Relationships among the parameters of sea-surface waves and underwater caustics caused by sunlight

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Abstract. We use the wave theory of light to study the brightness and the geometrical characteristics of bright stripes appearing on the bottom of a pool. The brightness of those stripes is linked to the distribution of refracted-light intensity in the vicinity of a caustics where the ray-optics approximation is inapplicable. The caustics arises whenever light is refracted on a wavy water surface. The relationships among the parameters of the surface waves and the width of the bright stripes (i.e., the caustic zone) are obtained. The correctness of our relationships is verified by the experiment carried out in a water pool. Our formulae can be used to develop the optical systems for determining the wave parameters (in particular, the sea-surface curvature on large scales) by recording the bright-stripes characteristics.

Keywords: surface waves, bright stripes, underwater caustics, caustic-zone width, surface curvature.

UDC: 535.4

1. Introduction

When waves appear on a surface of a water pool, a set of moving bright stripes can be observed at its bottom. The shape and the texture of these stripes depend on the regularity of surface waves. The clearer the water and the more regular the waves, the clearer and more ordered the movements of these bright stripes are. The reason for formation of the bright stripes is the refraction of a parallel beam of Sun rays on a rough (curved) water surface. The bright stripes are formed in the caustic directions, where the intensity of the refracted-light beam calculated in the geometrical-optics approximation goes to infinity. In this study we will apply the method for calculating the light intensity in the vicinity of a caustics for the reflected rays, which has been described in Ref. [1], to solve a similar problem for the refracted rays.

The caustics arising from the reflection and refraction are of the same nature. They are associated with infinitely large solutions arising along the direction of rainbows when one calculates the light scattering by a sphere [2] with the geometrical-optics approximation. Moreover, the latter singularities disappear whenever the wave nature of the light (i.e., the phase effects) is taken into account.

In optics, the caustic curve (or surface) is generally defined as an envelope of the rays reflected or refracted by a curved surface of a material medium. In other words, the reflected or refracted light rays are tangent lines at every point of the caustic curve. Then the envelope of the light rays is a curve where the rays are concentrated.

A wide variety of optical phenomena related to the caustics have been studied in a new branch of mathematics called a catastrophe theory [3]. Note also that the caustics are used in caustic engineering which describes the process of solving the inverse problems in computer graphics. The latter problems imply determining the surface which refracts or reflects light and thus forms a given image from the image characteristics [4]. The methods based upon neural networks are usually applied in this case, which creates an aesthetic picture of a caustics instead of a strict restoration of the surface that refracts the rays.

Although underwater caustics are frequently observed at the bottom of different water bodies (e.g., shallow waters of lakes or seas), still there are no strict and explicit mathematical expressions that relate the parameters of the waves and those of the caustics. The aim of the present work is to derive the above relationships. For simplicity, we will consider a two-dimensional case, when the appropriate formulae are relatively simple and a visual representation of the effect is evident.

2. Basic formulae

Let a curved cylindrical surface described by the equation $z=\zeta(x)$ in the Cartesian coordinate system xOz be irradiated by a parallel beam of light incident along the direction of the unit vector $\vec{s}_0=(s_{0x},s_{0z})$ (see Fig. 1). Then the unit vector of the refracted ray $\vec{s}_1=(s_{1x},s_{1z})$ determined by the law of refraction is expressed by the formula

$$\vec{s}_1=\frac{1}{m_w}\vec{s}_0+\left(\frac{1}{m_w}\cos\chi_0-\cos\chi_1\right)\cdot\vec{n}\,. \tag{1}$$

Here χ_0 and χ_1 are the local angles of respectively incidence and refraction, $\vec{n}$ denotes the normal at the point $M=(x,\zeta(x))$ of refraction, and m_w is the refractive index of water. For the incident-beam geometry and the coordinate system chosen in Fig. 1 we have the formulae

$$\theta_0=const,\theta_1=\theta_1(x),\ \theta_n=\theta_n(x)=-\arctan\zeta'(x),\chi_0=\chi_0(x),\ \chi_1=\chi_1(x)\,, \tag{2}$$

$$\vec{s}_0=const,\ \vec{s}_1=\vec{s}_1(x)\,,\ \ \vec{n}=\vec{n}(x)=\frac{-\zeta'(x)\vec{i}+\vec{k}}{\sqrt{1+\zeta'^2(x)}}\,, \tag{3}$$

with $\zeta'(x)=\dfrac{d\zeta(x)}{dx}$. Here $\vec{i}$ and $\vec{k}$ are the unit vectors of the xOz and xOz axes, respectively.

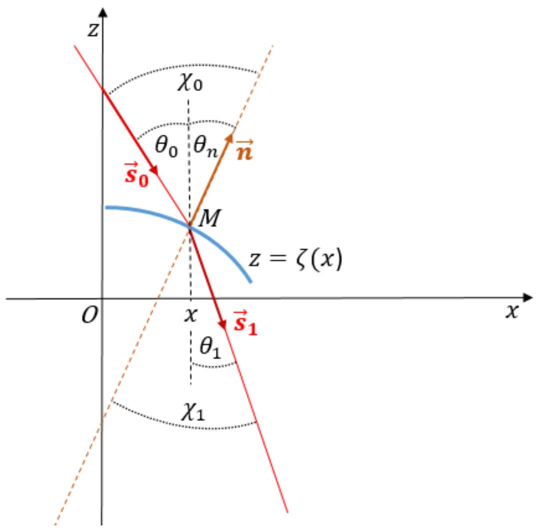

Fig. 1. Geometry of bright refraction at a rough 2D surface.

Let us find the divergence of the ray bundle after refraction occurring at the surface. Assume that the elementary section of the front of the incident plane light wave is given by M_0N at the moment $t=0$ and it transforms into Q_0Q after the time t has passed (see Fig. 2). If we introduce the notation

$NM = \delta(x)$ and $l(x) = MQ$ and the speed of light in air and water are given respectively by c and v, the relation $\frac{\delta(x)}{c} + \frac{l(x)}{v} = t$ should take place. By denoting $l_0 = l(x_0) = M_0Q_0$, we obtain

$$\delta(x) = s_{0x}(x - x_0) + s_{0z}(\zeta(x) - \zeta(x_0)), \tag{4}$$

$$l(x) = l_0 - \frac{\delta(x)}{m_w}, \tag{5}$$

where the refractive index of water is given by $m_w = \frac{c}{v} = 1.34$.

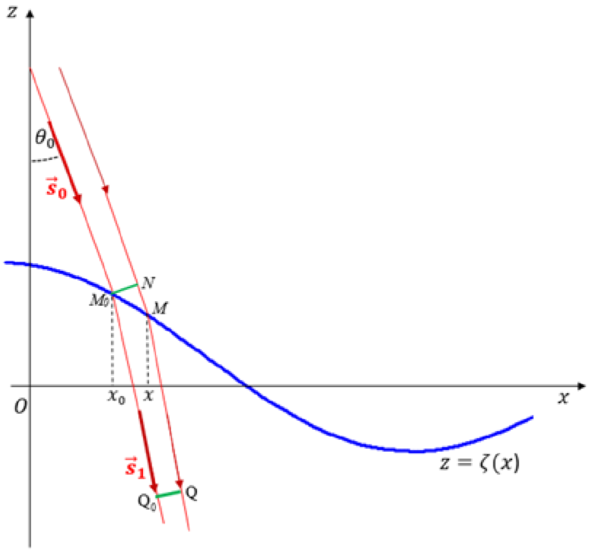

Fig. 2. Determination of the front of a refracted bright wave.

Now let us write out the parametric equation of the wave front at the distance l_0 from a fixed point $M_0 = (x_0, \zeta(x_0))$ at the surface. The equation of the wave front passing through the point Q_0 acquires the form

$$\begin{cases} X(x) = x + s_{1x}(x) \cdot l(x) \\ Z(x) = \zeta(x) + s_{1z}(x) \cdot l(x) \end{cases}, \tag{6}$$

where $X(x)$ and $Z(x)$ are the coordinates of the point Q. Then the divergence of the bundle of rays after refraction at the point M_0 reads as

$$\Gamma_1 = \frac{|Q_0Q|}{|M_0N|} = \frac{d\sigma_1}{d\sigma_0} = \frac{d\sigma_1}{dx}\frac{dx}{d\sigma_0} = \frac{\cos\theta_n}{\cos\chi_0}\frac{d\sigma_1}{dx} = \frac{\cos\theta_n}{\cos\chi_0}\sqrt{X'^2 + Z'^2}. \tag{7}$$

Using Eqs. (6) and (8)–(16), one gets

$$\frac{\sin\chi_0}{\sin\chi_1} = m_w, \tag{8}$$

$$\cos\chi_0 = \frac{s_{0x}\zeta'(x) - s_{0z}}{\sqrt{1 + \zeta'^2(x)}}, \quad \sin\chi_0 = \frac{s_{0x} + s_{0z}\zeta'(x)}{\sqrt{1 + \zeta'^2(x)}}, \quad \zeta'^2(x) = \left[\frac{d\zeta(x)}{dx}\right]^2, \tag{9}$$

$$\cos\theta_n = n_z = \frac{1}{\sqrt{1 + \zeta'^2(x)}}, \quad \sin\theta_n = n_x = \frac{-\zeta'(x)}{\sqrt{1 + \zeta'^2(x)}}, \tag{10}$$

$$\chi_0(x)=\theta_0+\theta_n(x),\ \ \chi_1(x)=\theta_1(x)+\theta_n(x)\ ,\ \frac{d\chi_0}{dx}=\frac{d\theta_n}{dx}=-\frac{\zeta''(x)}{1+\zeta'^2(x)}\ , \tag{11}$$

$$\theta_1(x)=\theta_0+\chi_1(x)-\chi_0(x)\ ,\ \frac{d\theta_1}{dx}=\frac{d\chi_1}{dx}-\frac{d\chi_0}{dx}\ , \tag{12}$$

$$\frac{d\theta_1}{dx}=\left(\frac{1}{m_w}\frac{\cos\chi_0}{\cos\chi_1}-1\right)\frac{d\chi_0}{dx}=\left(\frac{1}{m_w}\frac{\cos\chi_0}{\cos\chi_1}-1\right)\frac{d\theta_n}{dx}\ , \tag{13}$$

$$s_{0x}=\sin\theta_0,\ s_{0z}=-\cos\theta_0\ ,s_{1x}=\sin\theta_1\ ,s_{1z}=-\cos\theta_1\ , \tag{14}$$

$$s'_{1x}=-s_{1z}\cdot\theta'_1\ \ ,\ \ s'_{1z}=s_{1x}\cdot\theta'_1\ , \tag{15}$$

$$\rho=\frac{\left(1+\zeta'^2\right)^{3/2}}{\zeta''}\ ,\ \theta'_1=\left(\frac{\cos\chi_0}{m_w\cos\chi_1}-1\right)\theta'_n=\left(\frac{\cos\chi_0}{m_w\cos\chi_1}-1\right)\frac{\sqrt{1+\zeta'^2}}{-\rho}\ . \tag{16}$$

Basing on these relations, we obtain

$$X'^2+Z'^2=1+s'^2_{1x}l^2+s^2_{1x}l'^2+2s'_{1x}\,l+2s_{1x}l'+2s_{1x}s'_{1x}\,ll'$$
$$+\zeta'^2+s'^2_{1z}l^2+s^2_{1z}l'^2+2\zeta'\,s'_{1z}\,l+2\zeta' s_{1z}l'+2s_{1z}s'_{1x}\,ll'\ .$$

Let us take into account the relations

$$s_{1x}s'_{1x}+s_{1z}s'_{1z}=0\ ,\ s^2_{1x}+s^2_{1z}=1\ ,\ s_{1x}+s_{1z}\zeta'=\frac{1}{m}\left(s_{0x}+s_{0z}\zeta'\right),$$

$$-s_{1z}+s_{1x}\,\zeta'=-\frac{1}{m_w}\left(s_{0z}-s_{0x}\,\zeta'\right)-\left(\frac{1}{m_w}\cos\chi_0-\cos\chi_1\right)\sqrt{1+\zeta'^2}\ .$$

Then the formula

$$\begin{aligned}
X'^2+Z'^2&=1+\zeta'^2+l^2\,\theta'^2_1+\frac{1}{m_w{}^2}\left(s_{0x}+s_{0z}\zeta'\right)^2\\
&-2l\left(\frac{1}{m_w}\left(s_{0x}+s_{0z}\zeta'\right)+\left(\frac{1}{m_w}\cos\chi_0-\cos\chi_1\right)\sqrt{1+\zeta'^2}\right)\theta'_1-\frac{2}{m_w{}^2}\left(s_{0x}+s_{0z}\zeta'\right)^2\\
&=1+\zeta'^2+l^2\,\theta'^2_1-\frac{1}{m_w{}^2}\left(s_{0x}+s_{0z}\zeta'\right)^2\\
&-2l\left[\frac{1}{m_w}\left(s_{0x}+s_{0z}\zeta'\right)+\left(\frac{1}{m_w}\cos\chi_0-\cos\chi_1\right)\sqrt{1+\zeta'^2}\right]\theta'_1\\
&=1+\zeta'^2+l^2\,\theta'^2_1-\frac{\left(1+\zeta'^2\right)}{m_w{}^2}\sin^2\chi_0+2l\theta'_1\frac{\cos\chi_0}{m_w}\sqrt{1+\zeta'^2}\\
&-2l\left(\frac{1}{m_w}\cos\chi_0-\cos\chi_1\right)\theta'_1\sqrt{1+\zeta'^2}\\
&=\left(1+\zeta'^2\right)\left(1-\frac{\sin^2\chi_0}{m_w{}^2}\right)+l^2\,\theta'^2_1+2l\theta'_1\,\cos\chi_1\sqrt{1+\zeta'^2}\\
&=\left(1+\zeta'^2\right)\cos^2\chi_1+l^2\,\theta'^2_1+2l\theta'_1\,\cos\chi_1\sqrt{1+\zeta'^2}=\left(\sqrt{1+\zeta'^2}\cos\chi_1+l\theta'_1\right)^2
\end{aligned}$$

follows immediately. As a result, we arrive at the formula

$$\sqrt{X'^2+Z'^2}=\sqrt{1+\zeta'^2}\left|\cos\chi_1-\left(\frac{\cos\chi_0}{m_w\cos\chi_1}-1\right)\cdot\frac{l}{\rho}\right|\ . \tag{17}$$

The divergence Γ_1 of the beam refracted at the point M_0 at the distance $l_0 = l(x_0)$ reads as

$$\Gamma_1 = \frac{\cos\chi_1}{\cos\chi_0}\left|1-\left(\frac{\cos\chi_0}{m_w\,\cos\chi_1}-1\right)\cdot\frac{l_0}{\rho_0\cos\chi_1}\right|, \tag{18}$$

with $\rho_0 = \rho(x_0)$.

In agreement with the law of energy conservation under condition of no dissipation, the sum of the energies of the reflected and transmitted light waves must be equal to that of the incident one. Then the intensity $I_1(Q_0)$ of the refracted beam at the point Q_0 is determined from the relation $T(\chi_0)I_0 d\sigma_0 = I_1(Q_0)d\sigma_1$:

$$I_1(Q_0) = \frac{T(\chi_0)I_0}{\Gamma_1}, \tag{19}$$

where I_0 implies the intensity of the incident beam at the point M_0 and $T(\chi_0)$ is the Fresnel transmission coefficient [5]. If the modulus of the radius of curvature is equal to $|\rho_0| \ll l_0$, Eqs. (18) and (19) result in

$$\Gamma_1 = \left(1-\frac{\cos\chi_0}{m_w\,\cos\chi_1}\right)\cdot\frac{l_0}{|\rho_0|\cos\chi_0}, \tag{20}$$

$$I_1(Q_0) = \frac{T(\chi_0)}{\left(1-\dfrac{\cos\chi_0}{m_w\,\cos\chi_1}\right)}\cdot\frac{I_0}{l_0}\cdot|\rho_0|\cos\chi_0. \tag{21}$$

As follows from Eq. (20), we have the divergence $\Gamma_1 = 0$ at the refraction point M_* where the radius of curvature is equal to $\rho_* = \rho(x_*) = \infty$. Therefore the intensity of the refracted beam $I(Q_0)$ becomes infinite at the infinitely distant point Q_0 given by Eq. (21). It is seen from Eq. (16) that the radius of curvature amounts to $\rho_* = \infty$ at the inflection point M_* where $\zeta''(x_*) = 0$. In this case, the inclination of the surface θ_n and the deflection of the refracted ray θ_1 take their extreme values, $\theta_n'(x_*) = 0$ and $\theta_1'(x_*) = 0$. In other words, the refracted rays are concentrated around the ray refracted at the inflection point, which leads to infinite intensity (i.e., a caustics). Note that the symbol * is used to indicate the caustic point and the quantities related to it.

To find the intensity distribution $I_1(\Delta\theta_1) = I_1(Q)$ in the vicinity of the caustics, we use the Fresnel–Kirchhoff formula for the plane problem [6]:

$$U_1(Q) = \sqrt{\frac{ik_1}{2\pi}}\int\limits_{(F)} U_F(P)\frac{\exp(-ik_1 r)}{\sqrt{r}}\,d\sigma. \tag{22}$$

Here $U_1(Q)$ and $U_F(P)$ are the field values of the light waves respectively at the observation point Q and the point P of the surface (F), i.e. the wave front passes through the point $M_* = (x_*, \zeta(x_*))$. As for the rest of the notation, $k_1 = m_w k$ is the wave number of the light wave in water (with $k = \dfrac{2\pi}{\lambda}$ being the wave number of the light wave in air) and $r = |PQ|$ denotes the distance between the points P and Q (see Fig. 3).

The light intensity $I_1(Q)$ can be expressed in terms of $U_1(Q)$ according to the formula $I_1(Q)=U_1(Q)\cdot U_1^*(Q)$, where $U_1^*(Q)$ is a conjugate of $U_1(Q)$. Under the condition when the part of the wave front essential for the integration does not contain the inflection point, the calculation of the integral given by Eq. (22) by the stationary-phase method leads to a usual expression given by Eq. (21), which corresponds to the geometrical optics. In case when the observation point Q is located in the vicinity of the caustics, i.e. when the angle $\Delta\theta_1$ between the caustic direction and the direction of observation is small, the part of the wave front that makes a significant contribution to the integral would contain the inflection point.

To simplify the calculation of the integral, we choose a new coordinate system $\tilde{x}M_*\tilde{z}$ as shown in Fig. 3, so that the axis $M_*\tilde{z}$ is directed along the caustic direction $\vec{s}_1^{\,*}$. Then the equation of the wave front (F) in the coordinate system $\tilde{x}M_*\tilde{z}$ can be written as

$$\begin{cases} \tilde{X}(x)=-\left(X(x)-x_*\right)s_{1z}^*+\left(Z(x)-\zeta_*\right)s_{1x}^* \\ \tilde{Z}(x)=\left(X(x)-x_*\right)s_{1x}^*+\left(Z(x)-\zeta_*\right)s_{1z}^* \end{cases}.$$

In the neighbourhood of $\tilde{X}=0$, the Taylor expansion of (F) is given by

$$\tilde{Z}(\tilde{X})=\tilde{Z}(0)+\tilde{Z}'(0)\tilde{X}+\frac{\tilde{Z}''(0)}{2!}\tilde{X}^2+\frac{\tilde{Z}'''(0)}{3!}\tilde{X}^3+\dots . \tag{24}$$

Taking into account the relations $\tilde{Z}(0)=0$, $\tilde{Z}'(0)=0$ and $\tilde{Z}''(0)=0$, we arrive at the result

$$\tilde{Z}(\tilde{X})\approx b\tilde{X}^3, \tag{25}$$

where $b=\dfrac{\tilde{Z}'''(0)}{3!}$ and $\tilde{Z}'''(0)=\dfrac{d^3\tilde{Z}(\tilde{X}=0)}{d\tilde{X}^3}=\dfrac{\tilde{Z}'''(x_*)}{\tilde{X}'^3(x_*)}$. Using Eq. (23), one can find that $\tilde{X}_*'=-s_{1z}^*+s_{1x}^*\zeta_*'$ and $\tilde{Z}_*'''=\left(s_{1z}^*-\dfrac{1}{m_w}s_{0z}\right)\zeta_*'''$. As a consequence, we have

$$b=\frac{1}{6}\frac{\left(s_{1z}^*-\dfrac{1}{m_w}s_{0z}\right)\zeta_*'''}{\left(-s_{1z}^*+s_{1x}^*\zeta_*'\right)^3}. \tag{26}$$

The parameter b defines the behaviour of the refracted light-wave front in the neighbourhood of the point M_*. Performing these calculations for the intensity distribution $I_1(\Delta\theta_1)=I_1(Q)$ in the vicinity of the caustics, we finally obtain

$$I_1(\Delta\theta_1)=\frac{2\pi TI_0}{l_0}\frac{k_1^{1/3}}{(3b)^{2/3}}Ai^2\left(-\left(\frac{k_1^2}{3b}\right)^{1/3}\Delta\theta_1\right), \tag{27}$$

where $Ai(t)=\dfrac{1}{\pi}\displaystyle\int_0^\infty \cos\left(tx+\frac{x^3}{3}\right)dx$ is the Airy function [7] and

$$t=-\sqrt[3]{\frac{k_1^2}{3b}}\Delta\theta_1 . \tag{28}$$

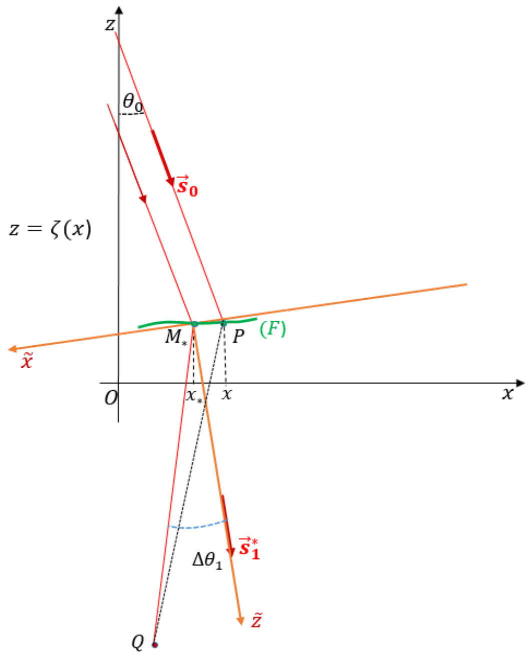

Fig. 3. Definition of a new coordinate system $\tilde{x}M_*\tilde{z}$ (see the text).

Thus, now we are in a position to identify the angle $\Delta\theta_1^c$ determining the angular width of the caustic zone. Its meaning is that the calculations of $I_1(Q)$ should be based on either Eq. (27) or Eq. (21) when we deal with the angles $\Delta\theta_1 < \Delta\theta_1^c$ or $\Delta\theta_1 \geq \Delta\theta_1^c$, respectively. In the caustic direction, the two rays appearing during refraction at the points located on the left and right of the caustic point are merged. Therefore, equating the doubled intensity in Eq. (21) with the intensity entering in Eq. (27) leads to the equality

$$\frac{1}{\sqrt[3]{3}\sqrt{2t}\ \pi} = Ai^2(t)\ . \tag{29}$$

The behaviours of the functions entering in the left- and right-hand sides of Eq. (29) are shown in Fig. 4. One can see that Eq. (29) has a number of roots. The first root, at which the signs of the slopes of the functions are the same, is given by $t_c = 1.79$. It lies in between the principal maximum and the first zero of the Airy function. Then the value $t_c = 1.79$ would determine the width of the caustic zone. This value agrees well with the estimations of the caustic zone found earlier for the analogical problems [2, 8, 9]. Moreover, the correctness of the value $t_c = 1.79$ chosen by us is confirmed by the equality for the energy flows:

$$\int_0^{t_c} \frac{1}{\sqrt[3]{3\pi}\sqrt{2t}}\,dt \approx \int_{-\infty}^{t_c} Ai^2(t)\,dt \ \Rightarrow\ \frac{\sqrt{2}}{\sqrt[3]{3\pi}}\sqrt{t_c} \approx \int_{-\infty}^{t_c} Ai^2(t)\,dt\ . \tag{30}$$

The inaccuracy of the equality given by Eq. (30) is about 11%.

Hence, we obtain the formula

$$\Delta\theta_1^c = t_c\left(\frac{3b}{k_1^2}\right)^{\frac{1}{3}} \tag{31}$$

for the angular width of the caustic zone $\Delta\theta_1^c$. Note that a more accurate calculation with the

Fresnel–Kirchhoff formula (22) gives a transition from the wave optics to the geometrical optics in the vicinity of $t_c = 1.79$, as shown in Fig. 4 by the green line.

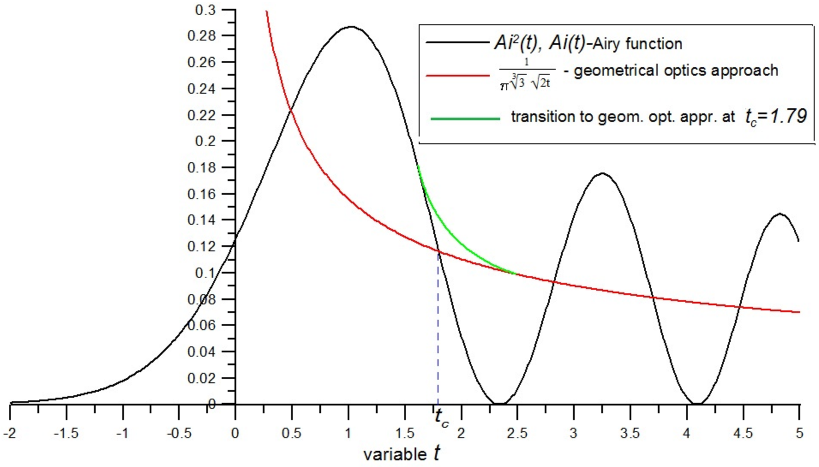

Fig. 4. Matching of wave-theory and geometrical-optics approaches (see the text).

We emphasize that Eq. (31) has been obtained for the parallel light beams. When the light source is the Sun, of which rays manifest a small angular divergence (about $\Delta\theta_0 = 0.5^0$), the corresponding angular increment of the caustic zone $\Delta\theta_1^0$ can be evaluated from the relation

$$\theta_1 = \theta_0 + \chi_1 - \chi_0 \ . \tag{32}$$

Assuming that the increments of the parameters θ_1, χ_0 and χ_1 at the caustic point $x_0 = x_*$ are caused by the increment $\Delta\theta_0$, we have

$$\Delta\theta_1^0 = \Delta\theta_0 + \Delta\chi_1^* - \Delta\chi_0^* \ . \tag{33}$$

From the relation $\chi_0^* = \theta_0 + \theta_n^*$ and Eq. (8) taken at the caustic point, we obtain

$$\Delta\chi_0^* = \Delta\theta_0 \quad , \ \left(\cos\chi_0^*\right)\Delta\chi_0^* = m_w\left(\cos\chi_1^*\right)\Delta\chi_1^* \ . \tag{34}$$

Taking into account the equality $\Delta\theta_1^0 = \Delta\chi_1^*$, one gets

$$\Delta\theta_1^0 = \frac{\cos\chi_0^*}{m_w \cos\chi_1^*}\Delta\theta_0 \, . \tag{35}$$

Some angular increment $\Delta\theta_1^d$ of the caustic zone can also occur due to the dispersion of light, i.e. due to the corresponding change in the refractive index Δm_w. It can be estimated from Eqs. (8) and (32) at the caustic point, while putting $\theta_0 = const$, $\chi_0 = const$ and $\Delta\theta_1^d = \Delta\chi_1^d$. In this case we obtain

$$\Delta\theta_1^d = -\frac{\sin\chi_0^*}{\cos\chi_1^*}\frac{\Delta m_w}{m_w{}^2} \, . \tag{36}$$

Finally, we have the formula

$$\Delta\theta_1^* = \Delta\theta_1^c + \Delta\theta_1^0 + \left|\Delta\theta_1^d\right| \tag{37}$$

for the angular width $\Delta\theta_1^*$ of the caustic zone. Consequently, the linear ‘size’ (or width) ΔL of the bright strip can be evaluated as follows:

$$\Delta L = \frac{h}{\cos\theta_1^*}\Delta\theta_1^* . \tag{38}$$

3. Model calculations

To reveal the essence of the caustic effect, we consider the refraction of a parallel light beam on the surface $z = \zeta(x) = a\cos Kx$, where a is the amplitude and $K = \frac{2\pi}{\Lambda}$ the wave number of the surface wave. Fig. 5 shows the paths of the light beams for the case of $a = 1$ and $K = 1$ at the incidence angle $\theta_0 = 0^0$. At the inflection points with the negative and positive slopes, the caustics appear in the directions given respectively by $\vec{s}_{1-}^*$ and $\vec{s}_{1+}^*$.

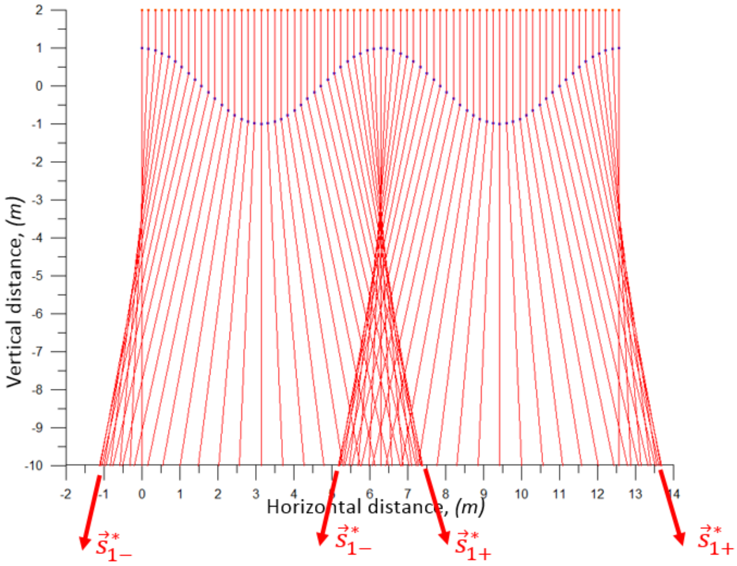

Fig. 5. Paths of the rays refracted at the surface $z = \cos x$, as calculated for the case of vertical incidence of a parallel light beam.

4. Natural experiment

Below we test experimentally the possibilities for retrieving the parameters of water-surface waves from the parameters of bright underwater stripes. A pool with the length, the width and the depth amounting respectively to 8, 4 and 1 m, has been filled with clean water (0.8 m thick). Almost sinusoidal waves with the amplitude $a = 0.045$ m and the wavelength $\Lambda = 0.52$ m have been generated mechanically, under practically no-wind condition.

At the time of our experiment, the zenith angle θ_0 of the Sun has been equal to 18^0 and the rays have fallen parallel to the side walls of the pool, i.e. perpendicular to the ridge of the generated surface wave (see Fig. 6). An instantaneous image of the bright underwater stripes has been captured by a camera with high spatial and temporal resolutions, from the height of 2 m above a calm water surface with a vertical sighting. Consequently, the waves can be considered as ‘frozen’ and the water environment under the waves can be taken as a part of a ‘lens of complex shape’.

With this formulation of the experiment, the problem under consideration is reduced to a two-dimensional one, so that utilization of the formulae obtained above is fully justified. The calculations yield in $\Delta\theta_1^c = 0.0066^0$, $\Delta\theta_1^0 = 0.31^0$, $|\Delta\theta_1^d| = 0.28^0$ and $\Delta\theta_1^* = 0.59^0$. Therefore we obtain $\Delta L = 0.82\,\text{cm}$ for the width of the bright strip. A comparison with the experimental width of the bright stripe seen from the photo in Fig. 6 testifies a good agreement, especially when considering that the side of the square of the checkerboard cell is 4 cm.

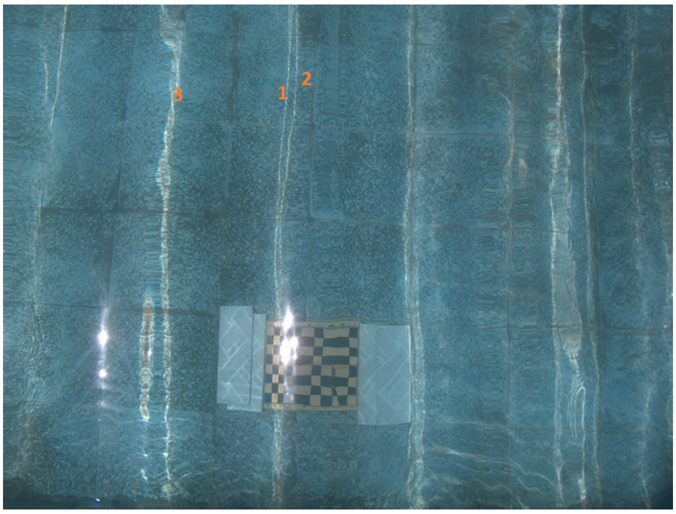

Fig. 6. Image of bright underwater stripes caused by an almost sinusoidal surface wave with the amplitude $a = 0.045\,\text{m}$ and the wavelength $\Lambda = 0.52\,\text{m}$. The stripes 1, 2 and 3 correspond to the points 1, 2 and 3 in Fig. 7.

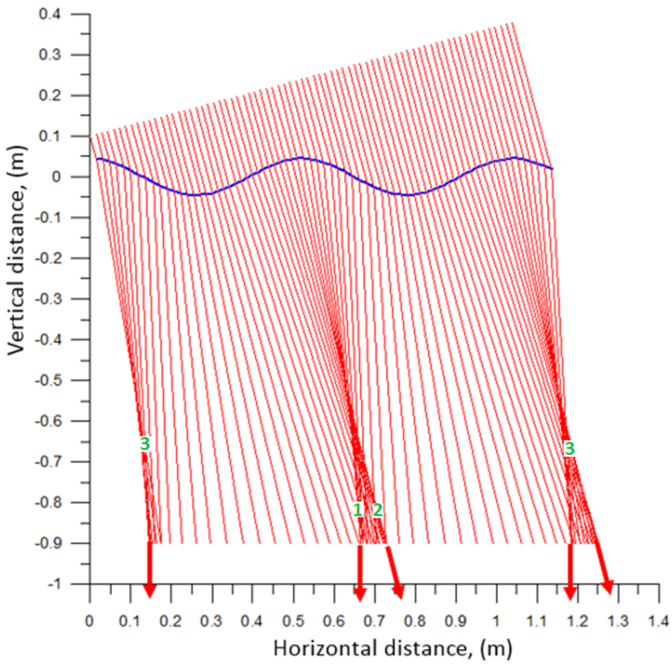

Fig. 7. Paths of the rays refracted at the surface $z = 0.045\cos 12.8x$, as calculated for the case of incidence of a parallel light beam at the angle $\theta_0 = 18^0$.

The paths of the refracted rays corresponding to our experiment (see Fig. 6) are indicated in Fig. 7. As seen from Fig. 7, there is an interval of depths (in the vicinity of the point 3) where the bright stripes are formed due to focusing of the refracted rays on the surface convexity. At the point 3, the divergence of the refracted beam determined by Eq. (18) is equal to zero, i.e. the equality

$$1-\left(\frac{\cos\chi_0}{m_w\cos\chi_1}-1\right)\cdot\frac{l_0}{\rho_0\cos\chi_1}=0 \tag{39}$$

holds true. Eq. (39) links the 'reduced focal length of the convex 2D lens' l_0 with the radius of curvature of the lens surface ρ_0 at the point $\left(x_0,\zeta\left(x_0\right)\right)$. It is easily seen that the distance between the bright stripes 1 and 3 equals approximately to the wavelength $\Lambda = 0.52$ m.

Notice that we consider the aquatic environment to be transparent, i.e. the scattering and the absorption of light in an aqueous medium is neglected. The theory of image formation through a wavy-sea surface, which is based on the optical-transfer function, is described in detail in the monograph [10], where the attenuations of the light beam caused by the scattering and the absorption are taken into account. However, the approach developed in Ref. [10] is applicable to accumulated images, in contrast to the instantaneous images we consider above. Despite the majority of problems concerned with the atmosphere–ocean surface interactions, common maritime works and navigation rules require the knowledge of only statistical characteristics of the waved-sea surface. On the other hand, the solution of some problems (e.g., recovering the instantaneous images of underwater objects distorted by surface waves [11]) demands the knowledge of the instantaneous state of the waved-sea surface. As shown in Ref. [11], the instantaneous relief of the sea surface can be constructed using the characteristics of glints of the Sun (or the other light sources). Then the glints as 'thumbprints' can identify uniquely the relief of the waved-sea surface. Using the statistical characteristics of the glints, one can also determine the statistical characteristics of the reflected-light intensity [12, 13]. Issuing from our results, a method based on recording a network of the bright underwater stripes can also be developed in order to determine both the instantaneous relief of the waved-sea surface and its characteristics.

5. Conclusion

In this work, we have derived the relationships among the parameters of harmonic water-surface waves and the width of the bright stripes on the surface (i.e., the underwater caustics). The correctness of our relationships has been verified by the experiments carried out with the real pool. The width of the bright stripes calculated with Eq. (38) and that measured in the experiment are in good agreement.

When carefully observing in-situ the bright stripes of the types 1 and 2 (see Fig. 6), one can see an iridescent colour. This fact points to the caustics caused by the most deflected beams, which corresponds to refraction at the inflection point. For the bright stripe of the type 3, the concentration of the refracted beams occurs due to a focusing effect of the surface convexity (see Eq. (39)). Because of mixing of the multi-coloured rays, the iridescent colour is not observed and only the white stripe is visible. This represents another indirect proof that the relationships obtained by us are correct.

In principle, the relationships among the parameters of the harmonic waves at the water surface and the underwater caustics can enable solving the inverse problem, i.e. determining the amplitude a and the wavelength Λ of the sinusoidal surface wave following from the measured widths of the bright underwater stripes.

Our relationships can also be used to develop new optical systems for determining the surface-wave parameters and, especially, the sea-surface curvatures on large scales by recording the bright-stripes characteristics. Finally, our approach can be improved to determine the statistical characteristics of complex surface waves on the basis of caustic-network images.

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Анотація. *Використано хвильову теорію світла для вивчення яскравості та геометричних характеристик яскравих смуг, що з'являються на дні басейну. Яскравість цих смуг пов'язана з розподілом інтенсивності заломленого світла в околі каустики, де наближення променевої оптики незастосовне. Каустика виникає щоразу, коли світло заломлюється на хвилястій поверхні води. Одержано залежності між параметрами хвиль на поверхні та шириною яскравих смуг (тобто зони каустики). Правильність одержаних виразів* *перевірено дослідом, проведеним у водному басейні. Наші формули можна використати для розробки оптичних систем для визначення параметрів хвиль (зокрема, кривизни морської поверхні на великих масштабах) шляхом реєстрації характеристик яскравих смуг.*

Ключові слова: *поверхневі хвилі, яскраві смуги, підводна каустика, ширина каустичної зони, кривизна поверхні.*