Cubic–quartic optical solitons having quadratic–cubic nonlinearity by sine–Gordon equation approach

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Abstract. We reveal cubic–quartic optical solitons with quadratic–cubic nonlinearity for the first time. Both polarization-preserving and birefringent optical fibres are considered. The study is also extended to include the perturbation terms of Hamiltonian type. The integration algorithm adopted in the present work is the method of sine-Gordon equation.

Keywords: solitons, quadratic–cubic nonlinearity, birefringence, perturbation

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1. Introduction

Studies of dynamics of optical solitons have transformed telecommunication engineering beyond comprehension. Today the appropriate technologies are widely applied in many braches, e.g. in Internet communication. Recent successes in quantum optics have also made some modern-day technological marvels a reality. A delicate balance between chromatic dispersion and self-phase modulation is one of the key factors that need to be sustained for a smooth transmission of soliton pulses across inter-continental distances. This is not always ensured. One possible reason is that the chromatic dispersion could run low. In such a situation, the common chromatic dispersion gets replaced by a combination of third- and fourth-order dispersions, which would together compensate for its low count.

In the present work, we will handle a governing nonlinear Schrödinger equation (NLSE) that comes with a combination of third- and fourth-order dispersions, which would yield cubic–quartic (CQ) solitons. A self-phase modulation in this model is set to be associated with a quadratic–cubic

(QC) form of the nonlinear refractive index. It should be marked that the optical solitons appearing under the condition of QC nonlinearity law have already been considered by using different integration methods [1–15].

The optical solitons with the QC nonlinearity and nonlinear perturbation terms have been studied with a semi-inverse variational principle [1], a method of undetermined coefficients [2] and a hypothesis of travelling waves [3]. Chirped optical Gausson solitons with the QC nonlinearity have been found in the presence of perturbation terms with the aid of collective-variables approach [4]. The structure and dynamics of one-dimensional binary Bose gases forming quantum droplets have been touched upon by solving a relevant amended Gross–Pitaevskii equation [5]. Optical-soliton cooling under QC nonlinearity has been reported in Ref. [6]. CQ optical solitons in the birefringent fibres with Kudryashov's law for the nonlinear refractive index, which have three special cases of the power-law nonlinearity parameters, have been addressed [7], using an extended trial-function algorithm. Chaotic solitons under managed QC nonlinearity have been yielded within the NLSE [8]. A single terahertz electromagnetic pulse has been found to have unique features in the presence of both second- and third-order nonlinearities [9]. Kink solitons in the QC nonlinear dispersive media have been discussed in Ref. [10]. Soliton solutions for the optical fibres made of QC nonlinear media have been obtained by virtue of a complex ansätze approach [11]. Moreover, optical solitons and conservation laws for the case of QC nonlinearity have been obtained [12]. CQ optical solitons have been found with the QC NLSE for four different non-Kerr nonlinearities, using a unified method of Riccati-equation expansion [13]. Bright, dark and singular optical solitons for the case of Kudryashov's sextic power-law nonlinearity of refractive index have been obtained with a unified Riccati-equation method, a mapping scheme and a supplemented Kudryashov's algorithm [14]. Finally, CQ optical solitons have been revealed in the polarization-preserving fibres with five different nonlinear refractive-index structures, using a perturbed complex Ginzburg–Landau equation and a supplemented Kudryashov's approach [15].

The QC form of the self-phase modulation mentioned above has been first proposed in 1994 and then resurfaced in 2011 (see Refs. [8–10]). Subsequently, a plethora of research results have been reported for such a nonlinear form of refractive index [1–7, 11, 12]. After that the concept of CQ solitons has come and gained a lot of attention of researchers. A number of prominent results have made it visible all across with a variety of models to address the soliton dynamics in optical fibres [13–15].

In the present work we combine the concept of CQ solitons to be studied with the QC nonlinearity. Namely, we formulate the governing NLSE and derive its solutions with the aid of sine-Gordon equation approach. This NLSE is modelled for the first time in the cases of polarization-preserving and birefringent fibres, both in the presence and absence of perturbation terms. Moreover, we report for the first time a complete spectrum of soliton solutions, which emerge from this integration scheme. The main results of our study will be displayed after a brief introduction to our model.

2. Polarization-preserving fibres

In this section we address the CQ-NLSE for the case of QC nonlinearity in the polarization-preserving fibres.

2.1. Unperturbed model

The unperturbed CQ-NLSE with the QC nonlinearity in the polarization-preserving fibres reads as

$$iq_t + iaq_{xxx} + bq_{xxxx} + \left(c_1|q| + c_2|q|^2\right)q = 0\,, \tag{1}$$

where x and t are the dimensionless distance and time, respectively. The first term in the Eq. (1) refers to the linear temporal evolution, the complex-valued function $q(x,t)$ represents the optical solitons in the polarization-preserving fibres, a and b are the constant coefficients of respectively third- and fourth-order dispersions, the constant coefficients c_1 and c_2 constitute the QC nonlinearity, and $i=\sqrt{-1}$.

To obtain the optical solitons from the unperturbed CQ-NLSE with the QC nonlinearity in the polarization-preserving fibres, we assume the travelling-wave transformation

$$\begin{aligned} &q(x,t) = U(\xi)e^{i\varphi(x,t)}, \quad \xi = x - vt, \\ &\varphi(x,t) = -\kappa x + \omega t + \theta_0, \end{aligned} \tag{2}$$

where the constants ω, v, κ and θ_0 represent the wave frequency, the velocity, the wave number and the phase constant, respectively. Here the real-valued functions $U(\xi)$ and $\varphi(x,t)$ stand respectively for the amplitude and phase components of the soliton. After substituting Eq. (2) into Eq. (1), one obtains a fourth-order ordinary differential equation

$$bU^{(iv)} + 6b\kappa^2U'' - \left(\omega + 3b\kappa^4\right)U + c_1U^2 + c_2U^3 = 0, \tag{3}$$

along with the constraints

$$a = 4b\kappa, \tag{4}$$

$$v = -8b\kappa^3, \tag{5}$$

where the notations $' = \dfrac{d}{d\xi}$, $'' = \dfrac{d^2}{d\xi^2}$ and $^{(iv)} = \dfrac{d^4}{d\xi^4}$ are used.

Eq. (3) can be integrated to determine the soliton profile, while Eq. (5) gives the soliton velocity. According to the approach of sine-Gordon equation, Eq. (3) yields a formal solution

$$U(\xi) = \sum_{i=1}^{N}\cos^{i-1}\left(V(\xi)\right)\left[B_i\sin\left(V(\xi)\right) + A_i\cos\left(V(\xi)\right)\right] + A_0 \tag{6}$$

along with the ordinary differential equation

$$V'(\xi) = \sin\left(V(\xi)\right), \tag{7}$$

and the exact solutions

$$\begin{aligned} &\sin\left(V(\xi)\right) = \operatorname{sech}(\xi), \quad \sin\left(V(\xi)\right) = \operatorname{icsh}(\xi), \\ &\cos\left(V(\xi)\right) = \tanh(\xi), \quad \cos\left(V(\xi)\right) = \coth(\xi). \end{aligned} \tag{8}$$

Here $V(\xi)$ is a new positive function of ξ, A_i and B_i are the constants, and the integer N can be identified by applying a balance principle between the nonlinear term and the highest-order derivative term. Balancing $U^{(iv)}$ with U^3 in Eq. (3) gives $N = 2$. Thus, Eq. (6) transforms into the solution

$$\begin{aligned} U(\xi) = {} & A_0 + B_1\sin\left(V(\xi)\right) + A_1\cos\left(V(\xi)\right) \\ & + \cos\left(V(\xi)\right)\left(B_2\sin\left(V(\xi)\right) + A_2\cos\left(V(\xi)\right)\right). \end{aligned} \tag{9}$$

After inserting Eqs. (9) and (7) into Eq. (3), one can arrive at the following results.

Case 1:

$$A_1 = 0,\ B_1 = 0,\ A_0 = \pm\sqrt{\frac{11b}{c_2}},\ A_2 = 0,$$
$$B_2 = \pm\sqrt{\frac{120b}{c_2}},\ c_1 = \pm 3\sqrt{11bc_2},\ \kappa = \pm\frac{\sqrt{15}}{3},\ \omega = -\frac{91b}{3}. \tag{10}$$

Substituting Eqs. (10) and (8) into Eq. (9) results in

$$q(x,t) = \pm\sqrt{\frac{b}{c_2}}\left\{\pm\sqrt{11} \pm \sqrt{120}\tanh\left(x + 8b\kappa^3 t\right)\operatorname{sech}\left(x + 8b\kappa^3 t\right)\right\} e^{i(-\kappa x + \omega t + \theta_0)}. \tag{11}$$

The solution given by Eq. (11) implies a combo dark-bright soliton, along with the corresponding constraint $bc_2 > 0$.

Case 2:

$$A_1 = 0, B_1 = 0, A_0 = \pm\sqrt{-\frac{120b}{c_2}}, A_2 = \pm\sqrt{-\frac{120b}{c_2}}, B_2 = 0,$$
$$c_1 = \pm\left(3\kappa^2 + 10\right)\sqrt{-\frac{6bc_2}{5}}, \omega = -b\left(3\kappa^4 - 24\kappa^2 - 16\right). \tag{12}$$

Inserting Eqs. (12) and (8) into Eq. (9) leads to

$$q(x,t) = \pm\sqrt{-\frac{120b}{c_2}}\left\{1 + \tanh^2\left(x + 8b\kappa^3 t\right)\right\} e^{i(-\kappa x + \omega t + \theta_0)}, \tag{13}$$

$$q(x,t) = \pm\sqrt{-\frac{120b}{c_2}}\left\{1 + \coth^2\left(x + 8b\kappa^3 t\right)\right\} e^{i(-\kappa x + \omega t + \theta_0)}. \tag{14}$$

The solutions given by Eqs. (13) and (14) display respectively dark and singular solitons, along with the constraint $bc_2 < 0$.

Case 3:

$$A_1 = 0, A_2 = \pm\sqrt{-\frac{120b}{c_2}},\ A_0 = \pm\frac{3b\kappa^4 + 12b\kappa^2 - 88b + \omega}{3\kappa^2 + 10}\sqrt{-\frac{5}{6bc_2}},$$
$$B_1 = 0,\ B_2 = 0,\ c_1 = \pm\frac{3b\kappa^4 - 120b\kappa^2 + 40b - 5\omega}{3\kappa^2 + 10}\sqrt{-\frac{3c_2}{10b}}, \tag{15}$$
$$117b^2\kappa^8 + 936b^2\kappa^6 + \left(-1176b^2 + 24b\omega\right)\kappa^4 + \left(4800b^2 - 120b\omega\right)\kappa^2$$
$$+4480b^2 + 280b\omega - 5\omega^2 = 0.$$

Substituting Eqs. (15) and (8) into Eq. (9) gives

$$q(x,t) = \left\{\pm\frac{3b\kappa^4 + 12b\kappa^2 - 88b + \omega}{3\kappa^2 + 10}\sqrt{-\frac{5}{6bc_2}} \pm \sqrt{-\frac{120b}{c_2}}\tanh^2\left(x + 8b\kappa^3 t\right)\right\} e^{i(-\kappa x + \omega t + \theta_0)}, \tag{16}$$

$$q(x,t) = \left\{\pm\frac{3b\kappa^4 + 12b\kappa^2 - 88b + \omega}{3\kappa^2 + 10}\sqrt{-\frac{5}{6bc_2}} \pm \sqrt{-\frac{120b}{c_2}}\coth^2\left(x + 8b\kappa^3 t\right)\right\} e^{i(-\kappa x + \omega t + \theta_0)}. \tag{17}$$

The solutions (16) and (17) represent dark and singular solitons, respectively, with the constraint $bc_2 < 0$.

Case 4:

$$A_1 = 0, B_1 = 0, A_0 = \pm\sqrt{-\frac{30b}{c_2}}, A_2 = \pm\sqrt{-\frac{30b}{c_2}}, B_2 = \pm\sqrt{\frac{30b}{c_2}},$$
$$c_1 = \pm\left(6\kappa^2 + 5\right)\sqrt{-\frac{3bc_2}{10}}, \omega = -b\left(3\kappa^4 - 6\kappa^2 - 1\right). \tag{18}$$

Inserting Eqs. (18) and (8) into Eq. (9) leads to

$$q(x,t) = \pm\sqrt{-\frac{30b}{c_2}}\left\{1 + \coth^2\left(x + 8b\kappa^3 t\right) + \coth\left(x + 8b\kappa^3 t\right)\operatorname{csch}\left(x + 8b\kappa^3 t\right)\right\} \times e^{i(-\kappa x + \omega t + \theta_0)}. \tag{19}$$

The solution (19) signifies a combo singular soliton with a constraint $bc_2 < 0$.

2.2. Perturbed model

The perturbed CQ-NLSE with the QC nonlinearity in the polarization-preserving fibres is structured as

$$iq_t + iaq_{xxx} + bq_{xxxx} + \left(c_1|q| + c_2|q|^2\right)q = i\left[\alpha\left(|q|^{2m}q\right)_x + \lambda\left(|q|^{2m}\right)_x q + \mu|q|^{2m}q_x\right], \tag{20}$$

where α is the coefficient of self-steepening nonlinearity, λ and μ denote the coefficients of higher-order dispersion effects, and m stands for the overall nonlinearity exponent. Substituting Eq. (2) into Eq. (20) leads to the ordinary differential equation

$$bU^{(iv)} + 6b\kappa^2 U'' - \left(\omega + 3b\kappa^4\right)U + c_1U^2 + c_2U^3 - \left(\kappa\alpha + \kappa\mu\right)U^{2m+1} = 0, \tag{21}$$

with the constraints

$$a = 4b\kappa, \tag{22}$$

$$v = -8b\kappa^3, \tag{23}$$

$$2\alpha m + \alpha + 2\lambda m + \mu = 0. \tag{24}$$

Eq. (21) admits Eq. (9). Inserting Eqs. (9) and (7) into Eq. (21) gives rise to the following results.

Case 1:

$$A_0 = 0,\ A_1 = 0,\ B_1 = 0,\ B_2 = 0,$$
$$m = 1,\ \kappa = \pm\frac{2\sqrt{3}}{3},\ \ \omega = \frac{200b}{3},$$
$$A_2 = \pm\sqrt{\frac{120b}{\alpha\kappa + \mu\kappa - c_2}},\quad c_1 = \pm\frac{16\sqrt{30b\left(\alpha\kappa + \mu\kappa - c_2\right)}}{5}. \tag{25}$$

Substitution of Eqs. (25) and (8) into Eq. (9) results in

$$q(x,t) = \pm\sqrt{\frac{120b}{\alpha\kappa + \mu\kappa - c_2}}\tanh^2\left(x + 8b\kappa^3 t\right)e^{i(-\kappa x + \omega t + \theta_0)}, \tag{26}$$

$$q(x,t) = \pm\sqrt{\frac{120b}{\alpha\kappa + \mu\kappa - c_2}}\coth^2\left(x + 8b\kappa^3 t\right)e^{i(-\kappa x + \omega t + \theta_0)}. \tag{27}$$

The solutions given by Eqs. (26) and (27) represent respectively dark and singular solitons, with the constraint $b\left(\alpha\kappa + \mu\kappa - c_2\right) > 0$.

Case 2:

$$\begin{gathered}B_1=0,\quad B_2=0,\quad m=1,\quad A_1=0,\\ A_0=\pm\sqrt{\frac{120b}{\alpha\kappa+\kappa\mu-c_2}},\quad A_2=\pm\sqrt{\frac{120b}{\alpha\kappa+\kappa\mu-c_2}},\\ \omega=-\left(3\kappa^4-24\kappa^2-16\right)b,\quad c_1=\pm\frac{\left(3\kappa^2+10\right)\sqrt{30b\left(\alpha\kappa+\kappa\mu-c_2\right)}}{5}.\end{gathered} \tag{28}$$

Inserting Eqs. (28) and (8) into Eq. (9) leads to

$$q(x,t)=\pm\sqrt{\frac{120b}{\alpha\kappa+\kappa\mu-c_2}}\left\{1+\tanh^2\left(x+8b\kappa^3t\right)\right\}e^{i\left(-\kappa x+\omega t+\theta_0\right)}, \tag{29}$$

$$q(x,t)=\pm\sqrt{\frac{120b}{\alpha\kappa+\kappa\mu-c_2}}\left\{1+\coth^2\left(x+8b\kappa^3t\right)\right\}e^{i\left(-\kappa x+\omega t+\theta_0\right)}. \tag{30}$$

The solutions given by Eq. (29) and (30) mean respectively dark and singular solitons, under the constraint $b\left(\alpha\kappa+\mu\kappa-c_2\right)>0$.

Case 3:

$$\begin{gathered}B_1=0,\quad A_1=0,\quad A_2=0,\quad m=1,\quad \kappa=\pm\frac{\sqrt{15}}{3},\\ A_0=\pm\sqrt{-\frac{11b}{\alpha\kappa+\kappa\mu-c_2}},\ B_2=\pm\sqrt{-\frac{120b}{\alpha\kappa+\kappa\mu-c_2}},\\ c_1=\pm3\sqrt{-11b\left(\alpha\kappa+\kappa\mu-c_2\right)},\quad \omega=-\frac{91b}{3}.\end{gathered} \tag{31}$$

Substitution of Eqs. (31) and (8) into Eq. (9) yields the results

$$\begin{gathered}q(x,t)=\pm\sqrt{-\frac{b}{\alpha\kappa+\kappa\mu-c_2}}\left\{\sqrt{11}+\sqrt{120}\tanh\left(x+8b\kappa^3t\right)\operatorname{sech}\left(x+8b\kappa^3t\right)\right\}\\ \times e^{i\left(-\kappa x+\omega t+\theta_0\right)}.\end{gathered} \tag{32}$$

The solution (32) is a combo dark-bright soliton appearing under the constraint $b\left(\alpha\kappa+\mu\kappa-c_2\right)<0$.

Case 4:

$$\begin{gathered}m=1,\ A_0=0,\ A_1=0,\ B_1=0,\\ \kappa=\pm\frac{2\sqrt{3}}{3},\ \omega=\frac{47b}{3},\ A_2=\pm\sqrt{\frac{30b}{\alpha\kappa+\kappa\mu-c_2}},\\ B_2=\pm\sqrt{-\frac{30b}{\alpha\kappa+\kappa\mu-c_2}},\ c_1=\pm17\sqrt{\frac{3b\left(\alpha\kappa+\kappa\mu-c_2\right)}{10}}.\end{gathered} \tag{33}$$

Inserting Eqs. (33) and (8) into Eq. (9) leads to

$$\begin{gathered}q(x,t)=\pm\sqrt{\frac{30b}{\alpha\kappa+\kappa\mu-c_2}}\left\{\coth^2\left(x+8b\kappa^3t\right)+\coth\left(x+8b\kappa^3t\right)\operatorname{csch}\left(x+8b\kappa^3t\right)\right\}\\ \times e^{i\left(-\kappa x+\omega t+\theta_0\right)}.\end{gathered} \tag{34}$$

The solution (34) signifies a combo singular soliton, with the constraint

$$b\left(\alpha\kappa+\mu\kappa-c_2\right)>0.$$

Case 5:

$$m=1,\ A_1=0,\ B_1=0,\ A_0=\pm\sqrt{\frac{30b}{\alpha\kappa+\kappa\mu-c_2}},$$
$$\omega=-\left(3\kappa^4-6\kappa^2-1\right)b,\ A_2=\pm\sqrt{\frac{30b}{\alpha\kappa+\kappa\mu-c_2}},\tag{35}$$
$$B_2=\pm\sqrt{-\frac{30b}{\alpha\kappa+\kappa\mu-c_2}},\ c_1=\pm\left(6\kappa^2+5\right)\sqrt{\frac{3b\left(\alpha\kappa+\kappa\mu-c_2\right)}{10}}.$$

After substituting Eqs. (35) and (8) into Eq. (9) one derives the solution

$$q(x,t)=\pm\sqrt{\frac{30b}{\alpha\kappa+\kappa\mu-c_2}}\left\{1+\coth^2\left(x+8b\kappa^3t\right)+\coth\left(x+8b\kappa^3t\right)\operatorname{csch}\left(x+8b\kappa^3t\right)\right\}$$
$$\times e^{i\left(-\kappa x+\omega t+\theta_0\right)}.\tag{36}$$

Eq. (36) implies a combo singular soliton, with the constraint $b\left(\alpha\kappa+\mu\kappa-c_2\right)>0$.

3. Birefringent fibres

Now we proceed to the CQ-NLSE with the QC nonlinearity appearing in the birefringent fibres.

3.1. Unperturbed model

Under conditions of the QC nonlinearity arising in the birefringent fibres with four-wave mixing, the unperturbed CQ-NLSE can be written in the form

$$iq_t+ia_1q_{xxx}+b_1q_{xxxx}+c_1q\sqrt{|q|^2+|r|^2+qr^*+q^*r}+\left(d_1|q|^2+e_1|r|^2\right)q+f_1r^2q^*=0,\tag{37}$$
$$ir_t+ia_2r_{xxx}+b_2r_{xxxx}+c_2r\sqrt{|r|^2+|q|^2+rq^*+r^*q}+\left(d_2|r|^2+e_2|q|^2\right)r+f_2q^2r^*=0,\tag{38}$$

where the complex-valued functions $q(x,t)$ and $r(x,t)$ account for the optical solitons in the birefringent fibres. At $l=1,2$, a_l and b_l stand for the coefficients of third- and fourth-order dispersions, respectively, d_l represent the coefficients of self-phase modulation, e_l are the coefficients of cross-phase modulation, and f_l are the coefficients of four-wave mixing. For the coefficients c_l, the first two terms purport respectively the self-phase modulation and the cross-phase modulation, while the last two terms depict the four-wave mixing inside the radical function.

To obtain the optical solitons in the birefringent fibres with the unperturbed CQ-NLSE and the QC nonlinearity, we assume the following travelling-wave transformations:

$$q(x,t)=U_1(\xi)e^{i\varphi(x,t)},\quad r(x,t)=U_2(\xi)e^{i\varphi(x,t)},$$
$$\xi=x-vt,\quad \varphi(x,t)=-\kappa x+\omega t+\theta_0.\tag{39}$$

Substitution of Eq. (39) into Eqs. (37) and (38) leads to the real part

$$b_lU_l^{(iv)}+\left(3\kappa a_l-6\kappa^2b_l\right)U_l''+\left(\kappa^4b_l-\omega-\kappa^3a_l\right)U_l$$
$$+c_lU_l^2+c_lU_lU_{\bar{l}}+d_lU_l^3+e_lU_lU_{\bar{l}}^2+f_lU_lU_{\bar{l}}^2=0,\tag{40}$$

and the imaginary part

$$\left(a_l-4\kappa b_l\right)U_l'''+\left(4\kappa^3b_l-v-3\kappa^2a_l\right)U_l'=0,\tag{41}$$

where $l=1,2$ and $\tilde{l}=3-l.$ Eqs. (40) and (41) can be reduced to the ordinary differential equation

$$b_l U_l^{(iv)}+6\kappa^2 b_l U_l''-\left(\omega+3\kappa^4 b_l\right)U_l+2c_l U_l^2+\left(d_l+e_l+f_l\right)U_l^3=0, \tag{42}$$

under the constraints

$$U_{\tilde{l}}=U_l, \tag{43}$$

$$a_l=4\kappa b_l, \tag{44}$$

$$v=-8\kappa^3 b_l. \tag{45}$$

Balancing $U_l^{(iv)}$ with U_l^3 in Eq. (42) gives the value $N=2$. Hence, Eq. (6) has the solution

$$U_l(\xi)=A_0+B_1\sin\left(V_l(\xi)\right)+A_1\cos\left(V_l(\xi)\right)+\cos\left(V_l(\xi)\right)\left(B_2\sin\left(V_l(\xi)\right)+A_2\cos\left(V_l(\xi)\right)\right). \tag{46}$$

Inserting Eqs. (46) and (7) into Eq. (42), one obtains the following results.

Case 1:

$$\begin{gathered}\kappa=\pm\frac{2\sqrt{3}}{3},\ \omega=\frac{200b_l}{3},\ A_0=0,\ A_1=0,\ B_1=0,\\ A_2=\pm\sqrt{-\frac{120b_l}{d_l+e_l+f_l}},\ B_2=0,\ c_l=\pm 8\sqrt{-\frac{6b_l\left(d_l+e_l+f_l\right)}{5}}.\end{gathered} \tag{47}$$

After substituting Eqs. (47) and (8) into Eq. (46) one gets

$$q(x,t)=\pm\sqrt{-\frac{120b_1}{d_1+e_1+f_1}}\tanh^2\left(x+8\kappa^3 b_1 t\right)e^{i(-\kappa x+\omega t+\theta_0)}, \tag{48}$$

$$r(x,t)=\pm\sqrt{-\frac{120b_2}{d_2+e_2+f_2}}\tanh^2\left(x+8\kappa^3 b_2 t\right)e^{i(-\kappa x+\omega t+\theta_0)}, \tag{49}$$

$$q(x,t)=\pm\sqrt{-\frac{120b_1}{d_1+e_1+f_1}}\coth^2\left(x+8\kappa^3 b_1 t\right)e^{i(-\kappa x+\omega t+\theta_0)}, \tag{50}$$

$$r(x,t)=\pm\sqrt{-\frac{120b_2}{d_2+e_2+f_2}}\coth^2\left(x+8\kappa^3 b_2 t\right)e^{i(-\kappa x+\omega t+\theta_0)}. \tag{51}$$

The both solutions (48) and (49) represent dark solitons, while the solutions given by Eqs. (50) and (51) are singular solitons, with the constraint $b_l\left(d_l+e_l+f_l\right)<0.$

Case 2:

$$\begin{gathered}\omega=-\left(3\kappa^4-24\kappa^2-16\right)b_l,\quad A_1=0,\quad B_1=0,\quad B_2=0,\\ A_0=\pm\sqrt{-\frac{120b_l}{d_l+e_l+f_l}},\ A_2=\pm\sqrt{-\frac{120b_l}{d_l+e_l+f_l}},\\ c_l=\pm\left(3\kappa^2+10\right)\sqrt{-\frac{3b_l\left(d_l+e_l+f_l\right)}{10}}.\end{gathered} \tag{52}$$

Inserting Eqs. (52) and (8) into Eq. (46) leads to

$$q(x,t)=\pm\sqrt{-\frac{120b_1}{d_1+e_1+f_1}}\left\{1+\tanh^2\left(x+8\kappa^3 b_1 t\right)\right\}e^{i(-\kappa x+\omega t+\theta_0)}, \tag{53}$$

$$r(x,t)=\pm\sqrt{-\frac{120b_2}{d_2+e_2+f_2}}\left\{1+\tanh^2\left(x+8\kappa^3 b_2 t\right)\right\}e^{i(-\kappa x+\omega t+\theta_0)}, \tag{54}$$

$$q(x,t)=\pm\sqrt{-\frac{120b_1}{d_1+e_1+f_1}}\left\{1+\coth^2\left(x+8\kappa^3 b_1 t\right)\right\}e^{i(-\kappa x+\omega t+\theta_0)}, \tag{55}$$

$$r(x,t)=\pm\sqrt{-\frac{120b_2}{d_2+e_2+f_2}}\left\{1+\coth^2\left(x+8\kappa^3 b_2 t\right)\right\}e^{i(-\kappa x+\omega t+\theta_0)}. \tag{56}$$

The solutions (53) and (54) signify dark solitons, while the solutions (55) and (56) represent singular solitons, provided that the constraint $b_l\left(d_l+e_l+f_l\right)<0$ is satisfied.

Case 3:

$$\begin{gathered}
A_0=\pm\sqrt{\frac{11b_l}{d_l+e_l+f_l}},\quad A_1=0,\quad A_2=0,\\
B_1=0,\quad B_2=\pm\sqrt{\frac{120b_l}{d_l+e_l+f_l}},\\
\kappa=\pm\frac{\sqrt{15}}{3},\quad \omega=-\frac{91b_l}{3},\quad c_l=\pm\frac{3\sqrt{11b_l\left(d_l+e_l+f_l\right)}}{2}.
\end{gathered} \tag{57}$$

If we substitute Eqs. (57) and (8) into Eq. (46), the relations

$$q(x,t)=\pm\sqrt{\frac{b_1}{d_1+e_1+f_1}}\left\{\sqrt{11}+\sqrt{120}\tanh\left(x+8\kappa^3 b_1 t\right)\operatorname{sech}\left(x+8\kappa^3 b_1 t\right)\right\}\times e^{i(-\kappa x+\omega t+\theta_0)}, \tag{58}$$

$$r(x,t)=\pm\sqrt{\frac{b_2}{d_2+e_2+f_2}}\left\{\sqrt{11}+\sqrt{120}\tanh\left(x+8\kappa^3 b_2 t\right)\operatorname{sech}\left(x+8\kappa^3 b_2 t\right)\right\}\times e^{i(-\kappa x+\omega t+\theta_0)}, \tag{59}$$

follow. The solutions (58) and (59) imply combo dark-bright solitons, with the constraint $b_l\left(d_l+e_l+f_l\right)>0$.

Case 4:

$$\begin{gathered}
A_0=0, A_1=0, A_2=\pm\sqrt{-\frac{30b_l}{d_l+e_l+f_l}},\\
B_1=0, B_2=\pm\sqrt{\frac{30b_l}{d_l+e_l+f_l}},\\
\kappa=\pm\frac{2\sqrt{3}}{3}, \omega=\pm\frac{47b_l}{3}, c_l=\pm\frac{17\sqrt{-30b_l\left(d_l+e_l+f_l\right)}}{20}.
\end{gathered} \tag{60}$$

Inserting Eqs. (60) and (8) into Eq. (46) leads to

$$q(x,t)=\pm\sqrt{-\frac{30b_1}{d_1+e_1+f_1}}\left\{\coth^2\left(x+8\kappa^3 b_1 t\right)+\coth\left(x+8\kappa^3 b_1 t\right)\operatorname{csch}\left(x+8\kappa^3 b_1 t\right)\right\}\times e^{i(-\kappa x+\omega t+\theta_0)}, \tag{61}$$

$$r(x,t)=\pm\sqrt{-\frac{30b_2}{d_2+e_2+f_2}}\left\{\coth^2\left(x+8\kappa^3 b_2 t\right)+\coth\left(x+8\kappa^3 b_2 t\right)\operatorname{csch}\left(x+8\kappa^3 b_2 t\right)\right\}\times e^{i(-\kappa x+\omega t+\theta_0)}. \tag{62}$$

The solutions (61) and (62) represent combo singular solitons, with the constraint $b_l\left(d_l+e_l+f_l\right)<0$.

Case 5:

$$A_0=\pm\sqrt{-\frac{30b_l}{d_l+e_l+f_l}},\quad A_1=0,\quad A_2=\pm\sqrt{-\frac{30b_l}{d_l+e_l+f_l}},$$
$$B_1=0,\quad B_2=\pm\sqrt{\frac{30b_l}{d_l+e_l+f_l}},\quad \omega=-\left(3\kappa^4-6\kappa^2-1\right)b_l, \tag{63}$$
$$c_l=\pm\left(6\kappa^2+5\right)\sqrt{-\frac{3b_l\left(d_l+e_l+f_l\right)}{40}}.$$

Substituting Eqs. (63) and (8) into Eq. (46) yields

$$q\left(x,t\right)=\pm\sqrt{-\frac{30b_1}{d_1+e_1+f_1}}\left\{1+\coth^2\left(x+8\kappa^3b_1t\right)+\coth\left(x+8\kappa^3b_1t\right)\operatorname{csch}\left(x+8\kappa^3b_1t\right)\right\}\times e^{i\left(-\kappa x+\omega t+\theta_0\right)}, \tag{64}$$

$$r\left(x,t\right)=\pm\sqrt{-\frac{30b_2}{d_2+e_2+f_2}}\left\{1+\coth^2\left(x+8\kappa^3b_2t\right)+\coth\left(x+8\kappa^3b_2t\right)\operatorname{csch}\left(x+8\kappa^3b_2t\right)\right\}\times e^{i\left(-\kappa x+\omega t+\theta_0\right)}. \tag{65}$$

The solutions (64) and (65) are combo singular solitons, with the constraint $b_l\left(d_l+e_l+f_l\right)<0$.

3.2. Perturbed model

Now we consider the perturbed CQ-NLSE with the QC nonlinearity in the birefringent fibres in which the four-wave mixing is present:

$$iq_t+ia_1q_{xxx}+b_1q_{xxxx}+c_1q\sqrt{|q|^2+|r|^2+qr^*+q^*r}+\left(d_1|q|^2+e_1|r|^2\right)q+f_1r^2q^*$$
$$=i\left[\alpha_1\left(|q|^2q\right)_x+\beta_1\left(|r|^2r\right)_x+\left\{\lambda_1\left(|q|^2\right)_x+\gamma_1\left(|r|^2\right)_x\right\}q+\left(\mu_1|q|^2+\delta_1|r|^2\right)q_x\right], \tag{66}$$

$$ir_t+ia_2r_{xxx}+b_2r_{xxxx}+c_2r\sqrt{|r|^2+|q|^2+rq^*+r^*q}+\left(d_2|r|^2+e_2|q|^2\right)r+f_2q^2r^*$$
$$=i\left[\alpha_2\left(|r|^2r\right)_x+\beta_2\left(|q|^2q\right)_x+\left\{\lambda_2\left(|r|^2\right)_x+\gamma_2\left(|q|^2\right)_x\right\}r+\left(\mu_2|r|^2+\delta_2|q|^2\right)r_x\right], \tag{67}$$

where α_l, β_l, λ_l, γ_l, μ_l and δ_l $\left(l=1,2\right)$ are the constant coefficients of the nonlinear terms. Substitution of Eq. (39) into Eqs. (66) and (67) leads to the real part

$$b_lU_l^{(iv)}+\left(3\kappa a_l-6\kappa^2b_l\right)U_l''+\left(\kappa^4b_l-\omega-\kappa^3a_l\right)U_l+c_lU_l^2$$
$$+c_lU_lU_{\tilde{l}}+\left(d_l-\kappa\alpha_l-\kappa\mu_l\right)U_l^3+\left(e_l+f_l-\kappa\delta_l\right)U_lU_{\tilde{l}}^2-\kappa\beta_lU_{\tilde{l}}^3=0, \tag{68}$$

and the imaginary part

$$\left(a_l-4\kappa b_l\right)U_l'''+\left(4\kappa^3b_l-v-3\kappa^2a_l\right)U_l'+c_lU_lU_{\tilde{l}}$$
$$-\left(3\alpha_l+2\lambda_l+\mu_l\right)U_l^2U_l'-3\beta_lU_{\tilde{l}}^2U_{\tilde{l}}'-2\gamma_lU_lU_{\tilde{l}}U_{\tilde{l}}'-\delta_lU_{\tilde{l}}^2U_l'=0. \tag{69}$$

Eqs. (68) and (69) reduce to the ordinary differential equation

$$b_lU_l^{(iv)}+6\kappa^2b_lU_l''-\left(\omega+3\kappa^4b_l\right)U_l+2c_lU_l^2$$
$$+\left(d_l+e_l+f_l-\kappa\alpha_l-\kappa\beta_l-\kappa\mu_l-\kappa\delta_l\right)U_l^3=0, \tag{70}$$

with the following constraints:

$$U_{\tilde{l}} = U_l, \tag{71}$$

$$a_l = 4\kappa b_l, \tag{72}$$

$$v = -8\kappa^3 b_l, \tag{73}$$

$$3\alpha_l + 3\beta_l + 2\lambda_l + 2\gamma_l + \mu_l + \delta_l = 0. \tag{74}$$

Eq. (70) admits Eq. (46). Insertion of Eqs. (46) and (7) into Eq. (70) gives rise to the following results.

Case 1:

$$\begin{aligned}
&\kappa = \pm\frac{2\sqrt{3}}{3},\ \omega = \frac{200 b_l}{3},\ A_0 = 0, A_1 = 0, B_1 = 0,\ B_2 = 0,\\
&A_2 = \pm\sqrt{\frac{120 b_l}{\alpha_l\kappa + \beta_l\kappa + \delta_l\kappa + \mu_l\kappa - d_l - e_l - f_l}},\\
&c_l = \pm 8\sqrt{\frac{6 b_l\left(\alpha_l\kappa + \beta_l\kappa + \delta_l\kappa + \mu_l\kappa - d_l - e_l - f_l\right)}{5}}.
\end{aligned} \tag{75}$$

Substituting Eqs. (75) and (8) into Eq. (46), one derives the relations

$$\begin{aligned}
q(x,t) = \pm&\sqrt{\frac{120 b_1}{\alpha_1\kappa + \beta_1\kappa + \delta_1\kappa + \mu_1\kappa - d_1 - e_1 - f_1}}\tanh^2\left(x + 8\kappa^3 b_1 t\right)\\
&\times e^{i\left(-\kappa x + \omega t + \theta_0\right)},
\end{aligned} \tag{76}$$

$$\begin{aligned}
r(x,t) = \pm&\sqrt{\frac{120 b_2}{\alpha_2\kappa + \beta_2\kappa + \delta_2\kappa + \mu_2\kappa - d_2 - e_2 - f_2}}\tanh^2\left(x + 8\kappa^3 b_2 t\right)\\
&\times e^{i\left(-\kappa x + \omega t + \theta_0\right)},
\end{aligned} \tag{77}$$

$$\begin{aligned}
q(x,t) = \pm&\sqrt{\frac{120 b_1}{\alpha_1\kappa + \beta_1\kappa + \delta_1\kappa + \mu_1\kappa - d_1 - e_1 - f_1}}\coth^2\left(x + 8\kappa^3 b_1 t\right)\\
&\times e^{i\left(-\kappa x + \omega t + \theta_0\right)},
\end{aligned} \tag{78}$$

$$\begin{aligned}
r(x,t) = \pm&\sqrt{\frac{120 b_2}{\alpha_2\kappa + \beta_2\kappa + \delta_2\kappa + \mu_2\kappa - d_2 - e_2 - f_2}}\coth^2\left(x + 8\kappa^3 b_2 t\right)\\
&\times e^{i\left(-\kappa x + \omega t + \theta_0\right)}.
\end{aligned} \tag{79}$$

The solutions (76) and (77) imply dark solitons, while the solutions given by Eqs. (78) and (79) represent singular solitons, with the constraint $b_l\left(\alpha_l\kappa + \beta_l\kappa + \delta_l\kappa + \mu_l\kappa - d_l - e_l - f_l\right) > 0$.

Case 2:

$$\begin{aligned}
&\omega = -\left(3\kappa^4 - 24\kappa^2 - 16\right) b_l,\ A_1 = 0,\ B_1 = 0,\ B_2 = 0,\\
&A_0 = \pm\sqrt{\frac{120 b_l}{\kappa\alpha_l + \kappa\beta_l + \kappa\delta_l + \kappa\mu_l - d_l - e_l - f_l}},\\
&A_2 = \pm\sqrt{\frac{120 b_l}{\kappa\alpha_l + \kappa\beta_l + \kappa\delta_l + \kappa\mu_l - d_l - e_l - f_l}},\\
&c_l = \pm\left(3\kappa^2 + 10\right)\sqrt{\frac{3 b_l\left(\kappa\alpha_l + \kappa\beta_l + \kappa\delta_l + \kappa\mu_l - d_l - e_l - f_l\right)}{10}}.
\end{aligned} \tag{80}$$

Inserting Eqs. (80) and (8) into Eq. (46) leads to

$$q(x,t)=\pm\sqrt{\frac{120b_1}{\kappa\alpha_1+\kappa\beta_1+\kappa\delta_1+\kappa\mu_1-d_1-e_1-f_1}}\times\left\{1+\tanh^2\left(x+8\kappa^3b_1t\right)\right\}e^{i(-\kappa x+\omega t+\theta_0)}, \tag{81}$$

$$r(x,t)=\pm\sqrt{\frac{120b_2}{\kappa\alpha_2+\kappa\beta_2+\kappa\delta_2+\kappa\mu_2-d_2-e_2-f_2}}\times\left\{1++\tanh^2\left(x+8\kappa^3b_2t\right)\right\}e^{i(-\kappa x+\omega t+\theta_0)}, \tag{82}$$

$$q(x,t)=\pm\sqrt{\frac{120b_1}{\kappa\alpha_1+\kappa\beta_1+\kappa\delta_1+\kappa\mu_1-d_1-e_1-f_1}}\times\left\{1+\coth^2\left(x+8\kappa^3b_1t\right)\right\}e^{i(-\kappa x+\omega t+\theta_0)}, \tag{83}$$

$$r(x,t)=\pm\sqrt{\frac{120b_2}{\kappa\alpha_2+\kappa\beta_2+\kappa\delta_2+\kappa\mu_2-d_2-e_2-f_2}}\times\left\{1+\coth^2\left(x+8\kappa^3b_2t\right)\right\}e^{i(-\kappa x+\omega t+\theta_0)}. \tag{84}$$

The solutions (81) and (82) are dark solitons, whereas the solutions (83) and (84) are singular solitons, with the constraint $b_l\left(\alpha_l\kappa+\beta_l\kappa+\delta_l\kappa+\mu_l\kappa-d_l-e_l-f_l\right)>0$.

Case 3:

$$\begin{gathered}\kappa=\pm\frac{\sqrt{15}}{3},\omega=-\frac{91b_l}{3},\ A_1=0,B_1=0,A_2=0,\\ A_0=\pm\sqrt{-\frac{11b_l}{\kappa\alpha_l+\kappa\beta_l+\kappa\delta_l+\kappa\mu_l-d_l-e_l-f_l}},\\ B_2=\pm\sqrt{-\frac{120b_l}{\kappa\alpha_l+\kappa\beta_l+\kappa\delta_l+\kappa\mu_l-d_l-e_l-f_l}},\\ c_l=\pm\frac{3}{2}\sqrt{-11b_l\left(\kappa\alpha_l+\kappa\beta_l+\kappa\delta_l+\kappa\mu_l-d_l-e_l-f_l\right)}.\end{gathered} \tag{85}$$

Substituting Eqs. (85) and (8) into Eq. (46) gives rise to the relations

$$q(x,t)=\pm\sqrt{-\frac{b_1}{\kappa\alpha_1+\kappa\beta_1+\kappa\delta_1+\kappa\mu_1-d_1-e_1-f_1}}\times\left\{\sqrt{11}+\sqrt{120}\tanh\left(x+8\kappa^3b_1t\right)\operatorname{sech}\left(x+8\kappa^3b_1t\right)\right\}\times e^{i(-\kappa x+\omega t+\theta_0)}, \tag{86}$$

$$r(x,t)=\pm\sqrt{-\frac{b_2}{\kappa\alpha_2+\kappa\beta_2+\kappa\delta_2+\kappa\mu_2-d_2-e_2-f_2}}\times\left\{\sqrt{11}+\sqrt{120}\tanh\left(x+8\kappa^3b_2t\right)\operatorname{sech}\left(x+8\kappa^3b_2t\right)\right\}\times e^{i(-\kappa x+\omega t+\theta_0)}. \tag{87}$$

The solutions given by Eqs. (86) and (87) signify combo dark-bright solitons, with the constraint $b_l\left(\alpha_l\kappa+\beta_l\kappa+\delta_l\kappa+\mu_l\kappa-d_l-e_l-f_l\right)<0$.

Case 4:

$$A_0=0,\quad A_1=0,\quad A_2=\pm\sqrt{\frac{30b_l}{\kappa\,\alpha_l+\kappa\,\beta_l+\kappa\,\delta_l+\kappa\,\mu_l-d_l-e_l-f_l}},$$
$$B_1=0,\quad B_2=\pm\sqrt{-\frac{30b_l}{\kappa\,\alpha_l+\kappa\,\beta_l+\kappa\,\delta_l+\kappa\,\mu_l-d_l-e_l-f_l}},\tag{88}$$
$$\kappa=\pm\frac{2\sqrt{3}}{3},\quad \omega=\frac{47b_l}{3},\quad c_l=\pm17\sqrt{\frac{3b_l\left(\kappa\,\alpha_l+\kappa\,\beta_l+\kappa\,\delta_l+\kappa\,\mu_l-d_l-e_l-f_l\right)}{40}}.$$

Inserting Eqs. (88) and (8) into Eq. (46) leads to

$$q(x,t)=\pm\sqrt{\frac{30b_1}{\kappa\alpha_1+\kappa\beta_1+\kappa\delta_1+\kappa\mu_1-d_1-e_1-f_1}}\times\left\{\coth^2\left(x+8\kappa^3b_1t\right)+\coth\left(x+8\kappa^3b_1t\right)\operatorname{csch}\left(x+8\kappa^3b_1t\right)\right\}e^{i(-\kappa x+\omega t+\theta_0)},\tag{89}$$

$$r(x,t)=\pm\sqrt{\frac{30b_2}{\kappa\alpha_2+\kappa\beta_2+\kappa\delta_2+\kappa\mu_2-d_2-e_2-f_2}}\times\left\{\coth^2\left(x+8\kappa^3b_2t\right)+\coth\left(x+8\kappa^3b_2t\right)\operatorname{csch}\left(x+8\kappa^3b_2t\right)\right\}e^{i(-\kappa x+\omega t+\theta_0)}.\tag{90}$$

The solutions (89) and (90) represent combo singular solitons, with the constraint $b_l\left(\alpha_l\kappa+\beta_l\kappa+\delta_l\kappa+\mu_l\kappa-d_l-e_l-f_l\right)>0$.

Case 5:

$$A_1=0,\quad B_1=0,\quad \omega=-\left(3\kappa^4-6\kappa^2-1\right)b_l,$$
$$A_0=\pm\sqrt{\frac{30b_l}{\kappa\alpha_l+\kappa\beta_l+\kappa\delta_l+\kappa\mu_l-d_l-e_l-f_l}},$$
$$A_2=\pm\sqrt{\frac{30b_l}{\kappa\alpha_l+\kappa\beta_l+\kappa\delta_l+\kappa\mu_l-d_l-e_l-f_l}},\tag{91}$$
$$B_2=\pm\sqrt{-\frac{30b_l}{\kappa\alpha_l+\kappa\beta_l+\kappa\delta_l+\kappa\mu_l-d_l-e_l-f_l}},$$
$$c_l=\pm\left(6\kappa^2+5\right)\sqrt{\frac{3b_l\left(\kappa\alpha_l+\kappa\beta_l+\kappa\delta_l+\kappa\mu_l-d_l-e_l-f_l\right)}{40}}.$$

Substituting Eqs. (91) and (8) into Eq. (46) gives

$$q(x,t)=\pm\sqrt{\frac{30b_1}{\kappa\alpha_1+\kappa\beta_1+\kappa\delta_1+\kappa\mu_1-d_l-e_l-f_l}}\times\left\{1+\coth^2\left(x+8\kappa^3b_1t\right)+\coth\left(x+8\kappa^3b_1t\right)\operatorname{csch}\left(x+8\kappa^3b_1t\right)\right\}e^{i(-\kappa x+\omega t+\theta_0)},\tag{92}$$

$$r(x,t)=\pm\sqrt{\frac{30b_2}{\kappa\alpha_2+\kappa\beta_2+\kappa\delta_2+\kappa\mu_2-d_2-e_2-f_2}}\times\left\{1+\coth^2\left(x+8\kappa^3b_2t\right)+\coth\left(x+8\kappa^3b_2t\right)\operatorname{csch}\left(x+8\kappa^3b_2t\right)\right\}e^{i(-\kappa x+\omega t+\theta_0)}.\tag{93}$$

The solutions (92) and (93) imply combo singular solitons, with the constraint $b_l\left(\alpha_l\kappa+\beta_l\kappa+\delta_l\kappa+\mu_l\kappa-d_l-e_l-f_l\right)>0$.

4. Conclusion

We have reported for the first time the CQ solitons emerging from the governing NLSE which maintains the QC nonlinearity. Our model has been considered in both polarization-preserving and birefringent fibres. Moreover, we consider the alternative cases of present or absent perturbation terms. These terms are of Hamiltonian type and, hence, they do not affect integrability of the model. A whole spectrum of the soliton solutions has emerged in the frame of sine-Gordon equation approach adopted by us. All of those solutions have been enumerated and discussed in brief.

The shortcoming of the above approach is that it has failed to retrieve a much-needed bright-soliton solution. Note that the latter would have served as a very important solution of the model, which can be potentially applied with reference to optical fibres, photonic-crystal fibres, meta-materials and some other waveguide types. Thus, our pending assignment is to retrieve the bright-soliton solutions of the model, using additional approaches which would enable moving further along with this model. These approaches should include establishing conservation laws, addressing the aspect of optical-soliton cooling, implementing a variational principle to recover dynamics of soliton parameters, applying a Laplace–Adomian decomposition scheme for studying the model numerically, applying a symmetry method, and some others techniques (see Refs. [16–25]).

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Анотація. *Вперше виявлено кубічно-квартичні оптичні солітони з квадратично-кубічної нелінійністю. Розглянуто як поляризаційні волокна, так і волокна з подвійним заломленням. Дослідження також розширено на випадок розгляду членів збурення гамільтонового типу. Алгоритмом інтегрування, прийнятим у цій роботі, є метод рівняння синус-Гордона.*