Polarization characteristics of a polychromatic wave

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Abstract. We consider polarization characteristics of a polychromatic wave associated with the Stokes parameters and the Jones vector. It is shown that the Stokes parameters can fail to describe adequately the polarization state of the wave formed as a result of superposition of the waves having different frequencies. To characterize this superposition, a modified Jones vector is introduced. Moreover, we demonstrate that the common Stokes parameters determine uniquely the spin moment of the resulting wave field.

Keywords: polarization, Stokes parameters, Jones vector, Poynting vector, spin angular momentum

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1. Introduction

In the most general case, a wave is considered to be polarized when the tip of its (either electric or magnetic) field-strength vector performs a deterministic oscillation, i.e. it moves along some trajectory determined in time (see, e.g., Ref. [1] *). If a coherent wave is dealt with, then the trajectory represents in general an ellipse (see, e.g., Ref. [2]). Note that, in most cases, such representations are automatically extended to a polychromatic wave, which can be considered as a superposition of some components having different frequencies.

However, the above interpretation of the polarization properties of polychromatic waves is not always correct [3]. Obviously, if we mean a superposition of waves with a continuous frequency spectrum, then this interpretation can be accepted with some reservations. A different situation arises if we have a superposition of coupled waves [4] with a rather narrow spectrum or, in general, a superposition of coherent waves with different frequencies [5–7]. In this case, the trajectory of the tip of the field vector becomes rather complicated [7].

Note that traditional polarization characteristics, such as Stokes parameters, can be measured for this superposition. It is evident that these time-averaged field characteristics are rigidly related to the coherent characteristics of a resulting wave [8, 9]. As a consequence, we will show that, even in case of a time-determined trajectory of the field-vector tip, these characteristics can correspond to an absolutely depolarized wave. Then the following question arises: What do they characterize in this case?

2. Stokes parameters of a polychromatic wave

It is known that the Stokes parameters can be expressed in terms of some combinations of the intensities corresponding to different polarization projections or as compositions of the

* Note that the number of references devoted to the polarization of electromagnetic waves is huge, with no exaggeration. Here and below, we cite only the most fundamental (or, perhaps, the most famous) sources, without detracting from any merits of the other authors.

corresponding elements of coherence matrix [1, 2, 8]. Then we suggest a following scheme in order to determine these parameters for a polychromatic wave that represents a superposition of elementary waves with different spectral compositions:

1. First, instantaneous-intensity components (i.e., instantaneous elements of the coherence matrix) are formed, which include both the quantities associated with a single spectral component and the corresponding intermodulation terms that contain information about the characteristics of various spectral components.

2. Second, the obtained relations are averaged over a sufficiently large time interval.

3. As a result, the intermodulation-intensity components of the polychromatic wave disappear. For example, it has been shown by Born and Wolf [8] that the resulting elements of the coherence matrix are sums of the elementary elements corresponding to each frequency. Due to this, the Stokes parameters can be considered as sums of the Stokes parameters of all spectral components of the wave [3, 10].

Let the paraxial approximation be adopted and the Cartesian components of the field of each spectral component be described by the relation

$$E_l = A_l(\omega,\vec{r})e^{j[\omega t+\Phi_l(\omega,\vec{r})]}, \tag{1}$$

where A_l and Φ_l are respectively the amplitude and the phase of a given component (with the quantity kz being assumed to be zero), and $l = x, y$.

If the resulting wave is formed as a wave with continuous spectrum, the sum of the elementary Stokes parameters $S_{i\text{sp}}(\omega)$ is transformed into the integral

$$S_i = \int_0^\infty \rho(\omega) S_{i\text{sp}}(\omega)\, d\omega, \tag{2}$$

where $\rho(\omega)$ stands for the spectral density that describes the contribution of each elementary spectral component.

Let us consider an example of superposition of two waves with different frequencies. We wish to demonstrate that the Stokes parameters of a polychromatic wave, in which the field vector performs a completely deterministic motion in time, can correspond to a completely depolarized wave.

Let we ascribe the Stokes parameters to the superposition of two arbitrarily polarized waves with different frequencies. To do this, we use the relationship between the Stokes parameters and the components of the coherence matrix (see, e.g., Ref. [8]). It is known that the components of the coherence matrix read as follows:

$$J = \begin{bmatrix} E_x E_x^* & E_x E_y^* \\ E_y E_x^* & E_y E_y^* \end{bmatrix} = \begin{bmatrix} J_{xx} & J_{xy} \\ J_{yx} & J_{yy} \end{bmatrix}, \tag{3}$$

where

$$\begin{aligned} E_x &= A_{1x} e^{j(\omega_1 t+\Phi_{1x})} + A_{2x} e^{j(\omega_2 t+\Phi_{2x})} \\ E_y &= A_{1y} e^{j(\omega_1 t+\Phi_{1y})} + A_{2y} e^{j(\omega_2 t+\Phi_{2y})} \end{aligned}. \tag{4}$$

Here Φ_{il}, $i = 1, 2$, $l = x, y$ are the initial phases of the wave components involved in the superposition.

Let us make some transformations and average the above relations over a significantly long time interval. The latter can be shown to correspond to the beat period [4]:

$$T_b = \frac{2\pi}{\Delta\omega} = \frac{\lambda_m^2}{c\Delta\lambda} + \frac{\lambda_m}{c}, \tag{5}$$

where $\Delta\omega$ denotes the difference between the circular frequencies of the waves, λ_m the smallest of the wavelengths, and $\Delta\lambda$ the corresponding difference in wavelengths. Then the resulting Stokes parameters can be written as

$$\begin{aligned} S_0 &= A_{1x}^2 + A_{2x}^2 + A_{1y}^2 + A_{2y}^2 \\ S_1 &= 0 \\ S_2 &= 2\left[A_{1x}A_{1y}\cos(\Delta_1) + A_{2x}A_{2y}\cos(\Delta_2)\right], \\ S_3 &= 2\left[A_{1x}A_{1y}\sin(\Delta_1) + A_{2x}A_{2y}\sin(\Delta_2)\right] \end{aligned} \tag{6}$$

where Δ_1 and Δ_2 are the phase differences between the orthogonal components of the first and second waves.

Below we will consider in a more detail some special cases following from Eqs. (6).

2.1. Linearly orthogonally polarized beams

Let one of the waves be polarized along the x-axis and the other along the y-axis:

$$\begin{aligned} A_{1x} &= A_1, \; A_{1y} = 0 \\ A_{2x} &= 0, \; A_{2y} = A_2 \end{aligned}. \tag{7}$$

In accordance with Eqs. (6), the Stokes parameters of this superposition acquire the form

$$\begin{aligned} S_0 &= A_1^2 + A_2^2 \\ S_1 &= 0 \\ S_2 &= 0 \\ S_3 &= 0 \end{aligned}. \tag{8}$$

In other words, the Stokes parameters of this superposition correspond to the parameters typical for a completely depolarized wave. Therefore, we have to apply another type of description of the polarization characteristics, the Jones matrix formalism. Moreover, we will have to introduce some changes to the latter approach, when compared with its common version.

In the definition of Jones vectors, the temporal part in the exponential form $e^{j\omega t}$ is omitted (see, e.g., Ref. [1]). However, since we now deal with superposition of the waves with different frequencies, we will preserve this exponential term and introduce a modified (time-dependent) Jones vector for each of the waves:

$$\vec{J} = e^{j\omega t}\begin{bmatrix} U_x \\ U_y \end{bmatrix}, \tag{9}$$

where $U_l = A_l e^{\Phi_l}$ and $l = x, y$ imply the complex amplitudes of the Cartesian wave components.

Hence, we introduce a modified Jones vector as a sum of vectors of the type of Eq. (9), which corresponds to superposition of the waves with different frequencies:

$$\vec{J}_{\text{mod}} = \sum_i^N \vec{J}_i = \sum_i^N e^{j\omega_i t}\begin{bmatrix} U_{ix} \\ U_{iy} \end{bmatrix}. \tag{10}$$

Here $\vec{J}_i$ are the modified Jones vectors corresponding to each of the waves with specific frequencies and N stands for the number of these waves.

Suppose that the first and second waves in our case are polarized linearly and orthogonally. To be specific, we assume

$$A_{1x} = A,\ A_{1y} = 0,\ \ A_{2x} = 0,\ A_{2y} = A, \tag{11a}$$

$$\Phi_{1x} = 0,\ \Phi_{2y} = \Delta, \tag{11b}$$

where Δ denotes the difference between the initial phases of the orthogonal waves. Then the modified Jones vector becomes either

$$\vec{J}_{\text{mod}} = Ae^{j\frac{\omega_1+\omega_2}{2}t}\begin{bmatrix} e^{j\frac{\Delta\omega}{2}t} \\ e^{-j\left(\frac{\Delta\omega}{2}t-\Delta\right)} \end{bmatrix} \tag{12a}$$

or

$$\vec{J}_{\text{mod}} = Ae^{j\frac{\omega_1+\omega_2}{2}t}e^{j\frac{\Delta\omega}{2}t}\begin{bmatrix} 1 \\ e^{-j(\Delta\omega t-\Delta)} \end{bmatrix}. \tag{12b}$$

Eq. (12b) can be interpreted as follows. The phase difference between the x- and y-components of the resulting wave varies with changing frequency and time accordingly to the factor $\Delta\omega t$. The azimuth of the vector remains constant, regardless of Δ, i.e. the 'instantaneous ellipticity' of the resulting wave changes under condition of invariable azimuth, and the tip of the electric-field strength vector really moves in a complicated manner.

An even more transparent interpretation of the modified Jones vector arises when the superposition of orthogonal circularly polarized waves is dealt with, as will be shown below in Subsection 2.2.

2.2. Circularly orthogonally polarized beams

For simplicity, we assume that the modules of amplitudes of the first and second waves are the same ($A_{il} = A$) and the phases of the x-components of these waves are zero. The second requirement is easily met by simply shifting time readings. Then the modified Jones vector of the resulting wave can be written as

$$\vec{J}_{\text{mod}} = Ae^{j\frac{\Delta_1+\Delta_2}{2}}e^{j\frac{\omega_1+\omega_2}{2}t}\begin{bmatrix} \cos\left(\frac{\Delta\omega}{2}t\right) \\ \cos\left(\frac{\Delta\omega}{2}t+\frac{\Delta_1-\Delta_2}{2}\right) \end{bmatrix}, \tag{13}$$

where $\Delta\omega = \omega_1 - \omega_2$ and Δ_i denotes the phase difference between the x- and y-components of the first and second waves.

If we take into account that $\frac{\Delta_1-\Delta_2}{2} = \pm\frac{\pi}{2}$ and $\frac{\Delta_1+\Delta_2}{2} = 0$, Eq. (13) can be transformed to

$$\vec{J}_{\text{mod}} = Ae^{j\frac{\omega_1+\omega_2}{2}t}\begin{bmatrix} \cos\left[\pm\beta(t)\right] \\ \sin\left[\pm\beta(t)\right] \end{bmatrix} \sim A\begin{bmatrix} \cos\left[\pm\beta(t)\right] \\ \sin\left[\pm\beta(t)\right] \end{bmatrix}, \tag{14}$$

with $\beta(t) = \frac{\Delta\omega}{2}t$.

Eq. (14) can be interpreted as the Jones vector of linearly polarized wave with the azimuthal angle $\pm\beta(t)$ and the frequency $\frac{\omega_1+\omega_2}{2}$. In other words, the instantaneous polarization azimuth $\pm\beta(t)$ changes over time. Then the field vector carries out clockwise (or counter-clockwise) circulation when the β term in Eq. (14) has the sign "+" (or "–").

Hence, the following general conclusion follows from our consideration: the common methods used for describing polarization characteristics in the analysis of superposition of coupled waves are not suitable if one has to describe adequately the polarization state of the resulting wave. The way out of this situation can be the approach based on the modified Jones vector. However, the question of what the Stokes parameters characterize in the case of superposition of waves with different frequencies still remains open. It will be addressed below.

3. Stokes parameters and spin angular momentum of a polychromatic wave

It is known [11–14] that, in the monochromatic case, the spin angular momentum of a wave is determined by the gradient of coordinate distribution of its local fourth Stokes parameter. Then the transverse components $\overline{P}_x$ and $\overline{P}_y$ of the time-averaged Poynting vector can be conditionally divided into orbital ($\overline{P}_{x\text{orb}}$ and $\overline{P}_{y\text{orb}}$) and spin ($\overline{P}_{x\text{spin}}$ and $\overline{P}_{y\text{spin}}$) parts [14, 15]:

$$\begin{cases} \overline{P}_x \approx \overline{P}_{x\text{orb}} + \overline{P}_{x\text{spin}} \\ \overline{P}_y \approx \overline{P}_{y\text{orb}} + \overline{P}_{y\text{spin}} \end{cases}, \tag{15}$$

where the terms responsible for the magnitude of the spin angular momentum are determined by the relation

$$\overline{P}_{x\text{spin}} = \frac{c}{16\pi k}\frac{\partial S_3}{\partial y},\ \overline{P}_{y\text{spin}} = \frac{c}{16\pi k}\frac{\partial S_3}{\partial x}. \tag{16}$$

Here k implies the wave number. In other terms, the spin component of the Poynting vector is also determined by the gradient of the fourth Stokes parameter.

For the polychromatic wave, the situation remains similar since, when the Poynting vector is averaged over any time interval longer than the beat period, the components of this vector are simple sums of the elementary components [16]:

$$\begin{cases} \overline{P}_x = \overline{P}_{x1} + \overline{P}_{x2} \\ \overline{P}_y = \overline{P}_{y1} + \overline{P}_{y2} \end{cases}, \tag{17}$$

where $\overline{P}_{l1}$ and $\overline{P}_{l2}$ ($l = x, y$) are the Poynting components of the first and second waves.

Basing on the relation (16), one can state that the spin components of the resulting Poynting vector read as

$$\begin{cases} \overline{P}_{x\text{spin}} = \frac{c}{16\pi}\left(\frac{1}{k_1}\frac{\partial S_{31}}{\partial y} + \frac{1}{k_2}\frac{\partial S_{32}}{\partial y}\right) \\ \overline{P}_{y\text{spin}} = \frac{c}{16\pi}\left(\frac{1}{k_1}\frac{\partial S_{31}}{\partial x} + \frac{1}{k_2}\frac{\partial S_{32}}{\partial x}\right) \end{cases}, \tag{18}$$

where $k_i = \frac{\omega_i}{c} = \frac{2\pi}{\lambda_i}$ and $\frac{\partial S_{3i}}{\partial l}$ ($i = 1, 2,\ l = x, y$) are the corresponding derivatives of the fourth Stokes parameter of the first and second waves.

The spin components of the polychromatic wave can be transformed into the integrals

$$\begin{cases} \overline{P}_{x\text{spin}} = \dfrac{c}{16\pi} \displaystyle\int_0^\infty \rho(\omega) \frac{1}{k(\omega)} \frac{\partial S_3(\omega)}{\partial y} d\omega \\ \overline{P}_{y\text{spin}} = \dfrac{c}{16\pi} \displaystyle\int_0^\infty \rho(\omega) \frac{1}{k(\omega)} \frac{\partial S_3(\omega)}{\partial x} d\omega) \end{cases}. \quad (19)$$

Hence, one can state that, in the case of the polychromatic wave, the Stokes parameters characterize the magnitude of the spin angular momentum of the wave rather that the polarization state of the latter. For instance, the results obtained in Section 2 for the superposition of linearly polarized waves with different frequencies (when the Stokes parameters correspond to a depolarized wave) indicate nothing but the fact that the spin moment of this superposition is zero.

4. Conclusions

Basing on the results obtained in the present work, one can arrive at the following main conclusions.

1. The common methods employed to describe the polarization characteristics of superpositions of the coupled waves (e.g., the approach based on application of the Stokes parameters) are not always suitable for adequate description of the polarization state of the resulting waves.

2. The way out of this situation can be the approach based on the modified Jones vectors.

3. The Stokes parameters of the polychromatic wave can fail when describing the polarization state of the wave. However, these parameters are uniquely related to the magnitude of the spin angular momentum of this wave. In this sense, their physical interpretation is hardly related to the trajectory described by the tip of the field-strength vector. Instead, the Stokes parameters can be ascribed to the nature of circulation the wave carries out.

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Анотація. *Розглянуто поляризаційні характеристики поліхроматичної хвилі, пов'язані з параметрами Стокса та вектором Джонса. Показано, що параметри Стокса не можуть адекватно описати стан поляризації хвилі, утвореної в результаті суперпозиції хвиль з різними частотами. Для характеризації такої суперпозиції введено модифікований вектор Джонса. З іншого боку, ми демонструємо, що загальновідомі параметри Стокса однозначно визначають спіновий момент результуючого хвильового поля.*