Bright and dark optical solitons for the concatenation model by the Laplace-Adomian decomposition scheme

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Abstract. This paper retrieves numerically the bright and dark 1-soliton solutions for the newly constructed concatenation model using the Laplace-Adomian decomposition technique. The errors analysis is also conducted, and they are of the order of 10^{-9}. The surface, sectional, and error plots are exhibited for bright and dark solitons.

Keywords: solitons; concatenation model; Adomian polynomials

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1. Introduction

Over sixty years ago, N. J. Zabusky and M. D. Kruskal created the term soliton, which refers to a stable solitary wave propagating in a nonlinear medium [1]. They were not the first to identify the extraordinary qualities of solitary waves, initially described in scientific literature as “a massive solitary elevation, a rounded, smooth, and well-defined mound of water,” originating back to John Scott Russell's 18th-century observation in a canal near Edinburgh. A few years later, in 1973, A. Hasegawa and F. Tappert published two fundamental works on nonlinear pulse transmission, which opened the door for optical solitons and nonlinear fiber optics [2, 3]. In the five decades since the publication of Hasegawa and Tappert's works, from applied mathematics and physics to chemistry and biology, theoretical and experimental explorations of solitary waves have proliferated and permeated a vast array of scientific fields. As universal models of propagation of solitons, many renowned equations appear in both their canonical and extended versions.

Nonlinear Schrödinger's equation (NLSE) is the most basic model explored to describe the dynamics of the propagation of solitons through optical fibers. There are several models that describe soliton dynamics that depend on various physical situations. The dispersive optical solitons are governed by the Schrödinger-Hirota equation, Fokas-Lenells equation, Radhakrishnan-Kundu-Lakshmanan equation, and many more. In birefringent fibers, the fundamental model is the Manakov equation. Apart from these familiar models, there are various other governing equations that have been studied in Quantum Optics. They are Lakshmanan-Porsezian-Daniel (LPD) model, the Sasa-Satsuma equation (SSE), the Kundu-Eckhaus equation, and various others.

In consequence, a recent trend in Quantum Optics is the formulation of a new concatenation model by combining these well-known models. This is the conjunction of the familiar NLSE, LPD, and SSE. Less than a decade ago, such a concatenation model was first proposed [4, 5]. Later, this model has been studied to carry out its Painlevé analysis and its full spectrum of 1-soliton solutions have been retrieved as well as its conserved quantities [6, 7, 8, 9]. It is now time to move further along.

In 1986, G. Adomian and R. Rach created the Laplace-Adomian decomposition method (LADM) to solve an extensive class of nonlinear differential equations [10]. Since 1986, LADM has been one of the most efficient mathematical methods for providing accurate numerical approximation solutions for a wide range of nonlinear problems. The method is well suited for physical problems since it may handle nonlinear problems without the need for linearization, perturbation, or discretization while requiring fewer calculations than traditional techniques.

Using LADM, this article analyzes the concatenation model numerically. Thus, a numerical analysis of bright and dark optical solitons for the model is recovered by the application of LADM. The error analysis is also conducted, and it shows that the absolute error is of the order of 10^{-9}. The details of the scheme and the corresponding numerical schemes are all exhibited in detail in the rest of the paper with perfect clarity.

The structure of the paper is as follows: In Section 2, we shall introduce the concatenation model briefly. In Section 3, both bright and dark solitons for the model are introduced as well as some restrictions necessary for their existence. In Section 4, the basic idea of LADM is explained step by step as well as its implementation to generate an algorithm that provides solutions to the concatenation model. The application of LADM on the concatenation model for different sets of coefficients and subject to different initial conditions is presented in Section 5. Finally, the conclusion is presented in Section 6.

2. The concatenation model

The concatenation model was first described in [4, 5] and is the result of combining two well-known models in optics, namely the Lakshmanan-Porsezian-Daniel (LPD) model [11] and the Sasa-Satsuma equation (SSE) [12]. This model is provided in its dimensionless form by:

$$\begin{aligned} & iq_t + aq_{xx} + b\left|q\right|^2 q \\ & + c_1\left[\sigma_1 q_{xxxx} + \sigma_2 (q_x)^2 q^* + \sigma_3 \left|q_x\right|^2 q + \sigma_4 \left|q\right|^2 q_{xx} + + \sigma_5 q^2 q_{xx}^* + \sigma_6 \left|q\right|^4 q\right] \\ & + ic_2\left[\sigma_7 q_{xxx} + \sigma_8 \left|q\right|^2 q_x + \sigma_9 q^2 q_x^*\right] = 0. \end{aligned} \tag{1}$$

where $q = q(x, t)$ is the wave profile, q_t gives the temporal dispersion, q_x is the spatial dispersion, q_{xx}, q_{xxx} and q_{xxxx} correspond to the higher-order dispersions, σ_i – represents the coefficients of nonlinearity, while a is the coefficient associated with second-order dispersion and b represents the coefficient of the nonlinear refractive index governed by the Kerr law. The coefficients of c_1 and c_2 are the portions from LPD and SSE, respectively.

The model of concatenation provided in (1) corresponds to its name. When $c_1 = 0$, (1) simplifies to the well-known SSE, but for $c_2 = 0$, (1) simplifies to the LPD model. However, if $c_1 = c_2 = 0$, it collapses to the well-known NLSE. Combining the Adomian decomposition technique with the well-known Laplace transform will display the bright and dark solitons for model (1) for the first time.

The several constraint conditions that will arise logically from the structure will also ensure the solitons existence requirements. In succeeding sections, the specifics are listed and illustrated.

3. The governing model for the bright and dark solitons

3.1. Bright solitons

The bright soliton solution to (1), studied through the technique of indeterminate coefficients in [6, 7], is denoted by:

$$q(x,t) = A_1 \operatorname{sech}\left[B_1 (x - vt)\right] e^{i(\kappa x + \omega_1 t + \theta_0)}, \tag{2}$$

where A_1 represents the amplitude of the bright soliton, while B_1 is the width of the bright soliton, κ represents the soliton wavenumber, ω_1 and θ_0 represent the frequency and phase constant, respectively, v is soliton velocity, x is coordinate, and t is time. The velocity v of the bright soliton is obtained as:

$$v = -2\kappa\left(a + 4c_1\sigma_1\kappa^2\right). \tag{3}$$

The amplitude A_1 and frequency ω_1 are represented with respect to the soliton width as follows:

$$A_1 = \sqrt{\frac{2\left(Z_3 + 10c_1\sigma_1 B_1^2\right)B_1^2}{Z_2 + \left(Z_4 + Z_5\right)B_1^2}}, \tag{4}$$

and

$$\omega_1 = -a\kappa^2 + Z_3 B_1^2 - c_1\sigma_1\left(3\kappa^4 - B_1^4\right), \tag{5}$$

while the width B_1 is given by:

$$B_1 = \frac{1}{2}\left[\frac{Z_3\left\{(Z_4 + Z_5)(2Z_4 + Z_5) - 40c_1^2\sigma_1\sigma_6\right\} - 2c_1\sigma_1(2Z_4 + 7Z_5)Z_2}{c_1\sigma_1\left(100c_1^2\sigma_1\sigma_6 - 4Z_4^2 - 3Z_4Z_5 + Z_5^2\right)} \pm \frac{\sqrt{\{Z_3(Z_4 + Z_5) - 10c_1\sigma_1 Z_4\}^2\{(2Z_4 + Z_5)^2 - 96c_1^2\sigma_1\sigma_6\}}}{c_1\sigma_1\left(100c_1^2\sigma_1\sigma_6 - 4Z_4^2 - 3Z_4Z_5 + Z_5^2\right)}\right]^{1/2}. \tag{6}$$

Thus, for simplicity, we have used the following notation:

$$Z_2 = b - c_1\kappa^2(\sigma_2 - \sigma_3 + \sigma_4 + \sigma_5) + \kappa c_2(\sigma_8 - \sigma_9), \tag{7}$$

$$Z_3 = a + 6c_1\sigma_1\kappa^2, \tag{8}$$

$$Z_4 = c_1(\sigma_4 + \sigma_5), \tag{9}$$

$$Z_5 = c_1(\sigma_2 + \sigma_3). \tag{10}$$

As can be seen in [6], the parameters Z_2, Z_3, Z_4, and Z_5 emerge in the derivation of the mathematical form of solitons using the technique of indeterminate coefficients.

Finally, the restrictions that ensure the existence of bright solitons are as follows:

$$c_2(\sigma_8 + \sigma_9) = 2\kappa c_1(\sigma_2 + \sigma_4 - \sigma_5), \tag{11}$$

$$c_2\sigma_7 = 4\kappa c_1\sigma_1, \tag{12}$$

$$\sigma_1 c_1^3\left[100\sigma_1\sigma_6 + (\sigma_2 + \sigma_3 + \sigma_4 + \sigma_5)\left\{\sigma_2 + \sigma_3 - 4(\sigma_4 + \sigma_5)\right\}\right] \neq 0, \tag{13}$$

$$\left\{2c_1(\sigma_2 + \sigma_3 + \sigma_4 + \sigma_5)\right\}^2 > 96c_1^2\sigma_1, \tag{14}$$

and

$$\left\{a+2c_1\sigma_1\left(3\kappa^2+5B_1^2\right)\right\}$$
$$\times\left\{b-c_1\kappa^2\left(\sigma_2-\sigma_3+\sigma_4+\sigma_5\right)+c_2\kappa\left(\sigma_8-\sigma_9\right)+c_1\left(\sigma_2+\sigma_3+\sigma_4+\sigma_5\right)B_1^2\right\}>0\ . \tag{15}$$

These latter restrictions are required from a mathematical standpoint in order for the coefficients A_1 and B_1 that involve radicals and quotients to be specified properly [6].

3.2. Dark solitons

The dark soliton solution to (1), studied through the technique of indeterminate coefficients in [6, 7], is denoted by:

$$q(x,t)=A_2\tanh\left[B_2(x-vt)\right]e^{i(\kappa x+\omega_2 t+\theta_0)}, \tag{16}$$

where the velocity v of the dark soliton, is given as in Eq. (3), the amplitude A_2 and frequency ω_2 are represented with respect to the soliton width B_2 as follows:

$$A_2=\sqrt{-\frac{2\left(Z_3+10c_1\sigma_1B_2^2\right)B_2^2}{Z_2+\left(Z_4+Z_5\right)B_2^2}}, \tag{17}$$

and

$$\omega_2=-\frac{2Z_5\left(Z_3-20c_1\sigma_1B_2^2\right)B_2^4}{Z_2-2\left(Z_4+Z_5\right)B_2^2}+\left(a\kappa^2+2Z_3B_2^2+c_1\sigma_1\left(3\kappa^4-16B_2^4\right)\right), \tag{18}$$

while the width B_2 is given by:

$$B_2=\pm\frac{1}{2}\left[\frac{2c_1\sigma_1\left(2Z_4+7Z_5\right)Z_2-Z_3\left\{\left(Z_4+Z_5\right)\left(2Z_4+Z_5\right)-40c_1^2\sigma_1\sigma_6\right\}}{2c_1\sigma_1\left(100c_1^2\sigma_1\sigma_6-4Z_4^2-3Z_4Z_5+Z_5^2\right)}\right.$$
$$\left.\pm\frac{\sqrt{\left\{Z_3\left(Z_4+Z_5\right)-10c_1\sigma_1Z_2\right\}^2\left\{\left(2Z_4+Z_5\right)^2-96c_1^2\sigma_1\sigma_6\right\}}}{2c_1\sigma_1\left(100c_1^2\sigma_1\sigma_6-4Z_4^2-3Z_4Z_5+Z_5^2\right)}\right]^{1/2}. \tag{19}$$

The same notation given by Eqs. (7)-(10) has also been adopted here. Finally, the restrictions that ensure the existence of dark solitons are as follows:

$$c_2\left(\sigma_8+\sigma_9\right)=2\kappa c_1\left(\sigma_2+\sigma_4-\sigma_5\right), \tag{20}$$

$$c_2\sigma_7=4\kappa c_1\sigma_1, \tag{21}$$

$$\sigma_1c_1^3\left[100\sigma_1\sigma_6+\left(\sigma_2+\sigma_3+\sigma_4+\sigma_5\right)\left\{\sigma_2+\sigma_3-4\left(\sigma_4+\sigma_5\right)\right\}\right]<0, \tag{22}$$

and

$$\left\{b-c_1\left(\sigma_2-\sigma_3+\sigma_4+\sigma_5\right)\kappa^2+c_2\left(\sigma_8-\sigma_9\right)\kappa\right\}\neq2c_1\left(\sigma_2+\sigma_3+\sigma_4+\sigma_5\right)B_2^2\ . \tag{23}$$

These latter restrictions are required from a mathematical standpoint in order for the coefficients A_2 and B_2 that involve radicals and quotients to be specified properly [6]. For further information about the concatenation model as well as its soliton-type solutions, the reader is recommended to the recent work published in [7].

4. Methodology: a brief overview

This section will provide a quick overview of the well-known Adomian decomposition technique and its enhancement arising from its combination with the Laplace transform [10, 13, 14]. Development is centered on attaining bright and dark solitons for the concatenated model (1).

In general, we may express Eq. (1) using operators as:

$$D_t q(x,t) + Rq(x,t) + Nq(x,t) = 0\,, \tag{24}$$

subject to the initial condition

$$q(x,t) = f(x), \tag{25}$$

where $D_t q(x,t) = iq_t$ is a standard temporal derivation operator, and $Rq(x,t)$ is a linear differential operator, which in this context is:

$$Rq(x,t) = aq_{xx} + c_1\sigma_1 q_{xxxx} + ic_2\sigma_7 q_{xxx}\,, \tag{26}$$

nevertheless, $Nq(x,t)$ is a non-linear operator that works as:

$$\begin{aligned} Nq(x,t) = b|q|^2 q + c_1\sigma_2 (q_x)^2 q^* + c_1\sigma_3 |q_x|^2 q + c_1\sigma_4 |q|^2 q_{xx} + c_1\sigma_5 q^2 q_{xx}^* \\ + c_1\sigma_6 |q|^4 q + ic_2\left(\sigma_8 |q|^2 q_x + \sigma_9 q^2 q_x^*\right) \end{aligned} \tag{27}$$

According to the classic Adomian decomposition approach, the unknown function q in any equation is decomposed into the sum of an infinite amount of summands defined by the decomposition series:

$$q(x,t) = \sum_{n=0}^{\infty} q_n(x,t), \tag{28}$$

where the $q_n(x,t)$ components must be computed recursively. Individually finding components q_1, q_2, $q_3,\ldots$ is the purpose of the decomposition technique, with being q_0 the initial condition.

Furthermore, the Adomian technique decomposes the nonlinear part as follows:

$$Nq(x,t) = \sum_{n=0}^{\infty} P_n(q_0,\ldots,q_n), \tag{29}$$

where P_n are all Adomian polynomials [15].

The nonlinear operator provided by Eq. (27) may be broken down as:

$$Nq(x,t) = (N_1 + N_2 + N_3 + N_4 + N_5 + N_6 + N_7 + N_8)\,q(x,t), \tag{30}$$

where

$$\begin{aligned} &N_1(q) = b|q|^2 q, \quad && N_2(q) = c_1\sigma_2 (q_x)^2 q^*, \quad && N_3(q) = c_1\sigma_3 |q_x|^2 q, \\ &N_4(q) = c_1\sigma_4 |q|^2 q_{xx}, && N_5(q) = c_1\sigma_5 q^2 q_{xx}^*, && N_6(q) = c_1\sigma_6 |q|^4 q, \\ &N_7(q) = ic_2\sigma_8 |q|^2 q_x, && N_8(q) = ic_2\sigma_9 q^2 q_x^*. && \end{aligned} \tag{31}$$

There exists a decomposition into an infinite series of Adomian polynomials for all nonlinear terms $N_1,\ldots,N_8$, as follows:

$$N_j(q) = \sum_{n=0}^{\infty} P_n^j(q_0, q_1 \ldots, q_n),\ j = 1,2,\ldots 8. \tag{32}$$

In (32), P_n^j symbolizes the Adomian polynomials for every $j = 1,2,\ldots,8,$ which are to be generated using the following formulae [16]:

$$P_n^j(q_0, q_1 \ldots, q_n) = \begin{cases} N_j(q_0), & n = 0 \\ \dfrac{1}{n}\displaystyle\sum_{k=0}^{n-1} (k+1)\, q_{k+1} \frac{\partial}{\partial q_0} P_{n-k-1}^j, & n = 1,2,3,\ldots \end{cases}. \tag{33}$$

The convergence of the series (32) is discussed in more detail in [17, 18].

From here on, we shall refer to the Laplace transform as $\mathcal{L}$ and the inverse operator as $\mathcal{L}^{-1}$. With $\mathcal{L}$ applied to both sides of the functional Eq. (24), we obtain:

$$\mathcal{L}\left\{D_t q(x,t) + Rq(x,t) + Nq(x,t)\right\} = 0. \tag{34}$$

With the initial condition that will be determined by the solitons first profiles f, we get:

$$\mathcal{L}\left\{q(x,t)\right\} = \frac{1}{s}\left[f(x) - \left(\mathcal{L}\left\{Rq(x,t)\right\} + \mathcal{L}\left\{Nq(x,t)\right\}\right)\right]. \tag{35}$$

where s is complex frequency domain parameter.

Substituting Eqs. (28) and (32) into Eq. (35), one obtains:

$$\mathcal{L}\left\{\sum_{n=0}^{\infty} q_n(x,t)\right\} = \frac{1}{s}\left[f(x) - \left(\mathcal{L}\left\{R\left(\sum_{n=0}^{\infty} q_n(x,t)\right)\right\} + \mathcal{L}\left\{\sum_{j=1}^{8}\sum_{n=1}^{\infty} P_n^j(q_0,\ldots,q_n)\right\}\right)\right]. \tag{36}$$

By comparing the two sides of Eq. (36), we get the Laplace transform of each portion of the solution, which is:

$$s\mathcal{L}\left\{q_0(x,t)\right\} = f(x) \tag{37}$$

In addition, the recursive relations are given for every $m \geq 1$ as:

$$s\mathcal{L}\left\{q_m(x,t)\right\} = -\left(\mathcal{L}\left\{Rq_{m-1}(x,t)\right\} + \mathcal{L}\left\{\sum_{j=1}^{8}\sum_{n=1}^{\infty} P_{m-1}^j(q_0,\ldots,q_n)\right\}\right). \tag{38}$$

Here, we will calculate several Adomian polynomials by considering the q-variable nonlinear operators N_j appearing in Eqs. (31) and by using the formula (33), we obtain:

$$P_0^1 = b q_0^2 q_0^*,$$

$$P_1^1 = b\left(q_1^* q_2^2 + 2q_0^* q_1 q_0\right),$$

$$P_2^1 = b\left(q_2^* q_0^2 + q_0^* q_1^2 + 2q_0^* q_2 q_0 + 2q_1^* q_1 q_0\right),$$

$$P_3^1 = b\left(q_3^* q_0^2 + q_1^* q_1^2 + 2q_2^* q_1 q_0 + 2q_1^* q_2 q_0 + 2q_0^* q_3 q_0 + 2q_0^* q_1 q_2\right),$$

$$P_4^1 = b\left(q_4^* q_0^2 + q_2^* q_1^2 + q_0^* q_2^2 + 2q_3^* q_1 q_0 + 2q_2^* q_2 q_0 + 2q_1^* q_3 q_0 + 2q_0^* q_4 q_0 + 2q_1^* q_1 q_2 + 2q_0^* q_1 q_3\right)$$

$$P_0^2 = c_1\sigma_2\ q_{0x}^2 q_0^*,$$

$$P_1^2 = c_1\sigma_2\left(q_1^* q_{0x}^2 + 2q_0^* q_{1x} q_{0x}\right),$$

$$P_2^2 = c_1\sigma_2\left(q_2^* q_{0x}^2 + q_0^* q_{1x}^2 + 2q_0^* q_{2x} q_{0x} + 2q_1^* q_{1x} q_{0x}\right),$$

$$P_3^2 = c_1\sigma_2(q_3^* q_{0x}^2 + q_1^* q_{1x}^2 + 2q_2^* q_{1x} q_{0x} + 2q_1^* q_{2x} q_{0x} + 2q_0^* q_{3x} q_{0x} + 2q_0^* q_{1x} q_{2x}),$$

$$\mathrm{P}_4^1 = b\left(q_4^* \mathrm{q}_0^2 + q_2^* q_1^2 + q_0^* q_2^2 + 2q_3^* q_1 q_0 + 2q_2^* q_2 q_0 + 2q_1^* q_3 q_0 + 2q_0^* q_4 q_0 + 2q_1^* q_1 q_2 + 2q_0^* q_1 q_3\right)$$

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$$P_0^3 = c_1\sigma_3 q_0 \overset{*}{q}_{0x} q_{0x},$$

$$P_1^3 = c_1\sigma_3 \left(q_0 \overset{*}{q}_{0x} q_{1x} + q_0 \overset{*}{q}_{1x} q_{0x} + q_1 \overset{*}{q}_{0x} q_{0x} \right),$$

$$P_2^3 = c_1\sigma_3 \left(q_0 \overset{*}{q}_{0x} q_{2x} + q_0 \overset{*}{q}_{1x} q_{1x} + q_0 \overset{*}{q}_{2x} q_{0x} + q_1 \overset{*}{q}_{0x} q_{1x} + q_1 \overset{*}{q}_{1x} q_{0x} + q_2 \overset{*}{q}_{0x} q_{0x} \right),$$

$$P_3^3 = c_1\sigma_3 \left(q_0 \overset{*}{q}_{0x} q_{3x} + q_0 \overset{*}{q}_{1x} q_{2x} + q_0 \overset{*}{q}_{2x} q_{1x} + q_0 \overset{*}{q}_{3x} q_{0x} + q_1 \overset{*}{q}_{0x} q_{2x} + q_1 \overset{*}{q}_{1x} q_{1x} \right.$$
$$\left. + q_1 \overset{*}{q}_{2x} q_{0x} + q_2 \overset{*}{q}_{0x} q_{1x} + q_2 \overset{*}{q}_{1x} q_{0x} + q_3 \overset{*}{q}_{0x} q_{0x} \right),$$

$$P_4^3 = c_1\sigma_3 \left(q_0 \overset{*}{q}_{0x} q_{4x} + q_0 \overset{*}{q}_{1x} q_{3x} + q_0 \overset{*}{q}_{2x} q_{2x} + q_0 \overset{*}{q}_{3x} q_{1x} + q_0 \overset{*}{q}_{4x} q_{0x} + q_1 \overset{*}{q}_{0x} q_{3x} + q_1 \overset{*}{q}_{1x} q_{2x} \right.$$
$$\left. + q_1 \overset{*}{q}_{2x} q_{1x} + q_1 \overset{*}{q}_{3x} q_{0x} + q_2 \overset{*}{q}_{0x} q_{2x} + q_2 \overset{*}{q}_{1x} q_{1x} + q_2 \overset{*}{q}_{2x} q_{0x} q_3 \overset{*}{q}_{0x} q_{1x} + q_3 \overset{*}{q}_{1x} q_{0x} + q_4 \overset{*}{q}_{0x} q_{0x} \right),$$

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$$P_0^4 = c_1\sigma_4 \overset{*}{q}_0 q_0 q_{0xx},$$

$$P_1^4 = c_1\sigma_4 \left(\overset{*}{q}_1 q_0 q_{0xx} + \overset{*}{q}_0 q_1 q_{0xx} + \overset{*}{q}_0 q_0 q_{1xx} \right),$$

$$P_2^4 = c_1\sigma_4 \left(\overset{*}{q}_2 q_0 q_{0xx} + \overset{*}{q}_1 q_1 q_{0xx} + \overset{*}{q}_0 q_2 q_{0xx} + \overset{*}{q}_1 q_0 q_{1xx} + \overset{*}{q}_0 q_1 q_{0xx} + \overset{*}{q}_0 q_0 q_{2xx} \right),$$

$$P_3^4 = c_1\sigma_4 \left(\overset{*}{q}_3 q_0 q_{0xx} + \overset{*}{q}_2 q_1 q_{0xx} + \overset{*}{q}_1 q_2 q_{0xx} + \overset{*}{q}_0 q_3 q_{0xx} + \overset{*}{q}_2 q_0 q_{1xx} + \overset{*}{q}_1 q_1 q_{1xx} \right.$$
$$\left. + \overset{*}{q}_0 q_2 q_{1xx} + \overset{*}{q}_1 q_0 q_{2xx} + \overset{*}{q}_0 q_1 q_{2xx} + \overset{*}{q}_0 q_0 q_{3xx} \right),$$

$$P_4^4 = c_1\sigma_4 \left(\overset{*}{q}_4 q_0 q_{0xx} + \overset{*}{q}_3 q_1 q_{0xx} + \overset{*}{q}_2 q_2 q_{0xx} + \overset{*}{q}_1 q_3 q_{0xx} + \overset{*}{q}_0 q_4 q_{0xx} + \overset{*}{q}_3 q_0 q_{1xx} \right.$$
$$+ \overset{*}{q}_2 q_1 q_{1xx} + \overset{*}{q}_1 q_2 q_{1xx} + \overset{*}{q}_0 q_3 q_{1xx} + \overset{*}{q}_2 q_0 q_{2xx} + \overset{*}{q}_1 q_1 q_{2xx}$$
$$\left. + \overset{*}{q}_0 q_2 q_{2xx} + \overset{*}{q}_1 q_0 q_{3xx} + \overset{*}{q}_0 q_1 q_{3xx} + \overset{*}{q}_0 q_0 q_{4xx} \right)$$

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$$P_0^5 = c_1\sigma_5\ q_0^2 \overset{*}{q}_{0xx},$$

$$P_1^5 = c_1\sigma_5 \left(q_1^2 \overset{*}{q}_{0xx} + 2 q_0 q_2 \overset{*}{q}_{0xx} \right),$$

$$P_2^5 = c_1\sigma_5 \left(q_1^2 \overset{*}{q}_{0xx} + 2 q_0 q_2 \overset{*}{q}_{0xx} + 2 q_0 q_1 \overset{*}{q}_{1xx} + q_0^2 \overset{*}{q}_{2xx} \right),$$

$$P_3^5 = c_1\sigma_5 \left(q_1^2 q_{1xx}^2 + 2 q_1 q_2 \overset{*}{q}_{0xx} + 2 q_0 q_3 \overset{*}{q}_{0xx} + 2 q_0 q_2 \overset{*}{q}_{1xx} + 2 q_0 q_1 \overset{*}{q}_{2xx} + q_0^2 \overset{*}{q}_{3xx} \right),$$

$$P_4^5 = c_1\sigma_5 \left(q_1^2 \overset{*}{q}_{2xx} + 2 q_1 q_2 \overset{*}{q}_{1xx} + q_2^2 \overset{*}{q}_{0xx} + 2 q_1 q_3 \overset{*}{q}_{0xx} + 2 q_0 q_4 \overset{*}{q}_{0xx} \right.$$
$$\left. + 2 q_0 q_3 \overset{*}{q}_{1xx} + 2 q_0 q_2 \overset{*}{q}_{2xx} + 2 q_0 q_1 \overset{*}{q}_{3xx} + q_0^2 \overset{*}{q}_{4xx} \right),$$

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$$P_0^6 = c_1\sigma_6 q_0^{*2} q_0^3,$$

$$P_1^6 = c_1\sigma_6 (2q_0^* q_1^* q_0^3 + 3 q_0^{*2} q_1 q_0^2),$$

$$P_2^6 = c_1\sigma_6 \left(q_1^{*2} q_0^3 + 2q_0^* q_2^* q_0^3 + 6q_0^* q_1^* q_1 q_0^2 + 3q_0^{*2} q_2 q_0^2 + 3q_0^{*2} q_1^2 q_0^2 \right),$$

$$P_3^6 = c_1\sigma_6 \Big(2q_1^* q_2^* q_0^3 + 2q_0^* q_3^* q_0^3 + 3q_1^{*2} q_1 q_0^2 + 6q_0^* q_2^* q_1 q_0^2 + 6q_0^* q_1^* q_2 q_0^2$$
$$+3q_0^{*2} q_3 q_0^2 + 6q_0^* q_1^* q_1^2 q_0 + 6q_0^{*2} q_1 q_2 q_0 + q_0^{*2} q_1^3 \Big)$$

$$P_4^6 = c_1\sigma_6 \Big(q_2^{*2} q_0^3 + 2q_1^* q_3^* q_0^3 + 2q_0^* q_4^* q_0^3 + 6q_1^* q_2^* q_1 q_0^2 + 6q_0^* q_3^* q_1 q_0^2 + 3q_1^{*2} q_2 q_0^2$$
$$+6q_0^* q_2^* q_2 q_0^2 + 6q_0^* q_1^* q_3 q_0^2 + 3q_0^{*2} q_4 q_0^2 + 3q_1^{*2} q_1^2 q_0 + 6q_0^* q_2^* q_1^2 q_0 + 3q_0^{*2} q_2^2 q_0$$
$$+12q_0^* q_1^* q_1 q_2 q_0 + 6q_0^{*2} q_1 q_3 q_0 + 2q_0^* q_1^* q_1^3 + 3q_0^{*2} q_1^2 q_2 \Big),$$

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$$P_0^7 = ic_2\sigma_8\, q_0 q_0^* q_{0x},$$

$$P_1^7 = ic_2\sigma_8 \left(q_1 q_0^* q_{0x} + q_0 q_1^* q_{0x} + q_0 q_0^* q_{1x} \right),$$

$$P_2^7 = ic_2\sigma_8 \left(q_2 q_0^* q_{0x} + q_1 q_1^* q_{0x} + q_0 q_2^* q_{0x} + q_1 q_0^* q_{1x} + q_0 q_1^* q_{1x} + q_0 q_0^* q_{2x} \right)$$

$$P_3^7 = ic_2\sigma_8 \Big(q_3 q_0^* q_{0x} + q_2 q_1^* q_{0x} + q_1 q_2^* q_{0x} + q_0 q_3^* q_{0x} + q_2 q_0^* q_{1x} + q_1 q_1^* q_{1x}$$
$$+q_0 q_2^* q_{1x} + q_1 q_2^* q_1 x + q_0 q_1^* q_2 x + q_0 q_0^* q_3 x \Big)$$

$$P_4^7 = ic_2\sigma_8 \Big(q_4 q_0^* q_{0x} + q_3 q_1^* q_{0x} + q_2 q_2^* q_{0x}$$
$$+q_1 q_3^* q_{0x} + q_0 q_4^* q_{0x} + q_3 q_0^* q_{1x} + q_2 q_1^* q_{1x} + q_1 q_2^* q_{1x} + q_0 q_3^* q_{1x}$$
$$+q_2 q_0^* q_{2x} + q_1 q_1^* q_{2x} + q_0 q_2^* q_{2x} + q_1 q_0^* q_{3x} + q_0 q_1^* q_{3x} + q_0 q_0^* q_{4x} \Big)$$

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$$P_0^8 = ic_2\sigma_9 q_{0x}^* q_0^2$$

$$P_1^8 = ic_2\sigma_9 \left(q_{1x}^* q_0^2 + q_{0x}^* q_1 q_0 \right),$$

$$P_2^8 = ic_2\sigma_9 \left(q_{2x}^* q_0^2 + 2q_{1x}^* q_1 q_0 + q_{0x}^* q_2 q_0 + q_{0x}^* q_1^2 \right)$$

$$P_3^8 = ic_2\sigma_9 \left(q_{3x}^* q_0^2 + 2q_{2x}^* q_1 q_0 + 2q_{1x}^* q_2 q_0 + 2q_{0x}^* q_3 q_0 + q_{1x}^* q_1^2 + 2q_{0x}^* q_1 q_2 \right)$$

$$P_4^8 = ic_2\sigma_9 \Big(q_{4x}^* q_0^2 + 2q_{3x}^* q_1 q_0 + 2q_{2x}^* q_2 q_0 + 2q_{1x}^* q_3 q_0 + 2q_{0x}^* q_4 q_0$$
$$+q_{2x}^* q_1^2 + 2q_{1x}^* q_1 q_2 + 2q_{0x}^* q_1 q_3 \Big),$$

for other Adomian polynomials, and so on.

Lastly, when the inverse Laplace transform $\mathcal{L}^{-1}$ is taken into account, the components $q_0, q_1, q_2, \ldots,$ are found by:

$$\begin{cases} q_0(x,t) = f(x), \\ q_1(x,t) = -\mathcal{L}^{-1}\left(\frac{1}{s}\mathcal{L}\{Rq_0(x,t)\} + \frac{1}{s}\left[\mathcal{L}\left\{\sum_{j=1}^{8} P_0^j(q_0,\ldots,q_n)\right\}\right]\right), \\ q_2(x,t) = -\mathcal{L}^{-1}\left(\frac{1}{s}\mathcal{L}\{Rq_1(x,t)\} + \frac{1}{s}\left[\mathcal{L}\left\{\sum_{j=1}^{8} P_1^j(q_0,\ldots,q_n)\right\}\right]\right), \\ \vdots \\ q_m(x,t) = -\mathcal{L}^{-1}\left(\frac{1}{s}\mathcal{L}\{Rq_{m-1}(x,t)\} + \frac{1}{s}\left[\mathcal{L}\left\{\sum_{j=1}^{8} P_{m-1}^j(q_0,\ldots,q_n)\right\}\right]\right), \quad m \geq 1, \end{cases} \tag{39}$$

where q_0 is referred to as the initial condition or the zero-th component.

The differential equation being studied was subsequently turned into an impressive determination of computable components when the Eq. (39) was applied. After identifying these components, we replace them with the expression (28) to have the solution in a series form.

Several researchers have shown explicitly that if a problem has an exact solution, the resulting series converges relatively quickly to that solution. Several authors carefully explored the convergence idea of the decomposition series to validate the quick convergence of the resultant series. In [17], Cherruault analyzed the convergence of Adomian's approach. In addition, Cherruault and Adomian [18] gave a novel convergence proof for the approach. For further information about the proofs offered to describe fast convergence, the reader is referred to the sources listed above and the references included within. See [19] for further information on the methodology and its special applicability to solitary waves. Recently, in [20], highly dispersive solitons, both dark and singular, have been successfully simulated using an enhancement of Adomian's technique.

5. Numerical calculation and graphical representation of the results

We will simulate the behavior of the solitons for the concatenation model using the approach described in the previous section.

5.1. Bright soliton simulation

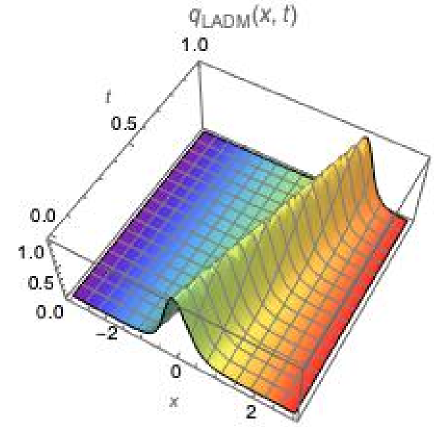
Consider the coefficients of the concatenation model (1) reported in Table 1 for the simulations for the two cases presented the condition at time $t = 0$ provided by:

$$f(x) = A_1 sech\left[B_1(x)\right] e^{i(\kappa x+\theta_0)}. \tag{40}$$

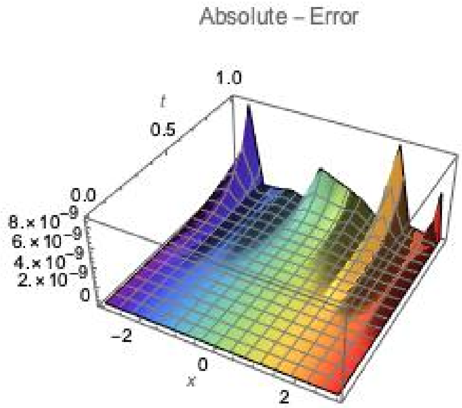
Figs. 1 and 2 show the results of simulations performed on the scenarios given in Table 1.

Table 1. Parameters for bright soliton simulation.

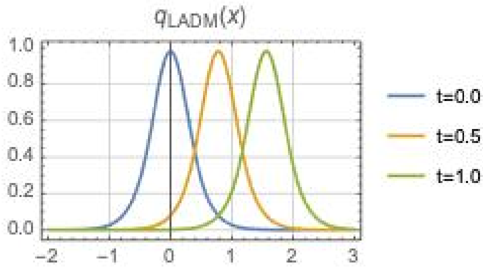
Cases	a	b	c_1	c_2 .	σ_1	σ_2	σ_3	σ_4	σ_5	σ_6	σ_7	σ_8	σ_9
1	1.1	0.2	9.2	0.6	4.3	-0.2	4.1	9.1	3.3	5.9	1.4	6.2	2.6
2	0.6	1.2	7.3	0.6	5.5	0.8	2.5	0.7	3.5	7.0	2.2	3.3	5.5



(a)

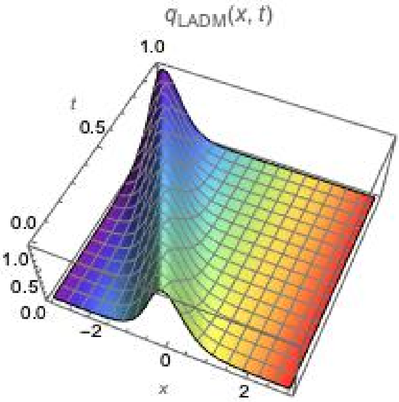


(b)

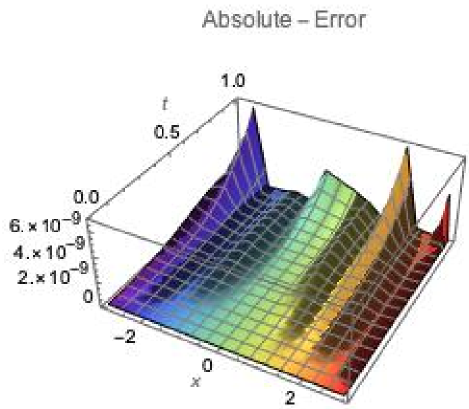


(c)

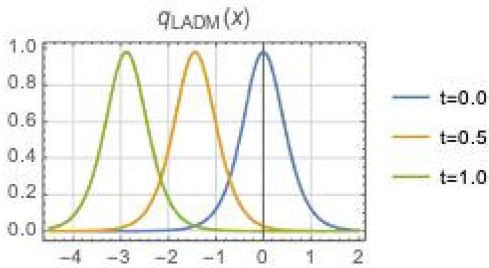
Fig. 1. Profile of soliton's behavior $q(x,t)$ for the spatial variable $x \in [-3,3]$, in relation to the parameters used in Case 1 (a). Error produced by LADM when the number of steps N=12 (b). 2D graph with $t = 0,\ t = 0.5,$ and $t = 1$ (c).



(a)



(b)



(c)

Fig. 2. Profile of Soliton's behavior behavior $q(x,t)$ for the spatial variable $x \in [-3,3]$, in relation to the parameters used in Case 2 (a). Error produced by LADM when the number of steps N=12 (b). 2D graph with $t = 0,\ t = 0.5,$ and $t = 1$ (c).

5.2. Dark soliton simulation

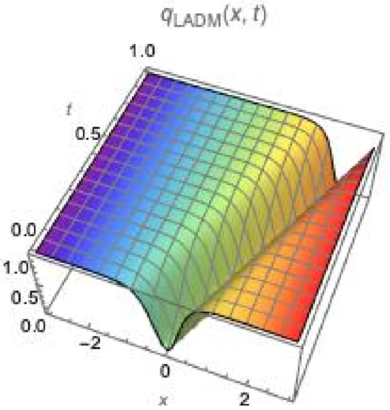
Consider the coefficients of the concatenation model (1) reported in Table 2 for the simulations for the two cases of the condition at time $t = 0$ provided by

$$f(x) = A_2 \tanh\left[B_2(x)\right] e^{i(\kappa x + \theta_0)}. \quad (41)$$

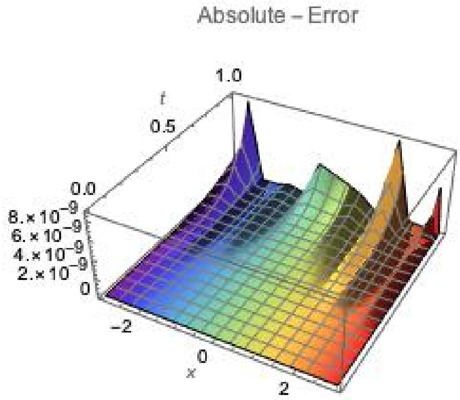
Fig. 3 and 4 show the results of simulations performed on the scenarios given in Table 2.

Table 2. Parameters for dark soliton simulation.

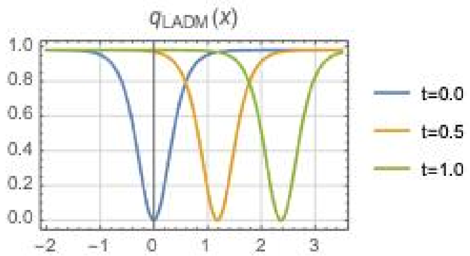
Cases	a	b	c_1	c_2 .	σ_1	σ_2	σ_3	σ_4	σ_5	σ_6	σ_7	σ_8	σ_9
3	2.2	0.8	-0.2	0.3	3.3	4.2	2.4	0.5	7.2	0.9	2.4	5.5	1.9
4	7.5	1.6	-8.2	3.3	0.4	3.8	0.2	1.1	0.8	1.8	5.6	2.9	6.4



(a)

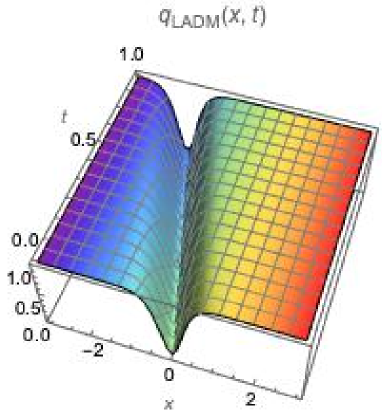


(b)

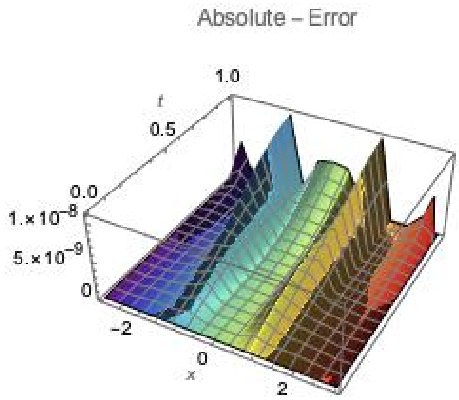


(c)

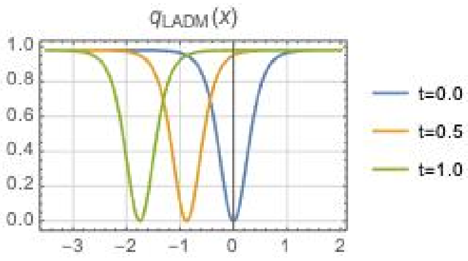
Fig. 3. Profile of Soliton's behavior behavior $q(x,t)$ for the spatial variable $x \in [-3,3]$, in relation to the parameters used in Case 3 (a). Error produced by LADM when the number of steps N=12 (b). 2D graph with $t=0$, $t=0.5$, and $t=1$ (c).



(a)



(b)



(c)

Fig. 4. Profile of Soliton's behavior $q(x,t)$ for the spatial variable $x \in [-3,3]$, in relation to the parameters used in Case 4 (a). Error produced by LADM when the number of steps N=12 (b). 2D graph with $t=0$, $t=0.5$, and $t=1$ (c).

6. Conclusions

The current paper studied the newly established concatenation model numerically by the LADM scheme. The surface plots and the error plots displayed the efficiency of the scheme. The scheme shows that the error is infinitesimally small for both bright and dark solitons. The results thus pave many ways for additional analysis of this model. Later additional numerical schemes will be implemented to handle the model. Some such proposed methods are the finite element method, finite difference scheme, improved Adomian decomposition scheme, numerical methods, boundary element method, and many others. The dynamical system of soliton parameters will also be displayed using the collective variables approach [11]. The results of such schemes will be disseminated with time.

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Анотація. *В цій статті чисельно отримані яскраві та темні 1-солітонні розв'язки для новоствореної моделі конкатенації з використанням техніки декомпозиції Лапласа-Адоміана. Також проведений аналіз похибок, які становлять порядку* 10^{-9}. *Для яскравих і темних солітонів представлені їхні графіки поверхонь, розрізів і похибок.*

Ключові слова: *солітони; модель конкатенації; многочлени Адоміана*