Optical solitons in the Sasa–Satsuma model with multiplicative noise via Itô calculus

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Abstract. We study for the first time perturbed optical solitons modelled using a Sasa–Satsuma equation involving a multiplicative noise. Two integration schemes retrieve soliton solutions to this model, which are described using parametric constraints.

Keywords: solitons, multiplicative noise, Itô calculus

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1. Introduction

Optical solitons molecules represent today’s technological marvel which is studied all across the globe, with a wide variety of far-reaching practical potentials. The above studies have mainly included integrability of a governing model, a soliton-perturbation theory, a quasi-monochromatic dynamics, a soliton-parameter dynamics, a collision-induced timing jitter, a four-wave mixing, and a number of numerical schemes for visualization. On the other hand, the aspect of stochasticity represents one of the features which have been rarely touched upon. As a result, there have been only a few studies addressing this issue [1–3]. Now it is time to shed more light in this direction.

Although there are a number of models utilized in fibre optics at the present, below we will concentrate on a well-known Sasa–Satsuma equation [4–8]. It is an integrable perturbed nonlinear Schrödinger’s equation, which represents a basic governing model for the studies of soliton propagation through optical fibres used on intercontinental distances. A stochastic analysis with the nonlinear Schrödinger’s equation addressed in the past has involved only an additive noise for solitons and Gaussons [1–3, 9, 10].

In the current work we will deal with the Sasa–Satsuma equation that includes multiplicative noise instead. A relevant Itô calculus will be implemented in order to recover soliton solutions.

The means employed by us are a unified Riccati-equation expansion and an enhanced Kudryashov's approach. Our main results will be demonstrated and their importance will be explained after a comprehensive analysis of mathematical procedures is done.

2. Governing model

We begin with writing out, for the first time, a dimensionless form of the stochastic Sasa–Satsuma equation with the inclusion of multiplicative noise in the Itô sense:

$$iu_t + au_{xx} + b|u|^2 u + i\left[\alpha u_{xxx} + \beta |u|^2 u_x + \gamma\left(|u|^2\right)_x u\right] + \sigma u \frac{dW(t)}{dt} = 0\,. \tag{1}$$

Here $u(x,t)$ is a complex-valued function, $i=\sqrt{-1}$, and x and t denote respectively the distance and the time in dimensionless form. The first term in the Eq. (1) governs a linear temporal evolution, a, b and α are the coefficients of respectively the chromatic dispersion, the Kerr law of nonlinearity and the third-order dispersion term, and β and γ imply the nonlinear dispersion terms. Finally, σ denotes the coefficient of noise strength and $W(t)$ refers to the standard Wiener process, such that d$W(t)$/dt is the white noise.

In order to solve the model equation (1), we first use the wave transformation

$$u(x,t) = \varphi(\xi)\exp i\left[\theta(x,t) + \sigma W(t) - \sigma^2 t\right], \tag{2}$$

$$\xi = x - ct, \quad \theta(x,t) = \kappa x - \Omega t\,, \tag{3}$$

where c, κ and Ω are nonzero real-valued constants, ξ denotes the wave variable, κ is the wave number, Ω is the frequency, c is the free-space velocity, $\varphi(\xi)$ is the amplitude component, and $\theta(x,t)$ the phase component.

Inserting the wave transformation given by Eq. (2) into the governing model (1) gives the ordinary differential equation (ODE)

$$(a - 3\kappa\alpha)\varphi'' + (b - \kappa\beta)\varphi^3 + \left(\Omega + \sigma^2 - a\kappa^2 + \alpha\kappa^3\right)\varphi = 0\,, \tag{4}$$

along with the wave number of the soliton

$$\kappa = \frac{a\beta + 2a\gamma - 3b\alpha}{3\left[\beta(\gamma - \alpha) + 2\gamma^2\right]}\,, \tag{5}$$

and its velocity

$$c = \frac{\left(2a\kappa - 3\kappa^2\alpha\right)(a - 3\kappa\alpha) - \alpha\left(\Omega + \sigma^2 - a\kappa^2 + \alpha\kappa^3\right)}{a - 3\kappa\alpha}\,. \tag{6}$$

Here a standard notation $'' = \frac{d^2}{d\xi^2}$ is used.

3. Unified Riccati-equation expansion

According to the above method, Eq. (4) has a formal solution

$$\varphi(\xi) = \alpha_0 + \alpha_1 F(\xi), \quad \alpha_1 \neq 0\,, \tag{7}$$

where α_0 and α_1 are real-valued constants. Here $F(\xi)$ represents a real-valued function that satisfies the Riccati equation:

$$F'(\xi) = C_0 + C_1 F(\xi) + C_2 F^2(\xi), C_2 \neq 0\,, \tag{8}$$

with C_j (j = 0, 1, 2) being real-valued constants. Substituting the formal solution given by Eq. (7) and the Riccati equation (8) into Eq. (4) yields the following result:

$$\Omega = \frac{1}{2}\Delta\left(a-3\alpha\kappa\right)+\kappa^2\left(a-\alpha\kappa\right)-\sigma^2,$$
$$\alpha_0 = \frac{C_1}{2}\sqrt{-\frac{2\left(a-3\alpha\kappa\right)}{b-\beta\kappa}}, \alpha_1 = C_2\sqrt{-\frac{2\left(a-3\alpha\kappa\right)}{b-\beta\kappa}}. \tag{9}$$

After inserting the parameters involved in Eq. (9) and the well-known solutions of the Riccati equation (8) into Eq. (7), one concludes that the model equation (1) has a dark-soliton solution

$$u(x,t) = \pm\sqrt{-\frac{\Delta\left(a-3\alpha\kappa\right)}{2\left(b-\beta\kappa\right)}}\tanh\left(\frac{\sqrt{\Delta}}{2}\xi\right)e^{i\left(\kappa x-\left[\frac{1}{2}\Delta\left(a-3\alpha\kappa\right)+\kappa^2\left(a-\alpha\kappa\right)\right]t+\sigma W(t)\right)}, \tag{10}$$

and a singular-soliton solution

$$u(x,t) = \pm\sqrt{-\frac{\Delta\left(a-3\alpha\kappa\right)}{2\left(b-\beta\kappa\right)}}\coth\left(\frac{\sqrt{\Delta}}{2}\xi\right)e^{i\left(\kappa x-\left[\frac{1}{2}\Delta\left(a-3\alpha\kappa\right)+\kappa^2\left(a-\alpha\kappa\right)\right]t+\sigma W(t)\right)}, \tag{11}$$

with $\left(a-3\alpha\kappa\right)\left(b-\beta\kappa\right)<0,\ \Delta = C_1^2-4C_0C_2>0$.

4. Enhanced Kudryashov's approach

Balancing $\varphi''(\xi)$ with $\varphi^3(\xi)$ in Eq. (4), we obtain

$$M+2p=3M \Rightarrow M=p. \tag{12}$$

Let us consider the following cases in our analysis.

Case 1: Choosing $p = 1$, one has $M = 1$. Then the formal solution follows immediately:

$$\varphi(\xi) = B_0 + B_1R(\xi), \quad B_1 \neq 0, \tag{13}$$

where B_0 and B_1 are real-valued constants and $R(\xi)$ is a real-valued function satisfying the ODE

$$R'^2(\xi) = R^2(\xi)\left[1-\chi R^2(\xi)\right]\ln^2 K, \quad 0<K\neq 1, \tag{14}$$

with χ denoting a nonzero real-valued constant. Substituting the formal solution given by Eq. (13) and the ODE (14) into Eq. (4), one retrieves the result

$$\Omega = -\left(a-3\alpha\kappa\right)\ln^2 K+\kappa^2\left(a-\alpha\kappa\right)-\sigma^2, \quad B_0 = 0, \quad B_1 = \sqrt{\frac{2\chi\left(a-3\alpha\kappa\right)\ln^2 K}{b-\beta\kappa}}. \tag{15}$$

Inserting the parameters given by Eqs. (15) and the well-known solutions of the ODE (14) into the formal solution (13), one reveals a bright-soliton solution

$$u(x,t) = \sqrt{\frac{2\left(a-3\alpha\kappa\right)\ln^2 K}{b-\beta\kappa}}\operatorname{sech}\left(\xi\ln K\right)e^{i\left(\kappa x+\left[\left(a-3\alpha\kappa\right)\ln^2 K-\kappa^2\left(a-\alpha\kappa\right)\right]t+\sigma W(t)\right)}, \tag{16}$$

with $\left(a-3\alpha\kappa\right)\left(b-\beta\kappa\right)>0$ (see Fig. 1). A singular-soliton solution

$$u(x,t) = \sqrt{-\frac{2\left(a-3\alpha\kappa\right)\ln^2 K}{b-\beta\kappa}}\operatorname{csch}\left(\xi\ln K\right)e^{i\left(\kappa x+\left[\left(a-3\alpha\kappa\right)\ln^2 K-\kappa^2\left(a-\alpha\kappa\right)\right]t+\sigma W(t)\right)} \tag{17}$$

also follows provided that the condition $\left(a-3\alpha\kappa\right)\left(b-\beta\kappa\right)<0$ holds true.

Case 2: Let us choose $p = 2$. Then we have $M = 2$. Now we recover the formal solution

$$\varphi(\xi) = B_0 + B_1R(\xi) + B_2R^2(\xi),\ B_2 \neq 0, \tag{18}$$

where B_0, B_1 and B_2 are real-valued constants and $R(\xi)$ implies a real-valued function that satisfies the ODE

$$R'^2(\xi) = R^2(\xi)\left[1-\chi R^4(\xi)\right]\ln^2 K, \quad 0 < K \neq 1, \tag{19}$$

with χ being a nonzero real-valued constant. Taking into account the formal solution (18), the ODE (19) and Eq. (4), one obtains

$$\Omega = -4(a-3\alpha\kappa)\ln^2 K + \kappa^2(a-\alpha\kappa) - \sigma^2, \quad B_0 = 0, \ B_1 = 0, \quad B_2 = 2\sqrt{\frac{2\chi(a-3\alpha\kappa)\ln^2 K}{b-\beta\kappa}}. \tag{20}$$

Substituting the parameters given by Eqs. (20) and the well-known solutions of the ODE (19) into the formal solution (18) leads again to the bright-soliton solution

$$u(x,t) = \sqrt{\frac{8(a-3\alpha\kappa)\ln^2 K}{b-\beta\kappa}}\operatorname{sech}(2\xi\ln K)\, e^{i\left(\kappa x+\left[4(a-3\alpha\kappa)\ln^2 K-\kappa^2(a-\alpha\kappa)\right]t+\sigma W(t)\right)}, \tag{21}$$

with $(a-3\alpha\kappa)(b-\beta\kappa) > 0$. The singular-soliton solution

$$u(x,t) = \sqrt{-\frac{8(a-3\alpha\kappa)\ln^2 K}{b-\beta\kappa}}\operatorname{csch}(2\xi\ln K)\, e^{i\left(\kappa x+\left[4(a-3\alpha\kappa)\ln^2 K-\kappa^2(a-\alpha\kappa)\right]t+\sigma W(t)\right)} \tag{22}$$

arises when we have $(a-3\alpha\kappa)(b-\beta\kappa) < 0$.

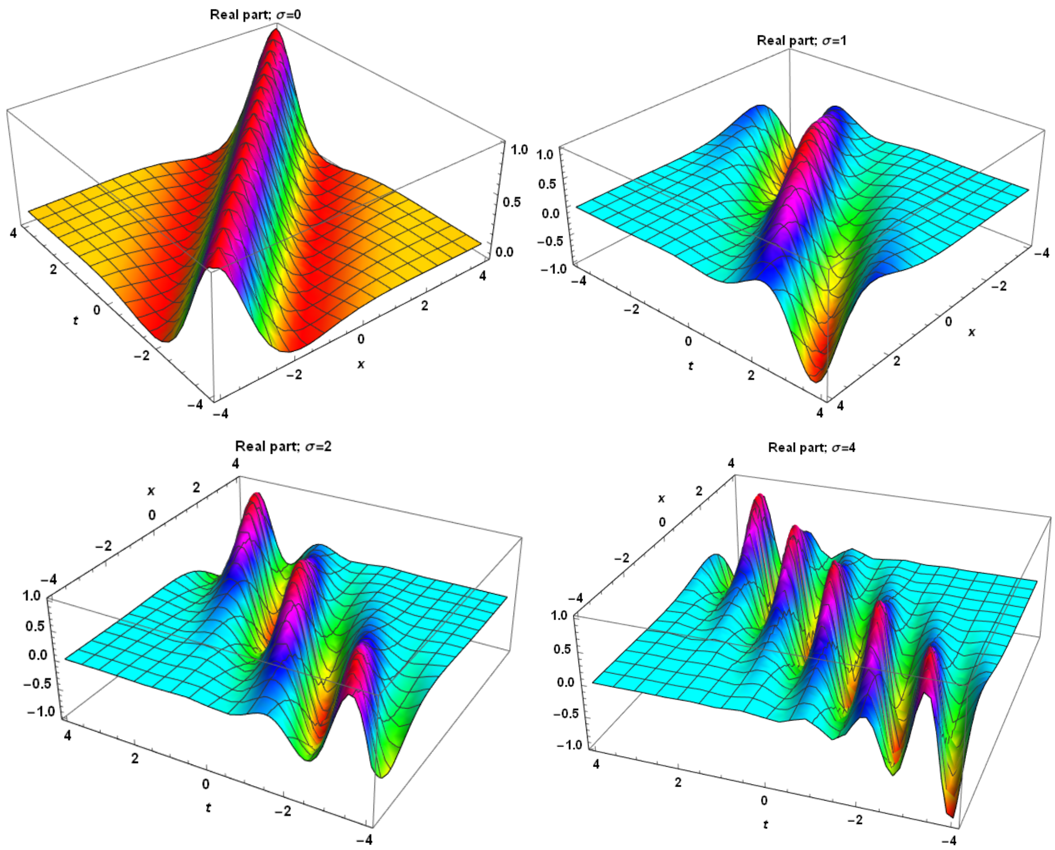

Fig. 1. Surface plots of a bright soliton (see Eq. (16)) obtained when the conditions $K = k = e$ hold true and the other parameters involved are unit. The values of σ parameter chosen in our calculations are displayed in the legend.

5. Discussion and concluding remarks

Hence, in the present work we have retrieved the soliton solutions to the Sasa–Satsuma equation that includes the multiplicative-noise effect. Two integration schemes, the Riccati-equation expansion and the enhanced Kudrayshov's approach, have led to the same soliton solutions, which are important for telecommunications industry. These solutions correspond closely to realistic situations since no notion of an optoelectronic system can be regarded as meaningful without proper consideration of its stochasticity.

Thus, our results are in great need of being explored further on. This would mean consideration of the noise in birefringent fibres and, eventually, in dispersion-flattened fibres. Thus, many more openings are in the pipeline, which would mean opening up a floodgate of opportunities. Such opportunities would be sequentially explored with time.

A following portion of discussion would be suitable in relation to the results obtained above and their possible impact. It is known that the influence of multiplicative noise on the stochastic differential equations in the Ito sense has already attracted the attention of many researches [1–3, 9, 10]. For instance, a (2+1)-dimensional stochastic chiral nonlinear Schrödinger equation that involves the multiplicative noise in the Itô sense has been employed in Ref. [1]. A stochastic Nizhnik–Novikov–Veselov system with the multiplicative noise in the Itô sense has also been considered [2]. A stochastic Burgers' equation, which is forced by the multiplicative noise in the Stratonovich sense, has been addressed in Ref. [3]. Moreover, a stochastic nonlinear Schrödinger equation with the multiplicative noise in the Ito sense has been discussed [9]. Finally, a stochastic Hirota–Maccari system involving the multiplicative noise in the Itô sense has also been investigated [10].

In this respect one can notice that the stochastic differential equations including either noise or fluctuations that depend on time represent more accurate mathematical models for many complex systems arising in various fields of applied science, e.g. in nonlinear optics. As a consequence, our first-time examination of the Sasa–Satsuma equation with the effect of multiplicative noise in the Ito sense is of a primary importance.

Note also that the Sasa–Satsuma equation without the effect of multiplicative noise has earlier been widely investigated in the literature [4–8]. For example, a multi-parameter family of solitons against the background solution to the Sasa-Satsuma equation has been reported in Ref. [4]. The perturbed Sasa–Satsuma equation has been discussed using a Laplace–Adomian decomposition method, which results in the both bright and dark optical solitons [5]. Moreover, the solitary waves arising within the approach of generalized Sasa–Satsuma equation with arbitrary refractive index have been discussed [6]. Envelope-travelling wave solutions within the Sasa-Satsuma equation have also been studied using a unified direct-integral method [7]. Finally, the initial-boundary value problem for the Sasa–Satsuma equation on the half-line has been examined using a unified-transform technique [8].

It is worthwhile that the Sasa–Satsuma equation with the effect of multiplicative noise in the Ito sense has been elaborated in the current work because the multiplicative noise involved in Eq. (1) describes a process where the phase of wave excitation is disturbed. In crystals, this type of noise corresponds to scattering of excitons by phonons due to thermal molecular vibrations.

Finally, the effect of the multiplicative noise on the bright-soliton solution given by Eq. (16) can be better understood and illustrated with the data displayed in Fig. 1 at $\sigma = 0$, 1, 2, 4. In particular, one can see that the noise intensity at $\sigma = 0$ reduces the model equation (1) to the canonical Sasa–Satsuma equation [4–8].

Disclosure

The authors declare that there is no conflict of interest.

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Анотація. *Ми вперше дослідили збурені оптичні солітони, змодельовані в рамках рівняння Саса–Сацуми, яке містить мультиплікативний шум. Для двох схем інтегрування одержано солітонні розв'язки моделі, які описано за допомогою параметричних обмежень.*