Optical solitons for the concatenation model with power-law nonlinearity: undetermined coefficients

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Abstract. In the current paper, a full spectrum of 1–soliton solutions to the concatenation model with the power–law of self–phase modulation has been recovered. The method of undetermined coefficients has permitted us to solve this problem successfully. The parameter constraints naturally emerge from the derivation and are also listed, guaranteeing these solitons' existence. It has been proved that dark solitons and singular solitons of a specific type would exist only when the power–law parameter collapses to unity.

Key words: solitons; power–law; concatenation; conservation laws.

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1. Introduction

One of the most interesting models that govern optical soliton propagation through waveguides stems from the concatenation of three well–known and well-studied equations. They are the nonlinear Schrödoimger’s equation (NLSE), the Lakshmanan–Porsezian–Daniel (LPD) equation, and the Sasa–Satsuma equation (SSE). Hence it is referred to as the concatenation model. This was first proposed in 2014 [1, 2]. Later, a lot of studies stemmed from this model. The rogue waves have been studied, the conservation laws have been extracted, the quiescent solitons of the model for nonlinear chromatic dispersion (CD) have been retrieved, and the Painleve analysis has been carried out. There are several approaches that revealed soliton solutions of a wider variety. Such integration schemes that made these happen are the usage of the undetermined coefficients, Kudryashov’s approaches, the trial equation method, and others [3–9]. The model was later studied in birefringent fibers and the soliton solutions were revealed in that case as well [4].

This paper turned the page. While all of the past results are with Kerr's law of self–phase modulation (SPM), the current work moves up ahead with the next form of SPM. This is the power–

law of nonlinearity. Thus, the concatenation model is studied in this paper with the power–law of nonlinearity. Again, the method of undetermined coefficients will retrieve the bright, dark, and singular 1–soliton solutions. The corresponding parameter constraints would naturally emerge during the course of the derivation of the soliton solutions. An interesting observation that would be made is that a dark 1–soliton solution and one of two forms of singular 1–soliton solutions would exist when the power–law nonlinearity parameter for the SPM would collapse to unity. The results along with the solution derivation are all discussed in detail and exhibited in the rest of the paper.

2. Governing model

The concatenation model with power law nonlinearity may be written as [3]:

$$\begin{aligned} & iq_t + aq_{xx} + b|q|^{2n} q \\ & +c_1\left[\sigma_1 q_{xxxx} + \sigma_2 (q_x)^2 q^* + \sigma_3 |q_x|^2 q + \sigma_4 |q|^{2n} q_{xx} + \sigma_5 q^2 q_{xx}^* + \sigma_6 |q|^{2n+2} q\right] \\ & +ic_2\left[\sigma_7 q_{xxx} + \sigma_8 |q|^{2n} q_x + \sigma_9 q^2 q_x^*\right] = 0. \end{aligned} \tag{1}$$

Where $q = q(x,t)$ is the wave profile, q_t gives the temporal dispersion, q_x is the spatial dispersion, q_{xx}, q_{xxx} and q_{xxxx} correspond to the higher-order dispersions, σ_i – represent the coefficients of nonlinearity. The coefficients of c_1 and c_2 are the portions from LPD and SSE, respectively. The concatenation model given by Eq. (1) is true to its name. For $c_1 = 0$, Eq. (1) reduces to the familiar SSE, while for $c_2 = 0$, equation (1) reduces to the LPD model. But for $c_1 = c_2 = 0$, (1) collapses to the familiar NLSE with power–law nonlinearity. The bright 1–soliton solution, from the method of undetermined coefficients, will reveal the conserved quantities. The several constraint conditions that will naturally emerge from the scheme will also guarantee the soliton existence criteria.

The model is an extended version of the familiar NLSE, SSE, and LPD models. Thus, the current model describes the propagation of solitons through an optical fiber in a much more accurate manner. CD and self–phase modulation provide the basic ingredients for the solitons to exist. In case CD runs low, this is compensated by the third–order dispersion and fourth–order dispersion effects that stem from the SSE and LPD components of the concatenation model. The details are enumerated and displayed in the subsequent sections.

3. Undetermined coefficients

To construct single soliton solutions for the system given by Eq. (1) we allow it to have solutions of the form [3]

$$q(x,t) = P(x,t)e^{i\Psi(x,t)} = P(\xi)e^{i\Psi(x,t)}, \tag{2}$$

for which the arguments of the amplitude factors and the phase factor respectively are:

$$\begin{aligned} & \xi(x,t) = x - vt, \quad \text{and} \\ & \Psi(x,t) = -\kappa x + \omega t + \theta_0, \end{aligned} \tag{3}$$

where $P(x,t)$ denotes the waveform according to each type of nonlinear wave, κ represents the soliton wavenumber, while ω and θ_0 represent the frequency and phase constant, respectively, v is soliton velocity, x is coordinate and t is time. By substituting Eq. (2) into Eq. (1) and separating it into real and imaginary parts, one gets:

$$\left(c_1\sigma_1\kappa^4 - c_2\sigma_7\kappa^3 - a\kappa^2 - \omega\right)P - \left\{c_1\left(\sigma_2 - \sigma_3 + \sigma_5\right)\kappa^2 + c_2\sigma_9\kappa\right\}P^3 + c_1\sigma_6 P^{2n+3}$$
$$+\left(b + c_2\sigma_8\kappa - c_1\sigma_4\kappa^2\right)P^{2n+1} + \left(a - 6c_1\sigma_1\kappa^2 + 3c_2\sigma_7\kappa\right)\frac{\partial^2 P}{\partial x^2} + c_1\sigma_1\frac{\partial^4 P}{\partial x^4} \tag{4}$$
$$+c_1\sigma_4 P^{2n}\left(\frac{\partial^2 P}{\partial x^2}\right) + c_1\sigma_5 P^2\frac{\partial^2 P}{\partial x^2} + c_1\left(\sigma_2 + \sigma_3\right)P\left(\frac{\partial P}{\partial x}\right)^2 = 0,$$

while the imaginary counterpart resulted to be:

$$\frac{\partial P}{\partial t} - \left(2a\kappa + 3c_2\sigma_7\kappa^2 - 4c_1\sigma_1\kappa^3\right)\frac{\partial P}{\partial x}$$
$$+\left(c_2\sigma_8 - 2c_1\sigma_4\kappa\right)P^{2n}\left(\frac{\partial P}{\partial x}\right)$$
$$+\left[c_2\sigma_9 - 2c_1\kappa\left(\sigma_2 - \sigma_5\right)\right]P^2\left(\frac{\partial P}{\partial x}\right) \tag{5}$$
$$+\left(c_2\sigma_7 - 4\kappa c_1\sigma_1\right)\frac{\partial^3 P}{\partial x^3} = 0.$$

From the imaginary part given by Eq.(5) the soliton speed comes out as,

$$v = -2\kappa\left(a + 4c_1\sigma_1\kappa^2\right), \tag{6}$$

whenever

$$c_2\sigma_8 = 2c_1\sigma_4\kappa, \tag{7}$$

$$c_2\sigma_9 = 2c_1\kappa\left(\sigma_2 - \sigma_5\right), \tag{8}$$

and

$$c_2\sigma_7 = 4\kappa c_1\sigma_1 \tag{9}$$

hold. In view of the above three conditions (7), (8) and (9), the real part equation (4) reduces to

$$-\left(\omega + a\kappa^2 + 3c_1\sigma_1\kappa^4\right)P + c_1\left(\sigma_3 + \sigma_5 - 3\sigma_2\right)\kappa^2 P^3 + c_1\sigma_6 P^{2n+3}$$
$$+\left(b + c_1\sigma_4\kappa^2\right)P^{2n+1} + \left(a + 6c_1\sigma_1\kappa^2\right)\frac{\partial^2 P}{\partial x^2} + c_1\sigma_1\frac{\partial^4 P}{\partial x^4} \tag{10}$$
$$+c_1\sigma_4 P^{2n}\left(\frac{\partial^2 P}{\partial x^2}\right) + c_1\sigma_5 P^2\left(\frac{\partial^2 P}{\partial x^2}\right) + c_1\left(\sigma_2 + \sigma_3\right)P\left(\frac{\partial P}{\partial x}\right)^2 = 0,$$

after simplification. Throughout the next four subsections, the last real part Eq.(10) will be utilized to construct four different types of soliton solutions for the concatenation model with power law nonlinearity given by Eq.(1).

3.1. Bright soliton

In this subsection, a bright soliton solution is constructed with the aid of the balancing principle. First, we assume the following profile for the waveform:

$$P(x,t) = A\ \operatorname{sech}^p \tau,$$
$$\tau = B(x - vt), \tag{11}$$

where A represents the amplitude of the soliton, p is a parameter to be determined, while B is the corresponding inverse width with v representing the soliton speed. The substitution of Eq.(11) into the real part (Eq. (10)) reduces the last to

$$
\begin{aligned}
&-\left[\left(\omega+a\kappa^{2}+3c_{1}\sigma_{1}\kappa^{4}\right)-p^{2}\left(a+6c_{1}\sigma_{1}\kappa^{2}\right)B^{2}-p^{4}c_{1}\sigma_{1}B^{4}\right]\operatorname{sech}^{p}\tau\\
&+c_{1}\left[\left(\sigma_{3}+\sigma_{5}-3\sigma_{2}\right)\kappa^{2}+p^{2}\left(\sigma_{2}+\sigma_{3}+\sigma_{5}\right)B^{2}\right]A^{3}\operatorname{sech}^{3p}\tau\\
&-p\left(p+1\right)\left[\left(a+6c_{1}\sigma_{1}\kappa^{2}\right)+2\left\{2+p\left(p+2\right)\right\}c_{1}\sigma_{1}B^{2}\right]AB^{2}\operatorname{sech}^{2+p}\tau\\
&+p\left(p+1\right)\left(p+2\right)\left(p+3\right)c_{1}\sigma_{1}AB^{4}\operatorname{sech}^{4+p}\tau\\
&-p\left[pc_{1}\left(\sigma_{2}+\sigma_{3}\right)+\left(p+1\right)c_{1}\sigma_{5}\right]A^{3}B^{2}\operatorname{sech}^{2+3p}\tau\\
&+\left[\left(b+c_{1}\sigma_{4}\kappa^{2}\right)+p^{2}c_{1}\sigma_{4}B^{2}\right]A^{2n+1}\operatorname{sech}^{(2n+1)p}\tau\\
&-p\left(p+1\right)c_{1}\sigma_{4}A^{2n+1}B^{2}\operatorname{sech}^{2+(2n+1)p}\tau+c_{1}\sigma_{6}A^{2n+3}\operatorname{sech}^{(2n+3)p}\tau=0,
\end{aligned}
\tag{12}
$$

after simplification. In this case, the balancing principle allows comparing exponents $\left(2n+1\right)p$ and $p+2$, leading to

$$
p=\frac{1}{n}. \tag{13}
$$

Equating the coefficients of the resulting linearly independent functions to zero, one gets for the soliton width

$$
B=n\kappa\sqrt{\frac{3\sigma_{2}-\sigma_{3}-\sigma_{5}}{\sigma_{2}+\sigma_{3}+\sigma_{5}}}, \tag{14}
$$

as long as

$$
\left(3\sigma_{2}-\sigma_{3}-\sigma_{5}\right)\left(\sigma_{2}+\sigma_{3}+\sigma_{5}\right)>0. \tag{15}
$$

In view of (14) the soliton frequency falls out as:

$$
\omega=\frac{2\kappa^{2}\left[\left(a+12c_{1}\sigma_{1}\kappa^{2}\right)\sigma_{2}^{2}-\left(a+4c_{1}\sigma_{1}\kappa^{2}\right)\left(\sigma_{3}+\sigma_{5}\right)^{2}\right]}{\left(\sigma_{2}+\sigma_{3}+\sigma_{5}\right)^{2}}, \tag{16}
$$

while the soliton amplitude results to be

$$
A=\left[\frac{\kappa^{2}\left(3\sigma_{2}-\sigma_{3}-\sigma_{5}\right)\left\{\sigma_{2}+\sigma_{3}+\left(n+1\right)\sigma_{5}\right\}}{\sigma_{6}\left(\sigma_{2}+\sigma_{3}+\sigma_{5}\right)}\right]^{\frac{1}{2n}}, \tag{17}
$$

provided

$$
\kappa^{2}\sigma_{6}\left(3\sigma_{2}-\sigma_{3}-\sigma_{5}\right)\left\{\sigma_{2}+\sigma_{3}+\left(n+1\right)\sigma_{5}\right\}\left(\sigma_{2}+\sigma_{3}+\sigma_{5}\right)>0. \tag{18}
$$

In addition, after considering (14) and (17), the identities

$$
\sigma_{4}\left\{\sigma_{2}+\sigma_{3}+\left(n+1\right)\sigma_{5}\right\}=\left(2n+1\right)\left(3n+1\right)\sigma_{1}\sigma_{6}, \tag{19}
$$

and

$$
\begin{aligned}
&\left(n+1\right)\sigma_{6}\left[\left\{a+12\left(n^{2}+n+1\right)c_{1}\sigma_{1}\kappa^{2}\right\}\sigma_{2}+a-4\left(n^{2}+n-1\right)c_{1}\sigma_{1}\kappa^{2}\left(\sigma_{3}+\sigma_{5}\right)\right]\\
&=\left[\sigma_{2}+\sigma_{3}+\left(n+1\right)\sigma_{5}\right]\left[\left(b+4c_{1}\sigma_{4}\kappa^{2}\right)+b\left(\sigma_{3}+\sigma_{5}\right)\right]
\end{aligned}
\tag{20}
$$

must be satisfied in order for the soliton to exist. Therefore, the bright soliton solution for the concatenation model with power–law nonlinearity (1) is

$$
q\left(x,t\right)=A\ \operatorname{sech}^{\frac{1}{n}}\left[B\left(x-vt\right)\right]e^{i\left(-\kappa x+\omega t+\theta_{0}\right)}, \tag{21}
$$

where the amplitude is given in (17) provided (18), the soliton width is expressed on (14) as long as (15) is satisfied, the speed was early constructed in (6) in view of the constraints (7)–(9), while the soliton frequency is depicted on (16). In addition, the identities (19) and (20) must be satisfied to preserve integrability.

3.2. Dark soliton

In this part, we construct a dark soliton solution following a similar procedure to the one used to construct the bright solution in the previous part. First, we assume the following profile for the waveform:

$$P(x,t) = A\tanh^{p}\tau, \qquad \tau = B(x - vt), \tag{22}$$

where, in this case, A and B are free parameters for the soliton, while p is a parameter whose value will be determined properly balancing nonlinearity and dispersion. The rest of the parameters have the same meaning as for bright soliton. Next, substituting Eq. (22) into Eq.(10), the real part Eq.(10) emerges as:

$$\begin{aligned}
&-\left\{\left(\omega + a\kappa^2 + 3c_1\sigma_1\kappa^4\right) + 2p^2\left[\left(a + 6c_1\sigma_1\kappa^2\right) - \left(5 + 3p^2\right)c_1\sigma_1B^2\right]B^2\right\}A \ \tanh^{p}\tau \\
&+c_1\left[\left(\sigma_3 + \sigma_5 - 3\sigma_2\right)\kappa^2 - 2p^2\left(\sigma_2 + \sigma_3 + \sigma_5\right)B^2\right]A^3 \ \tanh^{3p}\tau \\
&+p(p+1)[\left(a + 6c_1\sigma_1\kappa^2\right) - 4\{2 + p(p+2)\}c_1\sigma_1B^2\}AB^2 \ \tanh^{p+2}\tau \\
&+p(p+1)(p+2)(p+3)c_1\sigma_1AB^4 \ \tanh^{p+4}\tau \\
&+pc_1\left[\sigma_5 + p\left(\sigma_2 + \sigma_3 + \sigma_5\right)\right]A^3B^2 \ \tanh^{3p+2}\tau \\
&+p(p+1)c_1\sigma_4A^{2n+1}B^2 \ \tanh^{2+(2n+1)p}\tau \\
&+\left[\left(b + c_1\sigma_4\kappa^2\right) - 2p^2c_1\sigma_4B^2\right]A^{2n+1} \ \tanh^{(2n+1)p}\tau \ + \ c_1\sigma_6A^{2n+3} \ \tanh^{(2n+3)p}\tau \\
&+pc_1\left[p\left(\sigma_2 + \sigma_3 + \sigma_5\right) - \sigma_5\right]A^3B^2 \ \tanh^{3p-2}\tau \\
&+p(p-1)\left[\left(a + 6c_1\sigma_1\kappa^2\right) - 4\{2 + p(p-2)\}c_1\sigma_1B^2\right]AB^2 \ \tanh^{p-2}\tau \\
&+p(p-1)(p-2)(p-3)c_1\sigma_1AB^4 \ \tanh^{p-4}\tau \\
&+p(p-1)c_1\sigma_4A^{2n+1}B^2 \ \tanh^{(2n+1)p-2}\tau = 0.
\end{aligned} \tag{23}$$

As for bright soliton, by equating the exponents $(p+2)$ and $(2n+1)p,$ the value for the parameter as in Eq.(13) is retrieved. From the stand-alone coefficients of $\tanh^{p-4}[\tau]$ and $\tanh^{p-2}[\tau]$, we get $p = 1,$ implying that $n = 1,$ e.g., that the power law reduces to Kerr law in Eq.(1). The resulting values of p and n allow one to get from Eq.(23) the wave number

$$\omega = c_1\left\{\left(16B^4 - 12\kappa^2B^2 - 3\kappa^4\right)\sigma_1 + \left(\sigma_1 + \sigma_3\right)A^2B^2\right\} - a\left(\kappa^2 + 2B^2\right), \tag{24}$$

where the parameter A can be written in terms of B as

$$A = B\sqrt{\frac{2\left\{20c_1\sigma_1B^2 - \left(a + 6_1\sigma_1\kappa^2\right)\right\}}{c_1\left\{\kappa^2\left(\sigma_3 + \sigma_5 - 3\sigma_2\right) - 2\left(\sigma_2 + \sigma_3 + \sigma_5\right)B^2\right\} + \left\{b + c_1\sigma_4\left(\kappa^2 - 2B^2\right)\right\}}}, \tag{25}$$

as long as the radicand stays positive, while both free parameters must satisfy the identity

$$24\sigma_1B^4 + \left\{\sigma_2 + \sigma_3 + 2\left(\sigma_4 + \sigma_5\right)\right\}A^2B^2 + \sigma_6A^4 = 0. \tag{26}$$

Therefore, the single soliton solution for the concatenation model given by Eq.(1) can be written as

$$q(x,t) = A\ \ tanh\ \left[B(x-vt)\right]e^{i(-\kappa x+\omega t+\theta_0)} \tag{27}$$

where the parameters along with corresponding constraint were discussed above.

3.3. Singular soliton (type–I)

For the first-type of singular soliton, we embrace the following waveform:

$$P(x,t) = A\ \ csch^{p}\tau, \qquad \tau = B(x-vt), \tag{28}$$

To construct the type-1 of the singular soliton solution we first substitute Eq.(28) into Eq.(10) simplifying the last to:

$$\begin{aligned}
&-\left[\left(\omega + a\kappa^2 + 3c_1\sigma_1\kappa^4\right) - p^2\left(a + 6c_1\sigma_1\kappa^2\right)B^2 - p^4 c_1\sigma_1 B^4\right]\ csch^{p}\tau \\
&+c_1\left[\left(\sigma_3+\sigma_5-3\sigma_2\right)\kappa^2 + p^2\left(\sigma_2+\sigma_3+\sigma_5\right)B^2\right]A^3\ \ csch^{3p}\tau \\
&+p(p+1)\left[\left(a+6c_1\sigma_1\kappa^2\right) + 2\left\{2+p(p+2)\right\}c_1\sigma_1 B^2\right]AB^2\ \ csch^{p+2}\tau \\
&+p(p+1)(p+2)(p+3)c_1\sigma_1 AB^4\, csch^{4+p}\tau \\
&+p\left[pc_1\left(\sigma_2+\sigma_3\right) + (1+p)c_1\sigma_5\right]A^3B^2\ \ csch^{2+3p}\tau \\
&+\left[\left(b+c_1\sigma_4\kappa^2\right) + p^2 c_1\sigma_4 B^2\right]A^{2n+1}\ \ csch^{(2n+1)p}\tau \\
&+p(p+1)c_1\sigma_4 A^{2n+1}B^2\ \ csch^{2+(2n+1)p}\tau\ +\ c_1\sigma_6 A^{2n+3}\ \ csch^{(2n+3)p}\tau = 0.
\end{aligned} \tag{29}$$

Proper balancing allows again to equate the exponents $(2n+1)p$ and $(p+2)$, yielding the same value for the parameter p as in Eq.(13). Thus, inserting Eq.(13) into Eq.(29) and setting to zero the coefficients of the resulting independent linear functions one gets the width B as in Eq.(14), and the frequency ω as in Eq.(16), along with required constraint (15). For singular type-1 the parameter A is slightly different than in Eq.(17), it resulted to be

$$A = \left[-\frac{\kappa^2\left(3\sigma_2-\sigma_3-\sigma_5\right)\left\{\sigma_2+\sigma_3+(n+1)\sigma_5\right\}}{\sigma_6\left(\sigma_2+\sigma_3+\sigma_5\right)}\right]^{\frac{1}{2n}}. \tag{30}$$

Consequently, the direction of the corresponding constraint is inverted in Eq.(18), e.g

$$\sigma_6\left(3\sigma_2-\sigma_3-\sigma_5\right)\left\{\sigma_2+\sigma_3+(n+1)\sigma_5\right\}\left(\sigma_2+\sigma_3+\sigma_5\right) < 0. \tag{31}$$

For this type of soliton, the same identities (19) and (20) must be satisfied as well for the pulse to exist. Therefore, the type-1 singular soliton solution for the concatenation model with power law nonlinearity (1) is

$$q(x,t) = A\ \ csch^{\frac{1}{n}}\left[B(x-vt)\right]e^{i(-\kappa x+\omega t+\theta_0)}, \tag{32}$$

where the parameter A is given by Eq.(30) with corresponding constraint (31), while the rest of the parameters and solvability conditions are the same as for bright soliton.

3.4. Singular soliton (type–II)

The second type of singular soliton solution is the last kind of soliton we are going to discuss in this work. To retrieve type-I singular solitons from our concatenation system (1) a waveform having the following structure is assumed:

$$P(x,t) = A \ \ coth^{p}\tau, \\ \tau = B(x - vt), \tag{33}$$

where the parameters A and B represent the soliton amplitude and inverse width correspondingly, p is a parameter to be determined by balancing nonlinearity and dispersion, while v representing the soliton speed. Inserting Eq.(33) into the real part Eq. (10) leads to

$$\begin{aligned}
&-\left\{\left(\omega + a\kappa^2 + 3c_1\sigma_1\kappa^4\right) + 2p^2\left[\left(a + 6c_1\sigma_1\kappa^2\right) - \left(5 + 3p^2\right)c_1\sigma_1 B^2\right]B^2\right\}A \ \ coth^{p}\tau \\
&+c_1\left[\left(\sigma_3 + \sigma_5 - 3\sigma_2\right)\kappa^2 - 2p^2\left(\sigma_2 + \sigma_3 + \sigma_5\right)B^2\right]A^3 \ \ coth^{3p}\tau \\
&+p(p+1)\left[\left(a + 6c_1\sigma_1\kappa^2\right) - 4\left\{2 + p(2+p)\right\}c_1\sigma_1 B^2\right]AB^2 \ \ coth^{p+2}\tau \\
&+p(p+1)(p+2)(p+3)c_1\sigma_1 AB^4 \ \ coth^{p+4}\tau \\
&+pc_1\left[\sigma_5 + p\left(\sigma_2 + \sigma_3 + \sigma_5\right)\right]A^3B^2 \ \ coth^{3p+2}\tau \\
&+p(p+1)c_1\sigma_4 A^{2n+1}B^2 \ \ coth^{2+(2n+1)p}\tau \\
&+\left[\left(b + c_1\sigma_4\kappa^2\right) - 2p^2c_1\sigma_4 B^2\right]A^{2n+1} \ \ coth^{(2n+1)p}\tau + c_1\sigma_6 A^{2n+3} \ \ coth^{(2n+3)p}\tau \\
&+pc_1\left[p\left(\sigma_2 + \sigma_3 + \sigma_5\right) - \sigma_5\right]A^3B^2 \ \ coth^{3p-2}\tau \\
&+p(p-1)\left[\left(a + 6c_1\sigma_1\kappa^2\right) - 4\left(2 + p(p-2)\right)c_1\sigma_1 B^2\right]AB^2 \ \ coth^{p-2}\tau \\
&+p(p-1)(p-2)(p-3)c_1\sigma_1 AB^4 \ \ coth^{p-4}\tau \\
&+p(p-1)c_1\sigma_4 A^{2n+1}B^2 \ \ coth^{(2n+1)p-2}\tau = 0.
\end{aligned} \tag{34}$$

By inspection, the balancing yields the value for parameter p as in (13). However, the stand-alone elements dictate n=1, turning the power law nonlinearity into a cubic nonlinearity. Consequently, the value of p turns out to be one. Indeed, substituting the resulting value of p into Eq.(34) yields the same results as dark soliton with corresponding solvability conditions. Thus, the type-II singular soliton solution for system (1) is

$$q(x,t) = A \ \ coth \ \left[B(x - vt)\right]e^{i(-\kappa x + \omega t + \theta_0)}, \tag{35}$$

where all the parameters along with corresponding constraints are the same as for dark soliton discussed earlier on this manuscript.

4. Conclusions

The current paper recovered a full spectrum of 1–soliton solution to the concatenation model having power–law of nonlinear refractive index. The method of undetermined coefficients granted success with this scheme. The parameter constraints that naturally emerged during the derivation process are also enumerated for each soliton type. An interesting observation is that the dark solitons and one of the two forms of singular solitons would only exist for the power–law nonlinearity parameter to stay at unity. This is a very interesting observation of the concatenation model being made for the first time in this paper.

The results of this paper are a gateway to an avalanche of upcoming results that will follow through. Later, the model would retrieve the conservation laws. The quiescent solitons would also be recovered for the model using a variety of approaches. The model would be next considered with fractional temporal evolution that would mitigate the "Internet bottleneck effect".

Subsequently, with the conservation laws in place, the soliton perturbation theory would be implemented to obtain the quasi–monochromatic dynamics of such solitons.

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Анотація. *Ця стаття відновлює повний спектр 1-солітонних розв'язків моделі конкатенації зі степеневим законом самофазової модуляції. Метод невизначених коефіцієнтів дозволив успішно вирішити цю проблему. Обмеження наведених параметрів природним чином виникає із виведення, що у свою чергу гарантує існування цих солітонів. Доведено, що темні солітони та сингулярні солітони певного типу існували б лише тоді, коли порядок степеневого закону спадав би до одиниці.*

Ключові слова: *солітони; сила–закон; конкатенація; закони збереження.*