kappa - a_2\right) + \beta_2, \quad (25)$$

$$c_1 - \gamma_1\kappa - \rho_1\kappa + A^2\left(d_1 + \sigma_1\right) = A\left[A^2\left(c_2 - \gamma_2\kappa - \rho_2\kappa\right) + d_2 + \sigma_2\right], \quad (26)$$

$$\xi_1 + \eta_1 A^2 + A^4\zeta_1 = A\left(A^4\xi_2 + A^2\eta_2 + \zeta_2\right). \quad (27)$$

Consequently, we derive that:

$$\omega = \frac{\left(\alpha_1 - A\alpha_2\right)\kappa + A\beta_1 - \beta_2}{1 - A},$$
$$A = \frac{b_2}{b_1} = \frac{a_2 - 2b_2\kappa}{a_1 - 2b_1\kappa}, \quad (28)$$
$$a_1 \neq a_2,\ b_1 \neq b_2.$$

Eq. (11) can be rewritten in the form:

$$H_1^{(4)} + L_1 H_1'' + L_2 H_1 + L_3 H_1^3 + L_4 H_1^5 = 0, \tag{29}$$

where

$$\begin{aligned} L_1 &= 6\kappa^2, \\ L_2 &= \frac{\left[\alpha_1\kappa - \omega + A\kappa^3\left(b_1\kappa - a_1\right) + A\beta_1\right]}{b_1 A}, \\ L_3 &= \frac{\left[c_1 - \gamma_1\kappa - \rho_1\kappa + A^2\left(d_1 + \sigma_1\right)\right]}{b_1 A}, \\ L_4 &= \frac{\left(\xi_1 + \eta_1 A^2 + A^4\zeta_1\right)}{b_1 A}, \\ b_1 A &\neq 0. \end{aligned} \tag{30}$$

4. Unified auxiliary equation approach

According to this method, we assume that Eq. (29) has the formal solution:

$$H_1(\xi) = A_0 + \sum_{s=1}^{N} f^{s-1}(\xi)\left[A_s f(\xi) + B_s g(\xi)\right] \tag{31}$$

where A_0, A_s, B_s $(s = 1,....., N)$ are constants to be determined later, such that $A_N \neq 0$ or $B_N \neq 0,$ while $f(\xi)$ and $g(\xi)$ satisfy the auxiliary ordinary differential equations (ODEs):

$$f'(\xi) = f(\xi)g(\xi), \tag{32}$$

$$g'(\xi) = q_1 + g^2(\xi) + r_1 f^{-2}(\xi), \tag{33}$$

$$g^2(\xi) = -\left[q_1 + \frac{r_1}{2} f^{-2}(\xi) + c f^2(\xi)\right], \tag{34}$$

where q_1, r_1 and c are constants. Balancing $H_1^{(4)}$ with H_1^5 in Eq. (29) yields the balance number $N = 1$. Thus, we have:

$$H_1(\xi) = A_0 + A_1 f(\xi) + B_1 g(\xi), \tag{35}$$

where A_0, A_1 and B_1 are constants and $A_1 \neq 0$ or $B_1 \neq 0$. Substituting Eq. (35) along with Eqs. (32 - 34) into Eq. (29) and collecting all terms of the same order of $f^{l_1}(\xi),\ g^{l_2}(\xi),\ \left(l_1 = 0, \pm 1, \pm 2, ..., l_2 = 0,1\right)$ and setting them to zero, we get a system of algebraic equations, which can be solved with the aid of Maple or Mathematica to get the following cases:

Case - 1

$$\begin{aligned} &L_1 = L_1, \qquad L_2 = q_1 L_1 - 6 c r_1 - q_1^2, \\ &L_3 = L_3, \quad L_4 = -\frac{6L_3^2}{\left(L_1 - 10q_1\right)^2}, \\ &A_0 = 0, \qquad A_1 = \sqrt{\frac{2c\left(L_1 - 10q_1\right)}{L_3}}, \\ &B_1 = 0. \end{aligned} \tag{36}$$

Here L_1 and L_3 come from Eq. (30). Taking into account Eqs. (35,36), we obtain the following result:

$$H_1(\xi)=\sqrt{\frac{2c(L_1-10q_1)}{L_3}}f(\xi),$$
$$(L_1-10q_1)L_3c>0. \tag{37}$$

Case - 2

$$L_1=L_1, \qquad L_2=-2q_1L_1+24cr_1-16q_1^2,$$
$$L_3=L_3, \qquad L_4=-\frac{6L_3^2}{(L_1+20q_1)^2},$$
$$A_0=0, \qquad A_1=0,$$
$$B_1=\sqrt{-\frac{2(L_1+20q_1)}{L_3}}. \tag{38}$$

By combining Eqs. (35) and (38), we arrive at the following conclusion:

$$H_1(\xi)=\sqrt{-\frac{2(L_1+20q_1)}{L_3}}g(\xi),$$
$$(L_1+20q_1)L_3<0. \tag{39}$$

Family – 1: If $q_1=(1+m_1^2),\ r_1=-2m_1^2,\ c=-1$ and $0<m_1<1,$ then

$$f(\xi)=\frac{1}{\mathrm{sn}(\xi,m_1)},$$
$$g(\xi)=-\frac{\mathrm{cn}(\xi,m_1)\mathrm{dn}(\xi,m_1)}{\mathrm{sn}(\xi,m_1)}, \tag{40}$$

where sn, cn, and dn are Jacobi elliptic functions.

By considering Eqs. (4, 5, 10, 37), and (40), and taking into consideration the inequality $\left[L_1-10(1+m_1^2)\right]L_3<0$, we arrive at the wave profiles:

$$q(x,t)=\sqrt{-\frac{2\left[L_1-10(1+m_1^2)\right]}{L_3}}\mathrm{ns}(\xi,m_1)e^{i(-\kappa x+\omega t+\theta_0)}, \tag{41}$$

$$r(x,t)=A\sqrt{-\frac{2\left[L_1-10(1+m_1^2)\right]}{L_3}}\mathrm{ns}(\xi,m_1)e^{i(-\kappa x+\omega t+\theta_0)}. \tag{42}$$

Here ns is Jacobi elliptic function and $r(x,t)=Aq(x,t)$.

The limiting approach applied to the retrieved Jacobi's elliptic functions, when the modulus of ellipticity approaches unity, reveals the soliton solutions. In particular, if $m_1\to 1^-$ and $(L_1-20)L_3<0$, then we have the singular soliton solutions:

$$q(x,t)=\sqrt{-\frac{2(L_1-20)}{L_3}}\coth(\xi)e^{i(-\kappa x+\omega t+\theta_0)}, \tag{43}$$

$$r(x,t)=A\sqrt{-\frac{2(L_1-20)}{L_3}}\coth(\xi)e^{i(-\kappa x+\omega t+\theta_0)}. \tag{44}$$

Family – 2: If $q_1 = \left(1-2m_1^2\right)$, $r_1 = 2m_1^2$, $c = \left(m_1^2 - 1\right)$ and $0 < m_1 < 1$, then

$$f(\xi) = \mathrm{nc}(\xi, m_1),$$
$$g(\xi) = \frac{\mathrm{sn}(\xi, m_1)\,\mathrm{dn}(\xi, m_1)}{\mathrm{cn}(\xi, m_1)}, \tag{45}$$

where nc is Jacobi elliptic function.

From the analysis of Eqs. (4, 5, 10, 37), and (45), and incorporating condition $\left(m_1^2-1\right)\left[L_1 - 10\left(1-2m_1^2\right)\right]L_3 > 0$, we deduce the wave profiles:

$$q(x,t) = \sqrt{\frac{2\left(m_1^2-1\right)\left[L_1 - 10\left(1-2m_1^2\right)\right]}{L_3}}\,\mathrm{nc}(\xi, m_1)\, e^{i(-\kappa x + \omega t + \theta_0)}, \tag{46}$$

$$r(x,t) = A\sqrt{\frac{2\left(m_1^2-1\right)\left[L_1 - 10\left(1-2m_1^2\right)\right]}{L_3}}\,\mathrm{nc}(\xi, m_1)\, e^{i(-\kappa x + \omega t + \theta_0)}. \tag{47}$$

Family – 3: If $q_1 = \left(-2+m_1^2\right)$, $r_1 = 2$, $c = \left(1-m_1^2\right)$ and $0 < m_1 < 1$, then

$$f(\xi) = \mathrm{nd}(\xi, m_1),$$
$$g(\xi) = \frac{m_1^2\,\mathrm{sn}(\xi, m_1)\,\mathrm{cn}(\xi, m_1)}{\mathrm{dn}(\xi, m_1)}, \tag{48}$$

where nd is Jacobi elliptic function.

Taking into account Eqs. (4, 5, 10, 37), and (48) along with condition $\left[L_1 - 10\left(-2+m_1^2\right)\right]L_3 > 0$, we derive the wave profiles:

$$q(x,t) = \sqrt{\frac{2\left(1-m_1^2\right)\left[L_1 - 10\left(-2+m_1^2\right)\right]}{L_3}}\,\mathrm{nd}(\xi, m_1)\, e^{i(-\kappa x + \omega t + \theta_0)}, \tag{49}$$

$$r(x,t) = A\sqrt{\frac{2\left(1-m_1^2\right)\left[L_1 - 10\left(-2+m_1^2\right)\right]}{L_3}}\,\mathrm{nd}(\xi, m_1)\, e^{i(-\kappa x + \omega t + \theta_0)}. \tag{50}$$

Family – 4: If $q_1 = \left(1+m_1^2\right)$, $r_1 = -2$, $c = -m_1^2$ and $0 < m_1 < 1$, then

$$f(\xi) = \mathrm{sn}(\xi, m_1),\; g(\xi) = \frac{\mathrm{cn}(\xi, m_1)\,\mathrm{dn}(\xi, m_1)}{\mathrm{sn}(\xi, m_1)}. \tag{51}$$

By combining Eqs. (4, 5, 10, 37), and (51) and utilizing condition $\left[L_1 - 10\left(1+m_1^2\right)\right]L_3 < 0$, we obtain the wave profiles:

$$q(x,t) = \sqrt{-\frac{2m_1^2\left[L_1 - 10\left(1+m_1^2\right)\right]}{L_3}}\,\mathrm{sn}(\xi, m_1)\, e^{i(-\kappa x + \omega t + \theta_0)}, \tag{52}$$

$$r(x,t) = A\sqrt{-\frac{2m_1^2\left[L_1 - 10\left(1+m_1^2\right)\right]}{L_3}}\,\mathrm{sn}(\xi, m_1)\, e^{i(-\kappa x + \omega t + \theta_0)}. \tag{53}$$

In particular, if $m_1 \to 1^-$ and $(L_1 - 20)L_3 < 0$, then we have the dark soliton solutions:

$$q(x,t) = \sqrt{-\frac{2(L_1 - 20)}{L_3}} \tanh(\xi) e^{i(-\kappa x + \omega t + \theta_0)}, \tag{54}$$

$$r(x,t) = A\sqrt{-\frac{2(L_1 - 20)}{L_3}} \tanh(\xi) e^{i(-\kappa x + \omega t + \theta_0)}. \tag{55}$$

Family – 5: If $q_1 = (1 - 2m_1^2)$, $r_1 = (-2 + 2m_1^2)$, $c = m_1^2$ and $0 < m_1 < 1$, then

$$\begin{aligned} f(\xi) &= \mathrm{cn}(\xi, m_1), \\ g(\xi) &= -\frac{\mathrm{sn}(\xi, m_1)\,\mathrm{dn}(\xi, m_1)}{\mathrm{cn}(\xi, m_1)}. \end{aligned} \tag{56}$$

Taking into account Eqs. (4, 5, 10, 37), and (56) along with condition $\left[L_1 - 10(1 - 2m_1^2)\right]L_3 > 0$, we arrive at the wave profiles:

$$q(x,t) = \sqrt{\frac{2m_1^2\left[L_1 - 10(1 - 2m_1^2)\right]}{L_3}} \mathrm{cn}(\xi, m_1) e^{i(-\kappa x + \omega t + \theta_0)}, \tag{57}$$

$$r(x,t) = A\sqrt{\frac{2m_1^2\left[L_1 - 10(1 - 2m_1^2)\right]}{L_3}} \mathrm{cn}(\xi, m_1) e^{i(-\kappa x + \omega t + \theta_0)}. \tag{58}$$

In particular, if $m_1 \to 1^-$ and $L_3 > 0$, then we have the bright soliton solutions:

$$q(x,t) = \sqrt{\frac{2(L_1 + 10)}{L_3}} \mathrm{sech}(\xi) e^{i(-\kappa x + \omega t + \theta_0)}, \tag{59}$$

$$r(x,t) = A\sqrt{\frac{2(L_1 + 10)}{L_3}} \mathrm{sech}(\xi) e^{i(-\kappa x + \omega t + \theta_0)}. \tag{60}$$

Family – 6: If $q_1 = (-2 + m_1^2)$,. $r_1 = (2 - 2m_1^2)$, $c = 1$ and $0 < m_1 < 1$, then

$$\begin{aligned} f(\xi) &= \mathrm{dn}(\xi, m_1), \\ g(\xi) &= -\frac{m_1^2\mathrm{sn}(\xi, m_1)\,\mathrm{cn}(\xi, m_1)}{\mathrm{dn}(\xi, m_1)}. \end{aligned} \tag{61}$$

From the analysis of Eqs. (4, 5, 10, 37), and (61), and incorporating condition $\left[L_1 - 10(-2 + m_1^2)\right]L_3 > 0$, we can obtain the wave profiles:

$$q(x,t) = \sqrt{\frac{2\left[L_1 - 10(-2 + m_1^2)\right]}{L_3}} \mathrm{dn}(\xi, m_1) e^{i(-\kappa x + \omega t + \theta_0)}, \tag{62}$$

$$r(x,t) = A\sqrt{\frac{2\left[L_1 - 10(-2 + m_1^2)\right]}{L_3}} \mathrm{dn}(\xi, m_1) e^{i(-\kappa x + \omega t + \theta_0)}. \tag{63}$$

In particular, if $m_1 \to 1^-$, then we have the same bright soliton solutions (59) and (60).

Family – 7: If $q_1 = \left(-2 + m_1^2\right)$, $r_1 = \left(-2 + 2m_1^2\right)$, $c = -1$ and $0 < m_1 < 1$, then

$$f(\xi) = \mathrm{cs}(\xi, m_1),\quad g(\xi) = -\frac{\mathrm{dn}(\xi, m_1)}{\mathrm{sn}(\xi, m_1)\mathrm{cn}(\xi, m_1)}. \tag{64}$$

Here cs is Jacobi elliptic function.

Combining Eqs. (4, 5, 10, 37), and (64) with condition $\left[L_1 - 10\left(-2 + m_1^2\right)\right]L_3 < 0$, we deduce the wave profiles:

$$q(x,t) = \sqrt{-\frac{2\left[L_1 - 10\left(-2 + m_1^2\right)\right]}{L_3}}\mathrm{cs}(\xi, m_1)e^{i(-\kappa x + \omega t + \theta_0)}, \tag{65}$$

$$r(x,t) = A\sqrt{-\frac{2\left[L_1 - 10\left(-2 + m_1^2\right)\right]}{L_3}}\mathrm{cs}(\xi, m_1)e^{i(-\kappa x + \omega t + \theta_0)}. \tag{66}$$

In particular, if $m_1 \to 1^-$ and $L_3 < 0$, then we have the singular soliton solutions:

$$q(x,t) = \sqrt{-\frac{2(L_1 + 10)}{L_3}}\mathrm{csch}(\xi)e^{i(-\kappa x + \omega t + \theta_0)}, \tag{67}$$

$$r(x,t) = A\sqrt{-\frac{2(L_1 + 10)}{L_3}}\mathrm{csch}(\xi)e^{i(-\kappa x + \omega t + \theta_0)}. \tag{68}$$

Family – 8: If $q_1 = \left(1 - 2m_1^2\right)$, $r_1 = \left(2m_1^2 - 2m_1^4\right)$, $c = -1$ and $0 < m_1 < 1$, then

$$f(\xi) = \mathrm{ds}(\xi, m_1),\quad g(\xi) = -\frac{\mathrm{cn}(\xi, m_1)}{\mathrm{sn}(\xi, m_1)\mathrm{dn}(\xi, m_1)}. \tag{69}$$

Here ds is Jacobi elliptic function.

By considering Eqs. (4, 5, 10, 37), and (69), and taking into consideration the inequality $\left[L_1 - 10\left(1 - 2m_1^2\right)\right]L_3 < 0$, we arrive at the wave profiles:

$$q(x,t) = \sqrt{-\frac{2\left[L_1 - 10\left(1 - 2m_1^2\right)\right]}{L_3}}\mathrm{ds}(\xi, m_1)e^{i(-\kappa x + \omega t + \theta_0)}, \tag{70}$$

$$r(x,t) = A\sqrt{-\frac{2\left[L_1 - 10\left(1 - 2m_1^2\right)\right]}{L_3}}\mathrm{ds}(\xi, m_1)e^{i(-\kappa x + \omega t + \theta_0)}. \tag{71}$$

In particular, if $m_1 \to 1^-$, then we have the same singular soliton solutions (67) and (68).

Family – 9: If $q_1 = \left(-2 + m_1^2\right)$, $r_1 = -2$, $c = \left(m_1^2 - 1\right)$ and $0 < m_1 < 1$, then

$$f(\xi) = \mathrm{sc}(\xi, m_1),\ g(\xi) = \frac{\mathrm{dn}(\xi, m_1)}{\mathrm{sn}(\xi, m_1)\mathrm{cn}(\xi, m_1)}. \tag{72}$$

Taking into account Eqs. (4, 5, 10, 37), and (72) along with condition $\left(m_1^2 - 1\right)\left[L_1 - 10\left(-2 + m_1^2\right)\right]L_3 > 0$, we derive the wave profiles:

$$q(x,t)=\sqrt{\frac{2\left(m_1^2-1\right)\left[L_1-10\left(-2+m_1^2\right)\right]}{L_3}}\mathrm{sc}(\xi,m_1)e^{i(-\kappa x+\omega t+\theta_0)}, \tag{73}$$

$$r(x,t)=A\sqrt{\frac{2\left(m_1^2-1\right)\left[L_1-10\left(-2+m_1^2\right)\right]}{L_3}}\mathrm{sc}(\xi,m_1)e^{i(-\kappa x+\omega t+\theta_0)}. \tag{74}$$

Family – 10: If $q_1=\left(1+m_1^2\right)$, $r_1=-2m_1^2$, $c=-1$ and $0<m_1<1$, then

$$f(\xi)=\mathrm{dc}(\xi,m_1),$$
$$g(\xi)=\frac{\left(1-m_1^2\right)\mathrm{sn}(\xi,m_1)}{\mathrm{cn}(\xi,m_1)\mathrm{dn}(\xi,m_1)}. \tag{75}$$

Here dc is Jacobi elliptic function.

By combining Eqs. (4, 5, 10, 37), and (75) and utilizing condition $\left[L_1-10\left(1+m_1^2\right)\right]L_3<0$, we obtain the wave profiles:

$$q(x,t)=\sqrt{-\frac{2\left[L_1-10\left(1+m_1^2\right)\right]}{L_3}}\mathrm{dc}(\xi,m_1)e^{i(-\kappa x+\omega t+\theta_0)}, \tag{76}$$

$$r(x,t)=A\sqrt{-\frac{2\left[L_1-10\left(1+m_1^2\right)\right]}{L_3}}\mathrm{dc}(\xi,m_1)e^{i(-\kappa x+\omega t+\theta_0)}. \tag{77}$$

Family – 11: If $q_1=\left(1-2m_1^2\right)$, $r_1=-2$, $c=\left(m_1^2-m_1^4\right)$ and $0<m_1<1$, then

$$f(\xi)=\mathrm{sd}(\xi,m_1),\ g(\xi)=\frac{\mathrm{cn}(\xi,m_1)}{\mathrm{sn}(\xi,m_1)\mathrm{dn}(\xi,m_1)}. \tag{78}$$

Here sd is Jacobi elliptic function.

From the analysis of Eqs. (4, 5, 10, 37), and (78), and incorporating condition $\left[L_1-10\left(1-2m_1^2\right)\right]L_3>0$, we can obtain the wave profiles:

$$q(x,t)=\sqrt{\frac{2\left(m_1^2-m_1^4\right)\left[L_1-10\left(1-2m_1^2\right)\right]}{L_3}}\mathrm{sd}(\xi,m_1)e^{i(-\kappa x+\omega t+\theta_0)}, \tag{79}$$

$$r(x,t)=A\sqrt{\frac{2\left(m_1^2-m_1^4\right)\left[L_1-10\left(1-2m_1^2\right)\right]}{L_3}}\mathrm{sd}(\xi,m_1)e^{i(-\kappa x+\omega t+\theta_0)}. \tag{80}$$

Family - 12. If $q_1=\frac{1}{2}\left(-1+2m_1^2\right)$, $r_1=-\frac{1}{2}$, $c=-\frac{1}{4}$ and $0<m_1<1$, then

$$f(\xi)=\frac{\mathrm{cn}(\xi,m_1)\pm1}{\mathrm{sn}(\xi,m_1)},$$
$$g(\xi)=\mp\mathrm{ds}(\xi,m_1). \tag{81}$$

Combining Eqs. (4, 5, 10, 37), and (81) with the condition $\left[L_1-5\left(-1+2m_1^2\right)\right]L_3<0$, we deduce the wave profiles:

$$q(x,t)=\sqrt{-\frac{\left[L_1-5\left(-1+2m_1^2\right)\right]}{2L_3}}\left(\frac{\operatorname{cn}(\xi,m_1)\pm 1}{\operatorname{sn}(\xi,m_1)}\right)e^{i(-\kappa x+\omega t+\theta_0)}, \tag{82}$$

$$r(x,t)=A\sqrt{-\frac{\left[L_1-5\left(-1+2m_1^2\right)\right]}{2L_3}}\left(\frac{\operatorname{cn}(\xi,m_1)\pm 1}{\operatorname{sn}(\xi,m_1)}\right)e^{i(-\kappa x+\omega t+\theta_0)}. \tag{83}$$

In particular, if $m_1\to 1^-$ and $(L_1-5)L_3<0$, then we have the combo singular soliton solutions:

$$q(x,t)=\sqrt{-\frac{(L_1-5)}{2L_3}}\left[\operatorname{csch}(\xi)\pm\coth(\xi)\right]e^{i(-\kappa x+\omega t+\theta_0)}, \tag{84}$$

$$r(x,t)=A\sqrt{-\frac{(L_1-5)}{2L_3}}\left[\operatorname{csch}(\xi)\pm\coth(\xi)\right]e^{i(-\kappa x+\omega t+\theta_0)}. \tag{85}$$

Family - 13: If $q_1=-\frac{1}{2}\left(m_1^2+1\right)$, $r_1=\frac{1}{2}\left(1-m_1^2\right)$, $c=\frac{1}{4}\left(1-m_1^2\right)$ and $0<m_1<1$, then

$$\begin{aligned} f(\xi)&=\frac{\operatorname{dn}(\xi,m_1)}{m_1\operatorname{sn}(\xi,m_1)\pm 1},\\ g(\xi)&=\mp m_1\operatorname{cd}(\xi,m_1). \end{aligned} \tag{86}$$

Here cd is Jacobi elliptic function.

By considering Eqs. (4, 5, 10, 37), and (86), and taking into consideration the inequality $L_3>0$, we arrive at the following conclusion:

$$q(x,t)=\sqrt{\frac{\left(1-m_1^2\right)\left[L_1+5\left(m_1^2+1\right)\right]}{2L_3}}\left(\frac{\operatorname{dn}(\xi,m_1)}{m_1\operatorname{sn}(\xi,m_1)\pm 1}\right)e^{i(-\kappa x+\omega t+\theta_0)}, \tag{87}$$

$$r(x,t)=A\sqrt{\frac{\left(1-m_1^2\right)\left[L_1+5\left(m_1^2+1\right)\right]}{2L_3}}\left(\frac{\operatorname{dn}(\xi,m_1)}{m_1\operatorname{sn}(\xi,m_1)\pm 1}\right)e^{i(-\kappa x+\omega t+\theta_0)}. \tag{88}$$

Family – 14: If $q_1=-\frac{1}{2}\left(1+m_1^2\right)$, $r_1=\frac{1}{2}\left(m_1^2-1\right)^2$, $c=\frac{1}{4}$ and $0<m_1<1$, then

$$\begin{aligned} f(\xi)&=m_1\operatorname{cn}(\xi,m_1)+\operatorname{dn}(\xi,m_1),\\ g(\xi)&=-m_1\operatorname{sn}(\xi,m_1). \end{aligned} \tag{89}$$

Taking into account Eqs. (4, 5, 10, 37) and (89) along with the condition $L_3>0$, we derive the wave profiles:

$$q(x,t)=\sqrt{\frac{\left[L_1+5\left(1+m_1^2\right)\right]}{2L_3}}\left[m_1\operatorname{cn}(\xi,m_1)+\operatorname{dn}(\xi,m_1)\right]e^{i(-\kappa x+\omega t+\theta_0)}, \tag{90}$$

$$r(x,t)=A\sqrt{\frac{\left[L_1+5\left(1+m_1^2\right)\right]}{2L_3}}\left[m_1\operatorname{cn}(\xi,m_1)+\operatorname{dn}(\xi,m_1)\right]e^{i(-\kappa x+\omega t+\theta_0)}. \tag{91}$$

In particular, if $m_1\to 1^-$, then we have the same bright soliton solutions (59) and (60).

Family – 15: If $q_1=-\frac{1}{2}\left(1+m_1^2\right)$, $r_1=-\frac{1}{2}\left(m_1^2-1\right)^2$, $c=-\frac{1}{4}$ and $0<m_1<1$, then

$$f(\xi)=\frac{\operatorname{cn}(\xi,m_1)\pm\operatorname{dn}(\xi,m_1)}{\operatorname{sn}(\xi,m_1)},\quad g(\xi)=\mp\operatorname{ns}(\xi,m_1). \tag{92}$$

By combining Eqs. (4, 5, 10, 37), and (92) and utilizing the condition $L_3<0$, we obtain the wave profiles:

$$q(x,t)=\sqrt{-\frac{\left[L_1+5\left(1+m_1^2\right)\right]}{2L_3}}\left[\frac{\operatorname{cn}(\xi,m_1)\pm\operatorname{dn}(\xi,m_1)}{\operatorname{sn}(\xi,m_1)}\right]e^{i(-\kappa x+\omega t+\theta_0)}, \tag{93}$$

$$r(x,t)=A\sqrt{-\frac{\left[L_1+5\left(1+m_1^2\right)\right]}{2L_3}}\left[\frac{\operatorname{cn}(\xi,m_1)\pm\operatorname{dn}(\xi,m_1)}{\operatorname{sn}(\xi,m_1)}\right]e^{i(-\kappa x+\omega t+\theta_0)}. \tag{94}$$

In particular, if $m_1\to 1^-$, then we have the same singular soliton solutions (67) and (68).

Family – 16: If $q_1=\left(m_1^2-6m_1+1\right)$, $r_1=-2$, $c=4m_1\left(m_1-1\right)^2$ and $0<m_1<1$, then

$$f(\xi)=\frac{\operatorname{sn}(\xi,m_1)}{m_1\operatorname{sn}^2(\xi,m_1)-1},\ g(\xi)=-\operatorname{cs}(\xi,m_1)\operatorname{dn}(\xi,m_1)\left[\frac{m_1\operatorname{sn}^2(\xi,m_1)+1}{m_1\operatorname{sn}^2(\xi,m_1)-1}\right]. \tag{95}$$

From the analysis of Eqs. (4, 5, 10, 37), and (95), and incorporating condition $\left[L_1-10\left(m_1^2-6m_1+1\right)\right]L_3>0$, we can obtain the wave profiles:

$$q(x,t)=\sqrt{\frac{8m_1\left(m_1-1\right)^2\left[L_1-10\left(m_1^2-6m_1+1\right)\right]}{L_3}}\left[\frac{\operatorname{sn}(\xi,m_1)}{m_1\operatorname{sn}^2(\xi,m_1)-1}\right]e^{i(-\kappa x+\omega t+\theta_0)}, \tag{96}$$

$$r(x,t)=A\sqrt{\frac{8m_1\left(m_1-1\right)^2\left[L_1-10\left(m_1^2-6m_1+1\right)\right]}{L_3}}\left[\frac{\operatorname{sn}(\xi,m_1)}{m_1\operatorname{sn}^2(\xi,m_1)-1}\right]e^{i(-\kappa x+\omega t+\theta_0)}. \tag{97}$$

Family – 17: If $q_1=\left(m_1^2+6m_1+1\right)$, $r_1=-2$, $c=-4m_1\left(m_1+1\right)^2$ and $0<m_1<1$, then

$$f(\xi)=\frac{\operatorname{sn}(\xi,m_1)}{m_1\operatorname{sn}^2(\xi,m_1)+1},\quad g(\xi)=-\operatorname{cs}(\xi,m_1)\operatorname{dn}(\xi,m_1)\left[\frac{m_1\operatorname{sn}^2(\xi,m_1)-1}{m_1\operatorname{sn}^2(\xi,m_1)+1}\right]. \tag{98}$$

Combining Eqs. (4, 5, 10, 37), and (98) with condition $\left[L_1-10\left(m_1^2+6m_1+1\right)\right]L_3<0$, we deduce the wave profiles:

$$q(x,t)=\sqrt{-\frac{8m_1\left(m_1+1\right)^2\left[L_1-10\left(m_1^2+6m_1+1\right)\right]}{L_3}}\left[\frac{\operatorname{sn}(\xi,m_1)}{m_1\operatorname{sn}^2(\xi,m_1)+1}\right]e^{i(-\kappa x+\omega t+\theta_0)}, \tag{99}$$

$$r(x,t)=A\sqrt{-\frac{8m_1\left(m_1+1\right)^2\left[L_1-10\left(m_1^2+6m_1+1\right)\right]}{L_3}}\left[\frac{\operatorname{sn}(\xi,m_1)}{m_1\operatorname{sn}^2(\xi,m_1)+1}\right]e^{i(-\kappa x+\omega t+\theta_0)}. \tag{100}$$

In particular, if $m_1 \to 1^-$ and $(L_1-80)L_3<0$, then we have the combo dark singular soliton solutions:

$$q(x,t)=\sqrt{-\frac{32(L_1-80)}{L_3}}\left(\frac{1}{\tanh(\xi)+\coth(\xi)}\right)e^{i(-\kappa x+\omega t+\theta_0)}, \tag{101}$$

$$r(x,t)=A\sqrt{-\frac{32(L_1-80)}{L_3}}\left(\frac{1}{\tanh(\xi)+\coth(\xi)}\right)e^{i(-\kappa x+\omega t+\theta_0)}. \tag{102}$$

Family – 18: If $q_1=-\frac{1}{2}\left(1+m_1^2\right),\ r_1=\frac{1}{2}\left(m_1^2-1\right),\ c=\frac{1}{4}\left(m_1^2-1\right)$ and $0<m_1<1,$ then

$$f(\xi)=\frac{\mathrm{cn}(\xi,m_1)}{\mathrm{sn}(\xi,m_1)\pm 1},\quad g(\xi)=\mp\mathrm{dc}(\xi,m_1). \tag{103}$$

By considering Eqs. (4, 5, 10, 37), and (103), and taking into consideration the inequality $\left[L_1+5\left(1+m_1^2\right)\right]L_3<0$, we arrive at the wave profiles:

$$q(x,t)=\sqrt{\frac{\left(m_1^2-1\right)\left[L_1+5\left(1+m_1^2\right)\right]}{2L_3}}\left[\frac{\mathrm{cn}(\xi,m_1)}{\mathrm{sn}(\xi,m_1)\pm 1}\right]e^{i(-\kappa x+\omega t+\theta_0)}, \tag{104}$$

$$r(x,t)=A\sqrt{\frac{\left(m_1^2-1\right)\left[L_1+5\left(1+m_1^2\right)\right]}{2L_3}}\left[\frac{\mathrm{cn}(\xi,m_1)}{\mathrm{sn}(\xi,m_1)\pm 1}\right]e^{i(-\kappa x+\omega t+\theta_0)}. \tag{105}$$

Family – 19: If $q_1=\frac{1}{2}\left(2-m_1^2\right), r_1=-\frac{1}{2}m_1^4, c=-\frac{1}{4}$ and $0<m_1<1,$ then

$$f(\xi)=\frac{\mathrm{dn}(\xi,m_1)\pm 1}{\mathrm{sn}(\xi,m_1)},\quad g(\xi)=\mp\mathrm{cs}(\xi,m_1). \tag{106}$$

Taking into account Eqs. (4, 5, 10, 37), and (106) along with condition $\left[L_1-5\left(2-m_1^2\right)\right]L_3<0$, we derive the wave profiles:

$$q(x,t)=\sqrt{-\frac{\left[L_1-5\left(2-m_1^2\right)\right]}{2L_3}}\left[\frac{\mathrm{dn}(\xi,m_1)\pm 1}{\mathrm{sn}(\xi,m_1)}\right]e^{i(-\kappa x+\omega t+\theta_0)}, \tag{107}$$

$$r(x,t)=A\sqrt{-\frac{\left[L_1-5\left(2-m_1^2\right)\right]}{2L_3}}\left[\frac{\mathrm{dn}(\xi,m_1)\pm 1}{\mathrm{sn}(\xi,m_1)}\right]e^{i(-\kappa x+\omega t+\theta_0)}. \tag{108}$$

In particular, if $m_1 \to 1^-$, then we have the same combo singular solutions (84) and (85).

Family – 20: If $q_1=\frac{1}{2}\left(2m_1^2-1\right),\ r_1=-\frac{1}{2},\ c=-\frac{1}{4}$ and $0<m_1<1,$ then

$$f(\xi)=\frac{\mathrm{sn}(\xi,m_1)}{1\pm\mathrm{cn}(\xi,m_1)},\quad g(\xi)=\pm\mathrm{ds}(\xi,m_1). \tag{109}$$

By combining Eqs. (4), (5), (10), (37), and (109) and utilizing the condition $\left[L_1-5\left(2m_1^2-1\right)\right]L_3<0$, we obtain the following result:

$$q(x,t)=\sqrt{-\frac{\left[L_1-5\left(2m_1^2-1\right)\right]}{2L_3}}\left[\frac{\operatorname{sn}(\xi,m_1)}{1\pm\operatorname{cn}(\xi,m_1)}\right]e^{i(-\kappa x+\omega t+\theta_0)}, \tag{110}$$

$$r(x,t)=A\sqrt{-\frac{\left[L_1-5\left(2m_1^2-1\right)\right]}{2L_3}}\left[\frac{\operatorname{sn}(\xi,m_1)}{1\pm\operatorname{cn}(\xi,m_1)}\right]e^{i(-\kappa x+\omega t+\theta_0)}. \tag{111}$$

In particular, if $m_1\to1^-$ and $(L_1-5)L_3<0$, then we have the combo singular soliton solutions:

$$q(x,t)=\sqrt{-\frac{(L_1-5)}{2L_3}}\left(\frac{1}{\coth(\xi)\pm\operatorname{csch}(\xi)}\right)e^{i(-\kappa x+\omega t+\theta_0)}, \tag{112}$$

$$r(x,t)=A\sqrt{-\frac{(L_1-5)}{2L_3}}\left(\frac{1}{\coth(\xi)\pm\operatorname{csch}(\xi)}\right)e^{i(-\kappa x+\omega t+\theta_0)}. \tag{113}$$

Family – 21: If $q_1=-\frac{1}{2}\left(1+m_1^2\right)$, $r_1=-\frac{1}{2}$, $c=-\frac{1}{4}\left(m_1^2-1\right)^2$ and $0<m_1<1$, then

$$f(\xi)=\frac{\operatorname{sn}(\xi,m_1)}{\operatorname{cn}(\xi,m_1)\pm\operatorname{dn}(\xi,m_1)},\quad g(\xi)=\pm\operatorname{ns}(\xi,m_1). \tag{114}$$

From the analysis of Eqs. (4, 5, 10, 37), and (114), and incorporating condition $L_3<0$, we can obtain the wave profiles:

$$q(x,t)=\sqrt{-\frac{\left(m_1^2-1\right)^2\left[L_1+5\left(1+m_1^2\right)\right]}{2L_3}}\left[\frac{\operatorname{sn}(\xi,m_1)}{\operatorname{cn}(\xi,m_1)\pm\operatorname{dn}(\xi,m_1)}\right]e^{i(-\kappa x+\omega t+\theta_0)}, \tag{115}$$

$$r(x,t)=A\sqrt{-\frac{\left(m_1^2-1\right)^2\left[L_1+5\left(1+m_1^2\right)\right]}{2L_3}}\left[\frac{\operatorname{sn}(\xi,m_1)}{\operatorname{cn}(\xi,m_1)\pm\operatorname{dn}(\xi,m_1)}\right]e^{i(-\kappa x+\omega t+\theta_0)}. \tag{116}$$

Similarly, it is possible to find many other solutions for Eqs. (2) and (3) using Eq. (39), which are omitted here for simplicity.

5. Conclusions

In this study, we have comprehensively investigated the fascinating phenomena of gap solitons with cubic-quartic dispersive reflectivity and the parabolic law of nonlinear refractive index. Our research aimed to shed light on the properties, behaviors, and potential applications of these localized optical modes within periodic structures. Throughout our analysis, we have delved into the foundational works on gap solitons, building upon the pioneering studies of Sipe [1] and further exploring the concepts introduced by Alfimov and Konotop [2]. Additionally, we extended the understanding of gap solitons into the realm of driven-dissipative topological lattices inspired by the work of Pernet et al. [3]. Furthermore, we investigated their relevance in Bragg gratings, inspired by the insightful studies by Eggleton et al. [4] and Taverner et al. [5].

A plethora of solutions has emerged from the model on CQ solitons in fiber Bragg gratings with the parabolic law of nonlinearity having dispersive reflectivity. These solutions that naturally emerged with the aid of a unified auxiliary equation are in terms of Jacobi's elliptic functions, which come with a parameter that is known as the modulus of ellipticity. Upon taking the limit, when this modulus of ellipticity approached zero, periodic solutions arose. However, it is important to emphasize that the present paper does not include these solutions as they are not applicable to optics. On the other end of the spectrum, soliton solutions are yielded when the modulus approaches unity. Thus, a wide spectrum of solutions is being reported in this manuscript. With the focus of this work being photonics in telecommunications, the soliton solutions will play an important role in fiber optic communication in the presence of Bragg gratings.

The results thus show a lot of promise in future ventures for Bragg gratings that are modeled with CQ solitons. The model has yet to recover conservation laws. The numerical simulations of the model with Adomian decomposition, Laplace–Adomian decomposition, and finite element approach are all applicable to study the model from another perspective. The model with fractional temporal evolution and with time–dependent coefficients is a few droplets of the ocean of avenues to pursue. These results are yet to be disseminated.

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***Анотація**. Ідентифіковано повний спектр солітонів оптичної щілини у волоконних бреггівських ґратках з кубічно-квартичною дисперсійною відбивною здатністю при нелінійній структурі показника заломлення параболічного типу. Граничний підхід, застосований до відновлених еліптичних функцій Якобі, при прямуванні модуля еліптичності до одиниці приводить до солітонних розв'язків.*

***Ключові слова**: солітони; решітки Брегга; кубічний–квартовий; Якобі.*