Conditions for analytical description of anisotropy of acousto-optic figure of merit under consideration of polarization non-orthogonality of acoustic waves

Mys O., Adamenko D., Kostyrko M. and Vlokh R.

Vlokh Institute of Physical Optics, 23 Dragomanov Street, 79005, Lviv, Ukraine, vlokh@ifo.lviv.ua

Received: 06.11.2019

Abstract. We search for peculiar acousto-optic interaction planes in the crystals of all point symmetry groups, in which the strain tensor caused by acoustic waves (AWs) can be derived analytically with taking polarization non-orthogonality of the AWs into account. It is shown that the effective elasto-optic coefficients for the acousto-optic interactions in these peculiar planes can be obtained analytically. We ascertain that the analytical relations for arbitrary directions of AW propagation can be obtained only for the glass media described by the Curie symmetry groups.

Keywords: acousto-optic diffraction, anisotropy, crystal symmetry
UDC: 535.4+535.55

1. Introduction

Acousto-optic (AO) diffraction is a well-known phenomenon that consists in interaction of light radiation with acoustic wave (AW) propagating in a material medium [1, 2]. The efficiency of AO Bragg diffraction is given by the relation [2]

$$\eta = \frac{I}{I_0} = \sin^2\left(\frac{\pi}{\lambda \cos\theta_B}\sqrt{\mathrm{M}_2 P_a \frac{l}{2b}}\right), \tag{1}$$

where I and I_0 are the intensities of respectively diffracted and incident light, λ is the wavelength of optical radiation, θ_B the Bragg angle, P_a the AW power, M_2 the AO figure of merit (abbreviated as AOFM further on), l the interaction length, and b the height of the acoustic beam. The AOFM is defined as

$$\mathrm{M}_2 = \frac{n_i^3 n_d^3 p_{eff}^2}{\rho \upsilon_{ij}^3}. \tag{2}$$

Here n_i and n_d denote the refractive indices of respectively incident and diffracted waves, p_{eff} the effective elasto-optic coefficient, ρ the material density, and υ_{ij} the AW velocity. Note that the indices i and j correspond to the directions of AW propagation and polarization.

It is seen from Eq. (2) that the AOFM depends on a number of constitutive parameters. Most of these are tensor quantities, of which initial structure and number of nonzero components depend on the material symmetry. For example, the AW velocities are determined by a Christoffel equation through a fourth-rank elastic-stiffness tensor C_{ijkl} (or $C_{\lambda\mu}$ in the matrix notation). The effective elasto-optic coefficients are determined by complicated relations that include the components of a fourth-rank elasto-optic tensor $p_{\lambda\mu}$, while the refractive indices are given by a second-rank impermeability tensor B_μ [3]. Hence, the AOFM and the efficiency of AO diffraction depend on the acoustic and optical anisotropies of the material and, therefore, on the geometry of AO interactions. As a result, searching for the geometries for which the AOFM reaches its maximal values in a given crystal can potentially decrease the amplitude of a driving signal and so the total energy consumption.

In general, two methods for the analysis of AOFM anisotropy have been developed recently. These are the analytical method [4–7] and the method of numerical calculations [8, 9]. The analytical method enables one to obtain phenomenological relations for the AOFM, while the numerical method yields in a final result only, i.e. a spatial distribution of the AOFM. A number of glass-like and crystalline materials belonging to isotropic, cubic and optically uniaxial groups of symmetry have been studied using the analytical method. Note that the appropriate relations for the effective elasto-optic coefficient and the AOFM become more and more cumbersome for the crystalline materials with lower symmetries. Moreover, accounting for the effect of AW non-orthogonality often implies that it is impossible to obtain any analytical relations for the mechanical strains caused by the AWs. For example, recently we have used the analytical method [10] for the orthorhombic crystals of thallium arsenic sulphosalt, Tl_3AsS_4, and have taken the non-orthogonality of AW polarization into consideration. As a consequence, we have been able to analyze the AOFM anisotropy only within the crystallographic planes. Notice that the numerical method mentioned above has been adopted to address the AOFM anisotropy in orthorhombic SrB_4O_7 crystals for all of possible interaction geometries [11]. On the other hand, a clear disadvantage of the numerical approach consists in its inability to find out the reasons for a given behavior of the AOFM.

The problem for obtaining analytical relations for the components of AO-induced strain tensor lies in the fact that, with lowering crystalline symmetry, the displacement vector for the three acoustic eigenwaves usually lies out of the plane of AO interaction. We remind that the interaction plane is formed by the wave vectors of the incident and diffracted optical waves, and the AW vector. This leads to nonzero values of most of the off-diagonal Christoffel tensor components. As a result, it is impossible to obtain analytical solutions for the eigenvalues and eigenvectors of this tensor. However, as follows from our recent works (see, e.g., Ref. [10]), some peculiar interaction planes can still manifest the acoustic properties which ensure orientation of the displacement vector of quasi-longitudinal and one of quasi-transverse waves within the interaction plane. Then the polarization of the third eigenwave is purely transverse and its vector remains perpendicular to the interaction plane. The aim of the present work is to determine the orientation of these peculiar planes in the crystals belonging to different symmetry groups. As follows from the said above, this corresponds to finding out the conditions under which the AOFM anisotropy can be described analytically in the most complex though practical situation when the polarization non-orthogonality of the AWs is taken into account.

2. Method of analysis

It is well known that the eigenvectors of the Christoffel matrix $M_{il} = C_{ijkl} m_j m_k$ (with m_j and m_k being the components of the AW vector) correspond to the unit vectors of polarization (or displacement) of the acoustic eigenwaves [3]. In general, the angle of deviation of AW polarization from the purely transverse or longitudinal states (referred to as the angle of non-orthogonality) is given by the off-diagonal Christoffel tensor components. The appropriate formulae for the principal planes *XY*, *XZ* and *YZ* are as follows:

$$\tan 2\zeta = \frac{2M_{12}}{M_{11}-M_{22}}, \ \tan 2\zeta = \frac{2M_{31}}{M_{33}-M_{11}}, \ \tan 2\zeta = \frac{2M_{23}}{M_{22}-M_{33}}. \quad (3)$$

When the AW propagates, e.g., in the *XZ* plane which represents simultaneously the AO interaction plane, the appearance of a nonzero M_{31} component leads to non-orthogonality of a quasi-longitudinal AW and one of quasi-transverse AWs. The displacement vectors of these waves then lie in the *XZ* plane, while the third eigenwave is purely transverse, with its polarization being perpendicular to the *XZ* plane. If one of the components M_{23} or M_{12} is also nonzero, the displacement vectors of all the three eigenwaves go out from the interaction plane. The same concerns the other planes of AW propagation. Then the third-order equation by which one can determine the eigenvalues of the Christoffel tensor can be solved only numerically and, as a consequence, the components of the strain tensor cannot be obtained analytically.

Hence, it would be very useful to find those peculiar planes in the crystallographic system, which are characterized by a single off-diagonal component of the Christoffel tensor. In these planes, the eigenvectors of the Christoffel tensor are rotated around the direction perpendicular to the interaction plane. We have carried out this analysis for different point symmetry groups that characterize crystals and for different Curie symmetry groups (e.g., $\infty/\infty/mmm$) that characterize glass media. The interaction plane has been rotated around the principal axes *X*, *Y* and *Z* of the Cartesian coordinate system, which is based on the crystallographic system. These rotations have ensured passing through all the interaction planes that include at least one axis of coordinate system. Then the elastic-stiffness tensor has been rewritten in a new coordinate system formed by each rotation.

3. Results and discussion

3.1. Glass media (Curie symmetry groups $\infty/\infty/mmm$ and $\infty/\infty 2$)

Glass media are characterized by the elastic-stiffness matrix that contains only two independent coefficients C_{11} and C_{12} (i.e., we have $C_{44} = (C_{11} - C_{12})/2$). For the AW that propagates, e.g., in the *XZ* plane in a glass medium, the Christoffel tensor can be written as

$$M_{ij} = \begin{vmatrix} C_{11}\cos^2\theta + \frac{1}{2}(C_{11}-C_{12})\sin^2\theta & 0 & \left(C_{12}+\frac{1}{2}(C_{11}-C_{12})\right)\sin\theta\cos\theta \\ 0 & \frac{1}{2}(C_{11}-C_{12}) & 0 \\ \left(C_{12}+\frac{1}{2}(C_{11}-C_{12})\right)\sin\theta\cos\theta & 0 & \frac{1}{2}(C_{11}-C_{12})\cos^2\theta + C_{11}\sin^2\theta \end{vmatrix}, \quad (4)$$

where θ is the angle between the *X* axis and the wave-vector direction. Here the component M_{31} is not equal to zero. The orientation of the displacement vector, e.g., for the longitudinal AW is determined by the relation that follows from Eqs. (3):

$$\tan 2\zeta = \frac{2\left(C_{12}+\frac{1}{2}(C_{11}-C_{12})\right)\sin\theta\cos\theta}{\cos^2\theta\left(C_{11}-\frac{1}{2}(C_{11}-C_{12})\right)-\sin^2\theta\left(C_{11}-\frac{1}{2}(C_{11}-C_{12})\right)} = \tan 2\theta. \quad (5)$$

Then the equality $\zeta = \theta$ means that the wave remains purely longitudinal and the angle of non-orthogonality is equal to zero. The transverse waves also manifest pure types of polarization.

Thus, all the acoustic modes in the isotropic material media are nothing but the pure modes of the first kind (i.e., all the directions represent the longitudinal normals) [12].

Notice that two kinds of special directions can be distinguished [12, 13]: the first kind allows propagation of one longitudinal and two transverse modes (a longitudinal normal), while one transverse and two mixed modes propagate along the directions of the second kind (a transverse normal). A degenerated case for the glass media appears due to their high symmetry and the equality $C_{44} = (C_{11} - C_{12})/2$. Eq. (5) remains valid for arbitrary directions of AW propagation. Hence, the AWs propagating in the glass media in any direction do not reveal the non-orthogonality effect for the AW polarization. Accordingly, the analytical relations for the strain tensor components can be derived for all of the directions of AW propagation.

3.2. Crystals of cubic system (point symmetry groups m3m, 432, m3 and 23)

Here the matrix of elastic-stiffness coefficients contains three independent coefficients, C_{11}, C_{12}, and $C_{44} \neq (C_{11} - C_{12})/2$. For the AW propagating in the principal crystallographic planes {100} (e.g., in the XZ plane), the Christoffel tensor can be written as

$$M_{ij} = \begin{vmatrix} C_{11}\cos^2\theta + C_{44}\sin^2\theta & 0 & (C_{12}+C_{44})\sin\theta\cos\theta \\ 0 & C_{44} & 0 \\ (C_{12}+C_{44})\sin\theta\cos\theta & 0 & C_{44}\cos^2\theta + C_{11}\sin^2\theta \end{vmatrix}. \tag{6}$$

Then the angle between the X axis and the displacement vector becomes as follows:

$$\tan 2\zeta = \frac{(C_{12}+C_{44})}{(C_{11}-C_{44})}\tan 2\theta . \tag{7}$$

One can notice that the displacement vectors of two acoustic eigenwaves lie in the principal crystallographic planes whenever the AW propagates in these planes. They are the planes of propagation of acoustic modes of the second kind, in terms of Ref. [12]. The module of non-orthogonality angle increases when the factor $(C_{12}+C_{44})/(C_{11}-C_{44})$ deviates from unity. For instance, let the quasi-longitudinal AW propagate along the bisectors of the principal crystallographic axes (i.e., at $\theta = 45$ deg). Then we have $\zeta = 45$ deg for the arbitrary $(C_{12}+C_{44})/(C_{11}-C_{44})$ value. This implies that the non-orthogonality angle is zero for these propagation directions. The same is true for the cases when we have $\theta = 0$ or 90 deg.

Let us analyze the propagation of AWs in the planes {110}. The matrix of elastic-stiffness coefficients rewritten in the coordinate system $X'Y'Z$ rotated by 45 deg around the Z axis contains six independent coefficients: $C'_{11} = C'_{22}, C'_{12}, C'_{13} = C'_{23}, C'_{33}, C'_{44} = C'_{55}$ and C'_{66}. The Christoffel tensor for the AW propagation in the (110) plane can be written as

$$M_{ij} = \begin{vmatrix} C'_{11}\cos^2\theta + C'_{44}\sin^2\theta & 0 & (C'_{13}+C'_{44})\sin\theta\cos\theta \\ 0 & C'_{66}\cos^2\theta + C'_{44}\sin^2\theta & 0 \\ (C'_{13}+C'_{44})\sin\theta\cos\theta & 0 & C'_{44}\cos^2\theta + C'_{33}\sin^2\theta \end{vmatrix}. \tag{8}$$

It is easily seen that the displacement vectors for the two acoustic eigenwaves lie in the plane (110) and their orientations are given by

$$\tan 2\zeta = \frac{(C'_{13}+C'_{44})\sin 2\theta}{(C'_{11}-C'_{44})\cos^2\theta+(C'_{44}-C'_{33})\sin^2\theta}. \tag{9}$$

The non-orthogonality angle remains zero at $\theta = 0$ or 90 deg. Moreover, this angle is zero whenever the θ angle satisfies the condition

$$\theta = \mathrm{a}\tan\left(\frac{C'_{13}-C'_{11}+2C'_{44}}{C'_{13}-C'_{33}+2C'_{44}}\right)^{1/2}. \tag{10}$$

In fact, pure acoustic modes propagate at this angle [14], and this represents the special direction of the first type. This angle differs from 45 deg since we have $C'_{11} \neq C'_{33}$. In the other planes formed by rotating the interaction plane around, e.g., the principal axis Z by an arbitrary angle φ_Z, the elastic-stiffness components $C'_{26} = -C'_{16}$ and $C'_{62} = -C'_{61}$ become nonzero. This gives rise to additional nonzero off-diagonal component of the Christoffel tensor, $M_{12} = C'_{16}\cos^2\theta$. Then the AW displacement vector leaves the interaction plane.

Hence, the two conditions must be satisfied in order that the polarization non-orthogonality vanish for the AWs propagating in the interaction plane rotated by an arbitrary angle around the Z axis: (i) $M_{12} = 0$, which is fulfilled when we have $C'_{16} = 0.25(\mathrm{C}_{12}-\mathrm{C}_{11}+2\mathrm{C}_{44})\sin 4\varphi_Z$, and (ii) the condition defined by Eq. (10). It is seen that the condition (i) is fulfilled when $\mathrm{C}_{11}-\mathrm{C}_{12} = 2\mathrm{C}_{44}$, which is peculiar only for the glass media. For the cubic crystals there are six AW propagation directions within each octant of the Cartesian coordinate system, for which the angle of AW polarization non-orthogonality becomes zero. These are the three <110> directions and the three directions inside the {110} planes, which are oriented at the angles θ defined by Eq. (10). In addition, there are nine planes where the displacement vector lies inside these planes, namely the planes {100} and {110}.

3.3. Hexagonal system (point symmetry groups 6, $\bar{6}$, 6/m, 622, 6mm, $\bar{6}$m2 and 6/mmm, and Curie symmetry groups ∞, ∞/m, $\infty 2$, ∞mm and ∞/mmm)

The elastic-stiffness tensor for the hexagonal crystalline systems and the axial-symmetric Curie groups contains five independent coefficients: $C_{11} = C_{22}$, C_{12}, $C_{13} = C_{23}$, C_{33}, $C_{44} = C_{55}$ and $C_{66} = (C_{11} - C_{12})/2$. For the AWs propagating in the XZ plane, the Christoffel tensor is defined by the matrix given by Eq. (8), while the orientation of the displacement vector by Eq. (9). Thus, the angle of non-orthogonality is equal to zero at $\theta = 0$ and 90 deg, while the displacement vector lies in the XZ plane. The non-orthogonality angle becomes zero at the angle θ defined by Eq. (10). Since the six-fold (or infinity-fold) symmetry axis is parallel to the Z axis, the same is true for the arbitrary interaction plane containing the Z axis. Hence, the directions of AW propagation that form a cone with the apex angle 2θ are the directions of AW propagation with the pure transverse and longitudinal polarizations.

The Christoffel tensor for the AW propagation in the XY plane is as follows:

$$M_{ij} = \begin{vmatrix} C_{11}\cos^2\theta + C_{66}\sin^2\theta & (C_{66}+C_{12})\sin\theta\cos\theta & 0 \\ (C_{66}+C_{12})\sin\theta\cos\theta & C_{66}\cos^2\theta + C_{11}\sin^2\theta & 0 \\ 0 & 0 & C_{44}\cos^2\theta + C_{44}\sin^2\theta \end{vmatrix}. \tag{11}$$

The displacement vector orientation with respect to the X axis is given by the relation

$$\tan 2\zeta = \tan 2\theta \,. \tag{12}$$

Therefore the angle of non-orthogonality for the AWs propagating along arbitrary directions in the XY plane is equal to zero. When the interaction plane is rotated around the X axis by the angle φ_X , additional components of the elastic-stiffness tensor become nonzero, $C'_{14}, C'_{24} = C'_{34}$ and C'_{56} . This leads to nonzero values of all the Christoffel tensor components, i.e. we have

$$M_{ij} = \begin{vmatrix} C'_{11}\cos^2\theta + C'_{66}\sin^2\theta & (C'_{66}+C'_{12})\sin\theta\cos\theta & (C'_{14}+C'_{56})\sin\theta\cos\theta \\ (C'_{66}+C'_{12})\sin\theta\cos\theta & C'_{66}\cos^2\theta + C'_{22}\sin^2\theta & C'_{56}\cos^2\theta + C'_{24}\sin^2\theta \\ (C'_{14}+C'_{56})\sin\theta\cos\theta & C'_{56}\cos^2\theta + C'_{24}\sin^2\theta & C'_{55}\cos^2\theta + C'_{44}\sin^2\theta \end{vmatrix} . \tag{13}$$

If the AWs propagate in the ZY plane at the angle φ_X with respect to the Y axis, the angle θ becomes equal to 90 deg. In this case we have $M_{12} = M_{31} = 0$ and $M_{23} = C'_{24}$ in the matrix given by Eq. (13). However, since the AWs propagate inside the principal crystallographic plane, the equality $M_{23} = C'_{24} = 0$ holds true and the non-orthogonality angle is zero. The angle φ_X in this case plays a role of the angle θ defined by Eq. (10). This direction of AW propagation coincides with one of the generating lines of the cone with the apex angle 2θ .

As a result, the AWs with the pure transverse and longitudinal polarizations in the crystals of hexagonal system propagate along the principal axes X, Y and Z, along the generating lines of the cone oriented around the Z axis, and along arbitrary directions in the XY plane. Moreover, in all the interaction planes that include the Z axis, two of the eigenwaves are characterized by the displacement vector which does not leave these planes.

3.4. Tetragonal crystals (point symmetry groups 422, 4mm, $\bar{4}$2m and 4/mmm)

The matrix of elastic-stiffness coefficients for these symmetry groups contains six independent coefficients: $C_{11} = C_{22}$, C_{12}, $C_{13} = C_{23}$, C_{33}, $C_{44} = C_{55}$ and C_{66}. Therefore, the Christoffel tensor for the AW propagation inside the principal XZ (or YZ) plane is defined by Eq. (8). The displacement vectors of two acoustic eigenwaves lie in the interaction plane for all the directions of AW propagation within these planes. The orientation of this vector is described by Eq. (9). The non-orthogonality angle is equal to zero for the directions given by Eq. (10). When the coordinate system is rotated around the Z axis by the angle φ_Z , an additional component $C'_{26} = -C'_{16}$ of elastic-stiffness tensor appears. Then the component $M_{12} = C'_{16}\cos^2\theta$ arises in the Christoffel tensor (with $C'_{16} = 0.25(C_{12} - C_{11} + 2C_{66})\sin 4\varphi_Z$) and the displacement vector leaves the interaction plane. At the angles $\varphi_Z = 0$, 45 or 90 deg, the C'_{16} and M_{12} components are zero and the displacement vectors of two of the eigenwaves lie inside the interaction plane. Moreover, the non-orthogonality in these planes vanishes at the angle θ determined by Eq. (10). In the XY plane, the only nonzero diagonal component M_{12} is present in the Christoffel tensor:

$$M_{12} = (C_{66} + C_{12})\sin\theta\cos\theta \,. \tag{14}$$

Then the non-orthogonality angle can be written as

$$\tan 2\zeta = \frac{(C_{12} + C_{66})}{(C_{11} - C_{66})}\tan 2\theta \,. \tag{15}$$

In other words, the non-orthogonality angle is zero only when we have $C_{66} = (C_{11} - C_{12})/2$. When the interaction plane is rotated around the *X* axis by the angle φ_X , new nonzero components appear in the rewritten elastic-stiffness tensor, C'_{14}, C'_{24} C'_{34} and C'_{56}. As a result, all the off-diagonal components of the Christoffel tensor become nonzero and the displacement vectors for all eigenwaves lie out of the interaction plane.

3.5. Tetragonal crystals (point symmetry groups 4, $\bar{4}$ and 4/m)

These crystals are characterized by the elastic-stiffness tensor containing seven components, of which six are invariant. In comparison with the tetragonal systems mentioned in subsection 3.4, here we have additional component $C_{26} = -C_{16}$. Under rotation of the coordinate system around the *Z* axis by the angle

$$\varphi_Z = \frac{1}{4} \mathrm{a}\tan \frac{4C_{16}}{C_{11} - C_{12} - 2C_{66}}, \tag{16}$$

the C'_{16} component vanishes and the structure of the elastic-stiffness tensor becomes the same as that described in subsection 3.4 for the higher-symmetry tetragonal groups. Hence, in the crystals of lower-symmetry tetragonal groups there are two mutually perpendicular planes that contain the *Z* axis, in which the non-orthogonality angle behaves in the same manner as in the principal planes *XZ* and *YZ* of higher-symmetry tetragonal groups. One of these planes is rotated with respect to the principal plane *XZ* by the angle φ_Z determined by Eq. (16). In the *XY* plane, the displacement vector lie in this plane and the angle of non-orthogonality is given by the relation

$$\tan 2\zeta = \frac{(C_{12} + C_{66})\sin 2\theta + 2C_{16}\cos 2\theta}{(C_{11} - C_{66})\cos 2\theta + 2C_{16}\sin 2\theta}. \tag{17}$$

When the interaction plane is rotated around the *X* axis by an arbitrary angle φ_X , all the elastic-stiffness components in the rewritten system are nonzero. As a consequence, all of the off-diagonal components of the Christoffel tensor become nonzero and the displacement vectors of the eigenwaves lie out of the interaction plane.

Hence, the displacement vectors of two eigenwaves in the crystals of high-symmetry groups of the tetragonal system lie in the principal crystallographic plane or in the plane rotated by 45 deg around the *Z* axis. At a certain angle θ determined by Eq. (10), the non-orthogonality effect vanishes in the *XZ*, *YZ* and (110) planes. In particular, the non-orthogonality angle is zero when the AWs propagate along the principal axes *X*, *Y* and *Z*. In the crystals of low-symmetry groups of the tetragonal system, the displacement vectors of two eigenwaves either lie in the interaction plane (in the case of *XY* plane) or are rotated around the *Z* axis by the angle defined by Eq. (16) (in the two mutually perpendicular planes). In the latter case, the non-orthogonality angle is equal to zero when the AWs propagate along the *Z* axis, along the X' and Y' directions given by Eq. (16), and along the directions given by Eq. (10) in the $X'Z$ and $Y'Z$ planes.

3.6. Trigonal crystals (point symmetry groups 32, 3m, $\bar{3}m$, 3 and $\bar{3}$)

The elastic-stiffness tensor for the crystals that belong to the symmetry groups 32, 3m and $\bar{3}m$ contains six independent components, $C_{11} = C_{22}$, C_{12}, $C_{13} = C_{23}$, C_{33}, $C_{44} = C_{55}$, $C_{66} = (C_{11} - C_{12})/2$, $C_{56} = C_{14}$ and $C_{24} = -C_{14}$. Here the three-fold symmetry axis represents the acoustic axis, and the transverse AWs propagating along this direction have the same velocity and the pure polarizations.

The Christoffel tensor for the AWs propagating in the YZ plane (which is simultaneously the symmetry mirror plane) can be written as

$$M_{ij}=\begin{vmatrix} \begin{pmatrix} C_{66}\cos^2\theta+ \\ C_{44}\sin^2\theta+C_{56}\sin 2\theta \end{pmatrix} & 0 & 0 \\ 0 & \begin{pmatrix} C_{11}\cos^2\theta+ \\ C_{44}\sin^2\theta+C_{24}\sin 2\theta \end{pmatrix} & \begin{pmatrix} C_{24}\cos^2\theta+ \\ 0.5(C_{13}+C_{44})\sin 2\theta \end{pmatrix} \\ 0 & \begin{pmatrix} C_{24}\cos^2\theta \\ +0.5(C_{13}+C_{44})\sin 2\theta \end{pmatrix} & C_{44}\cos^2\theta+C_{33}\sin^2\theta \end{vmatrix}. \quad (18)$$

Then the displacement vectors of the two eigenwaves lie in the YZ plane, while the non-orthogonality angle reads as

$$\tan 2\zeta=\frac{2C_{24}\cos^2\theta+(C_{13}+C_{44})\sin 2\theta}{(C_{11}-C_{44})\cos^2\theta+(C_{44}-C_{33})\sin^2\theta+C_{24}\sin 2\theta}. \quad (19)$$

Due to symmetry conditions, the interaction planes formed as a result of rotation of the YZ plane around the Z axis by the angle $\varphi_Z=n\times 120$ deg (with n being an integer number) are the same, and Eqs. (18) and (19) remain valid for the three symmetry-equivalent planes. All the components of the Christoffel tensor for the AWs propagating in the XZ and XY planes are nonzero. This means that the displacement vectors of all the eigenwaves do not belong to these interaction planes.

The elastic-stiffness tensor for the crystals that belong to the symmetry groups 3 and $\bar{3}$ contains seven components (and six invariant ones): $C_{11}=C_{22}$, C_{12}, $C_{13}=C_{23}$, C_{33}, $C_{44}=C_{55}$, $C_{66}=(C_{11}-C_{12})/2$, $C_{56}=C_{14}$, $C_{24}=-C_{14}$ and $C_{25}=C_{46}=-C_{15}$. There is a single AW propagation direction, the Z axis, in the crystals belonging to these point groups, along which the non-orthogonality effect vanishes. The displacement vector for the AWs propagating inside the principal crystallographic planes does not belong to these planes. However, when the coordinate system is rotated around the Z axis with respect to the YZ plane by a certain angle,

$$\varphi_Z=n\times 120+\frac{1}{3}\mathrm{a}\tan\frac{C_{25}}{C_{14}}, \quad (20)$$

the C'_{25} component vanishes and the structure of the elastic-stiffness tensor becomes the same as that typical for the symmetry groups 32, 3m and $\bar{3}m$ of trigonal system. In other terms, there are three planes rotated by the angle given by Eq. (20) with respect to the YZ plane, in which two eigenwaves propagate with the polarization vectors belonging to these planes.

3.7. Orthorhombic crystals (point symmetry groups mmm, mm2 and 222)

These point symmetry groups are subgroups of the tetragonal groups described in subsection 3.4. The matrix of elastic-stiffness coefficients for these groups contains nine independent coefficients: C_{11}, C_{22}, C_{33}, C_{12}, C_{13}, C_{23}, C_{44}, C_{55} and C_{66}. Here the behaviour of the displacement vector is somewhat similar to that peculiar for the tetragonal symmetry groups mentioned above. In particular, the displacement vectors of two eigenwaves belong to the interaction plane, whenever these planes coincide with the principal crystallographic planes. However, the angle of AW propagation at which they acquire pure polarizations differs for different crystallographic planes. For the *XZ* plane, this angle with respect to the *X* axis can be written as

$$\theta = \mathrm{a}\tan\left(\frac{C'_{13} - C'_{11} + 2C'_{55}}{C'_{13} - C'_{33} + 2C'_{55}}\right)^{1/2} . \tag{21}$$

For the *YZ* plane, the angle with respect to the *Y* axis at which a pure polarization is reached is given by

$$\theta = \mathrm{a}\tan\left(\frac{C'_{23} - C'_{22} + 2C'_{44}}{C'_{23} - C'_{33} + 2C'_{44}}\right)^{1/2} . \tag{22}$$

Finally, for the *XY* plane this angle with respect to the *X* axis reads as

$$\theta = \mathrm{a}\tan\left(\frac{C'_{12} - C'_{11} + 2C'_{66}}{C'_{12} - C'_{22} + 2C'_{66}}\right)^{1/2} . \tag{22}$$

Thus, the special directions of the second type for the orthorhombic crystals belong to the principal crystallographic planes.

3.8. Monoclinic crystals (point symmetry groups 2/m, 2 and m)

The special directions of the second type in the monoclinic crystals belong to the plane that coincides with the mirror symmetry plane or is perpendicular to the two-fold axis. The Christoffel tensor for this case (e.g., for the *XZ* plane, with $m \| XZ$ and $2 \perp XZ$) can be written as

$$M_{ij} = \begin{vmatrix} \begin{pmatrix} C_{11}\cos^2\theta + C_{55}\sin^2\theta \\ +C_{15}\sin 2\theta \end{pmatrix} & 0 & \begin{pmatrix} C_{15}\cos^2\theta + C_{35}\sin^2\theta \\ +0.5(C_{13} + C_{55})\sin 2\theta \end{pmatrix} \\ 0 & \begin{pmatrix} C_{66}\cos^2\theta + C_{44}\sin^2\theta \\ +C_{46}\sin 2\theta \end{pmatrix} & 0 \\ \begin{pmatrix} C_{15}\cos^2\theta + C_{35}\sin^2\theta \\ +0.5(C_{13} + C_{55})\sin 2\theta \end{pmatrix} & 0 & \begin{pmatrix} C_{55}\cos^2\theta + C_{33}\sin^2\theta \\ +C_{35}\sin 2\theta \end{pmatrix} \end{vmatrix} . \tag{23}$$

Then the relation for the orientation angle of the displacement vector in the *XZ* plane with respect to the *X* axis acquires the form

$$\tan 2\zeta = \frac{2C_{15}\cos^2\theta + 2C_{35}\sin^2\theta + (C_{13} + C_{55})\sin 2\theta}{C_{11}\cos^2\theta - C_{33}\sin^2\theta - C_{55}\cos 2\theta + (C_{15} - C_{35})\sin 2\theta} . \tag{24}$$

This means that all directions in this plane represent the special directions of the second type. In other words, the quasi-longitudinal and quasi-transverse waves polarized in this plane should be mixed waves, while the transverse wave polarized perpendicular to this plane is characterized by the pure polarization type.

4. Conclusions

Let us summarize the main results obtained above. The AWs that propagate in the glass media along arbitrary directions do not reveal the effect of polarization non-orthogonality. In the crystals of cubic system, there are six directions of AW propagation inside each octant of the Cartesian coordinate system, for which the angle of polarization non-orthogonality for the AWs is equal to zero. These are the three directions <110> and the three directions lying in the {110} planes, which are oriented at specific angles θ with respect to the principal crystallographic axes.

Besides, there are nine planes in which the displacement vectors of two acoustic eigenwaves lie inside these planes. These are the planes $\{100\}$ and $\{110\}$. In the crystals of hexagonal system, the AWs with purely transverse or longitudinal polarizations propagate along the principal axes *X*, *Y* and *Z*, as well as along the generating lines of the cone oriented along the *Z* axis and along arbitrary directions in the *XY* plane. Moreover, in all the interaction planes which include the *Z* axis, the two propagating eigenwaves are characterized by the displacement vectors which do not leave these planes.

In the crystals of tetragonal system described by the higher-symmetry groups 422, 4mm, $\bar{4}$2m and 4/mmm, the displacement vectors of two eigenwaves lie in the principal crystallographic planes and in the planes rotated by 45 deg around the *Z* axis. At a certain orientation of propagation direction with respect to the *X* and *Y* axes, which is given by a specific angle θ, or along the [110] direction, the non-orthogonality vanishes in the *XZ*, *YZ* and (110) planes . The angle of non-orthogonality is equal to zero when the AWs propagate along the principal axes *X*, *Y* and *Z*. In the crystals of tetragonal system described by the lower-symmetry groups 4, $\bar{4}$ and 4/m, the displacement vectors of two acoustic eigenwaves lie in the interaction plane for the cases of propagation in the *XY* plane and in the two mutually perpendicular planes rotated around the *Z* axis by the angle, which is determined by the ratio of elastic-stiffness coefficients. In the latter case, the angle of non-orthogonality is zero when the AWs propagate along the *Z* axis, along the X' and Y' directions determined by the ratio mentioned above, as well as along the directions given by a specific angle θ in the $X'Z$ and $Y'Z$ planes.

Our next subject has been the trigonal crystals belonging to the symmetry groups 32, 3m and $\bar{3}m$. When the AWs propagate in the *YZ* plane or in the planes formed by rotation of the *YZ* plane around the *Z* axis by the angle $\varphi_Z = n \times 120$ deg (with *n* being integer), the polarization vectors of two of the eigenwaves belong to the interaction plane. The same is true for the symmetry groups 3 and $\bar{3}$. However, these planes in the latter case are rotated by the angle $\varphi_Z = n \times 120 + [\mathrm{a}\tan(C_{25} / C_{14})]/3$ with respect to the *YZ* plane.

The special directions of the second type in the orthorhombic crystals belong to the principal crystallographic planes. In the monoclinic crystals, there is a single plane in which two eigenwaves propagate with the polarization belonging to this plane. This special plane is perpendicular to the two-fold axis and parallel to the symmetry mirror plane. Finally, the third AW is transverse and purely polarized along the direction perpendicular to the above plane.

Hence, we have found all the interaction planes for the crystals of all point symmetry groups, in which the strain tensor caused by the AWs can be derived analytically under condition that the effect of non-orthogonality of the AW polarization is taken into account. This means that the effective elasto-optic coefficients for the AO interactions in these planes can be obtained analytically. On the other hand, the analytical relations for arbitrary directions of AW propagation can be obtained only for the glass-like media that belong to the Curie symmetry groups. Finally, one can easily testify that the results of our work agree well with the data [12] in the part of longitudinal normals, though our method of analysis is much simpler than that adopted in Ref. [12].

Acknowledgement

The authors acknowledge financial support of the present study from the Ministry of Education and Science of Ukraine (the Projects ## 0118U003899 and 0117U000802).

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Mys O., Adamenko D., Kostyrko M. and Vlokh R. 2019. Conditions for analytical description of anisotropy of acousto-optic figure of merit under consideration of polarization non-orthogonality of acoustic waves. Ukr.J.Phys.Opt. 20: 175 – 185. doi: 10.3116/16091833/20/4/175/2019

Анотація. *Знайдено особливі площини акустооптичної взаємодії в кристалах усіх точкових симетрійних груп, в яких тензор напружень, викликаний акустичними хвилями (АХ), можна одержати аналітично з урахуванням явища неортогональності поляризацій АХ. Показано, що ефективні пружнооптичні коефіцієнти для акустооптичних взаємодій у цих особливих площинах можна одержати аналітично. Встановлено, що аналітичні співвідношення для довільних напрямків поширення АХ можна одержати лише для склоподібних середовищ, що описуються групами симетрії Кюрі.*