Fifth-rank axial tensor describing the gradient piezogyration effect

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Abstract. We have derived a fifth-rank axial tensor with the internal symmetry $\varepsilon[V^2]^2V$ that describes the gradient piezogyration effect for all the point symmetry groups, including continuous-symmetry groups. It has been found that twelve different structures of such a tensor can be distinguished. The gradient piezogyration effect is analyzed for the cases of torsion and bending of crystals and crystalline textures.

Keywords: optical activity, gradient piezogyration, fifth-rank tensor

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1. Introduction

Recently we have shown [1–4] that mechanical torsion can induce an optical rotation in crystals possessing no natural optical activity or a common piezogyration effect [5–8]. This phenomenon described as a gradient piezogyration effect is caused by a spatial gradient of mechanical stresses rather than the stresses themselves. The tensorial relation describing the gradient piezogyration effect is as follows:

$$\Delta g_{ln} = \beta_{lnkmv} \partial\sigma_{km} / \partial X_v , \quad (1)$$

where Δg_{ln} denotes the induced increment of the gyration tensor, $\partial\sigma_{km} / \partial X_v$ the coordinate derivative of the stress tensor, and β_{lnkmv} a fifth-rank axial tensor. Further on we will use a standard matrix notation explained in detail, e.g., in the textbook [9]. Namely, we have $\beta_{lnkmv} = \beta_{\lambda\mu v}$ when $l,n \leftrightarrow \lambda = 1,...,6$, $km \leftrightarrow \mu = 1, 2, 3$ and $\beta_{lnkmv} = 2\beta_{\lambda\mu v}$ when $l,n \leftrightarrow \lambda = 1,...,6$ $k,m \leftrightarrow \mu = 4,5,6$. Notice that the fifth-rank axial tensor remains nonzero in the material media including second-order symmetry operations such as a centre of symmetry, mirror planes or inversion axes. Therefore the effect mentioned above is not forbidden even in centrosymmetric media.

In our recent work [4] we have analyzed manifestations of the gradient piezogyration effect under condition when an inhomogeneous stress field is caused by a torsion moment applied to crystals not possessing the natural optical activity. Nonetheless, one can notice that the torsion-induced optical activity (see, e.g., the study [3] where it has been verified experimentally) represents, maybe, the simplest case of the gradient piezogyration. It is obvious that the optical rotation can be produced by different kinds of coordinate dependences of the stress tensor components. In order to analyze in detail the effect, one needs to know in advance the forms of the fifth-rank axial tensor for all the point symmetry groups, including the infinite-order Curie groups. As far as we know, the tensor with the internal symmetry $\varepsilon[V^2]^2V$ has not been derived before.

Therefore the present work is aimed at obtaining the structure of this tensor for different point symmetry groups.

2. Fifth-rank axial tensor for different point groups

To derive the tensor with the internal symmetry $\varepsilon[V^2]^2V$ for most of the point groups, it is sufficient to use a method by F. G. Fumi (a so-called method of direct inspection) [10]. While deriving the tensor for the trigonal and hexagonal systems, as well as for the Curie groups (i.e., those including rotation axes of infinite order), one has to apply a method of cyclic coordinates [9]. Finally, the number of independent components in the tensor has been checked using a group-theoretical technique (see Refs. [9, 11]).

Employing the techniques mentioned, we have obtained 12 different structures of the tensor for different symmetry systems. The latter systems are such that they include the following symmetry groups: (1) $\infty/\infty/\mathrm{mmm}$ and $\infty/\infty 2$; (2) m3m, 432 and $\bar{4}3\mathrm{m}$; (3) m3 and 23; (4) ∞/mmm, $\infty\mathrm{mm}$, $\infty 2$, 6/mmm, 622, 6mm and $\bar{6}\mathrm{m}2$; (5) ∞/m, ∞, 6/m, 6 and $\bar{6}$; (6) 4/mmm, 422, 4mm and $\bar{4}2\mathrm{m}$; (7) 4/m, 4 and $\bar{4}$; (8) 32, 3m and $\bar{3}\mathrm{m}$; (9) 3 and $\bar{3}$; (10) mmm, 222 and mm2; (11) 2/m, 2 and m; and (12) 1 and $\bar{1}$. The corresponding matrices of the tensor written in the principal coordinate system associated with the optical Fresnel ellipsoid are represented in Tables 1–12.

Table 1. Structure of tensor with the internal symmetry $\varepsilon[V^2]^2V$ for the groups $\infty/\infty/\mathrm{mmm}$ and $\infty/\infty 2$.

	$\frac{\partial\sigma_1}{\partial X_1}$	$\frac{\partial\sigma_2}{\partial X_1}$	$\frac{\partial\sigma_3}{\partial X_1}$	$\frac{\partial\sigma_4}{\partial X_1}$	$\frac{\partial\sigma_5}{\partial X_1}$	$\frac{\partial\sigma_6}{\partial X_1}$	$\frac{\partial\sigma_1}{\partial X_2}$	$\frac{\partial\sigma_2}{\partial X_2}$	$\frac{\partial\sigma_3}{\partial X_2}$	$\frac{\partial\sigma_4}{\partial X_2}$	$\frac{\partial\sigma_5}{\partial X_2}$	$\frac{\partial\sigma_6}{\partial X_2}$	$\frac{\partial\sigma_1}{\partial X_3}$	$\frac{\partial\sigma_2}{\partial X_3}$	$\frac{\partial\sigma_3}{\partial X_3}$	$\frac{\partial\sigma_4}{\partial X_3}$	$\frac{\partial\sigma_5}{\partial X_3}$	$\frac{\partial\sigma_6}{\partial X_3}$
Δg_1	0	0	0	0	0	0	0	0	0	0	$-\beta_{152}$	0	0	0	0	0	0	β_{152}
Δg_2	0	0	0	β_{152}	0	0	0	0	0	0	0	0	0	0	0	0	0	$-\beta_{152}$
Δg_3	0	0	0	$-\beta_{152}$	0	0	0	0	0	0	β_{152}	0	0	0	0	0	0	0
Δg_4	0	$-\beta_{152}$	β_{152}	0	0	0	0	0	0	0	0	β_{152}	0	0	0	0	$-\beta_{152}$	0
Δg_5	0	0	0	0	0	$-\beta_{152}$	β_{152}	0	$-\beta_{152}$	0	0	0	0	0	0	β_{152}	0	0
Δg_6	0	0	0	0	β_{152}	0	0	0	0	$-\beta_{152}$	0	0	$-\beta_{152}$	β_{152}	0	0	0	0

As seen from Table 1, the tensor with the internal symmetry $\varepsilon[V^2]^2V$ is nonzero even for the isotropic media. Here it includes a single independent component β_{152}. As already mentioned in the work [4], the torsion-induced optical activity has to appear in these media if the light propagates along the torsion axis. On the other hand, the stress-component gradients $\partial\sigma_2/\partial X_1$ (or $\partial\sigma_3/\partial X_1$) appearing, e.g., under bending would induce the only gyration tensor component g_4. The latter can be measured under very inconvenient experimental conditions: the light beam should be incident oblique with respect to a surface of sample of a rectangular shape, with the surfaces being perpendicular to the coordinate axes. The same is true for all of the derivatives $\partial\sigma_\mu/\partial X_\nu$, where $\mu,\nu = 1, 2, 3$. This is also the property for the symmetry groups ∞/mmm, $\infty\mathrm{mm}$, $\infty 2$, m3m, 432, $\bar{4}3\mathrm{m}$, m3, 23, 6/mmm, 622, 6mm, $\bar{6}\mathrm{m}2$, 4/mmm, 422, 4mm, and $\bar{4}2\mathrm{m}$. In other words, the torsion applied around the principal axes would lead to appearance of the optical rotation along the direction of torsion axis, while the bending would not (see Tables 2, 3, 4 and 6).

Nonetheless, the bending stresses $\partial\sigma_\mu/\partial X_3$ ($\mu = 1, 2$) applied to materials belonging to the symmetry groups ∞/m, ∞, 6/m, 6, $\bar{6}$, 4/m, 4 and $\bar{4}$, would induce the optical activity along all of the principal axes (see Tables 5 and 7).

Table 2. Structure of tensor with the internal symmetry $\varepsilon[\mathrm{V}^2]^2\mathrm{V}$ for the groups m3m, 432 and $\bar{4}3\mathrm{m}$.

	$\frac{\partial\sigma_1}{\partial X_1}$	$\frac{\partial\sigma_2}{\partial X_1}$	$\frac{\partial\sigma_3}{\partial X_1}$	$\frac{\partial\sigma_4}{\partial X_1}$	$\frac{\partial\sigma_5}{\partial X_1}$	$\frac{\partial\sigma_6}{\partial X_1}$	$\frac{\partial\sigma_1}{\partial X_2}$	$\frac{\partial\sigma_2}{\partial X_2}$	$\frac{\partial\sigma_3}{\partial X_2}$	$\frac{\partial\sigma_4}{\partial X_2}$	$\frac{\partial\sigma_5}{\partial X_2}$	$\frac{\partial\sigma_6}{\partial X_2}$	$\frac{\partial\sigma_1}{\partial X_3}$	$\frac{\partial\sigma_2}{\partial X_3}$	$\frac{\partial\sigma_3}{\partial X_3}$	$\frac{\partial\sigma_4}{\partial X_3}$	$\frac{\partial\sigma_5}{\partial X_3}$	$\frac{\partial\sigma_6}{\partial X_3}$
Δg_1	0	0	0	0	0	0	0	0	0	0	β_{152}	0	0	0	0	0	0	$-\beta_{152}$
Δg_2	0	0	0	$-\beta_{152}$	0	0	0	0	0	0	0	0	0	0	0	0	0	β_{152}
Δg_3	0	0	0	β_{152}	0	0	0	0	0	0	$-\beta_{152}$	0	0	0	0	0	0	0
Δg_4	0	β_{421}	$-\beta_{421}$	0	0	0	0	0	0	0	0	β_{462}	0	0	0	0	$-\beta_{462}$	0
Δg_5	0	0	0	0	0	$-\beta_{462}$	$-\beta_{421}$	0	β_{421}	0	0	0	0	0	0	β_{462}	0	0
Δg_6	0	0	0	0	β_{462}	0	0	0	0	$-\beta_{462}$	0	0	β_{421}	$-\beta_{421}$	0	0	0	0

Table 3. Structure of tensor with the internal symmetry $\varepsilon[\mathrm{V}^2]^2\mathrm{V}$ for the groups m3 and 23.

	$\frac{\partial\sigma_1}{\partial X_1}$	$\frac{\partial\sigma_2}{\partial X_1}$	$\frac{\partial\sigma_3}{\partial X_1}$	$\frac{\partial\sigma_4}{\partial X_1}$	$\frac{\partial\sigma_5}{\partial X_1}$	$\frac{\partial\sigma_6}{\partial X_1}$	$\frac{\partial\sigma_1}{\partial X_2}$	$\frac{\partial\sigma_2}{\partial X_2}$	$\frac{\partial\sigma_3}{\partial X_2}$	$\frac{\partial\sigma_4}{\partial X_2}$	$\frac{\partial\sigma_5}{\partial X_2}$	$\frac{\partial\sigma_6}{\partial X_2}$	$\frac{\partial\sigma_1}{\partial X_3}$	$\frac{\partial\sigma_2}{\partial X_3}$	$\frac{\partial\sigma_3}{\partial X_3}$	$\frac{\partial\sigma_4}{\partial X_3}$	$\frac{\partial\sigma_5}{\partial X_3}$	$\frac{\partial\sigma_6}{\partial X_3}$
Δg_1	0	0	0	β_{141}	0	0	0	0	0	0	β_{152}	0	0	0	0	0	0	β_{163}
Δg_2	0	0	0	β_{163}	0	0	0	0	0	0	β_{141}	0	0	0	0	0	0	β_{152}
Δg_3	0	0	0	β_{152}	0	0	0	0	0	0	β_{163}	0	0	0	0	0	0	β_{141}
Δg_4	β_{411}	β_{421}	β_{431}	0	0	0	0	0	0	0	0	β_{462}	0	0	0	0	β_{453}	0
Δg_5	0	0	0	0	0	β_{453}	β_{431}	β_{411}	β_{421}	0	0	0	0	0	0	β_{462}	0	0
Δg_6	0	0	0	0	β_{462}	0	0	0	0	β_{453}	0	0	β_{421}	β_{431}	β_{411}	0	0	0

Table 4. Structure of tensor with the internal symmetry $\varepsilon[\mathrm{V}^2]^2\mathrm{V}$ for the groups ∞/mmm, $\infty\mathrm{mm}$, $\infty 2$, 6/mmm, 622, 6mm and $\bar{6}\mathrm{m}2$ *.

	$\frac{\partial\sigma_1}{\partial X_1}$	$\frac{\partial\sigma_2}{\partial X_1}$	$\frac{\partial\sigma_3}{\partial X_1}$	$\frac{\partial\sigma_4}{\partial X_1}$	$\frac{\partial\sigma_5}{\partial X_1}$	$\frac{\partial\sigma_6}{\partial X_1}$	$\frac{\partial\sigma_1}{\partial X_2}$	$\frac{\partial\sigma_2}{\partial X_2}$	$\frac{\partial\sigma_3}{\partial X_2}$	$\frac{\partial\sigma_4}{\partial X_2}$	$\frac{\partial\sigma_5}{\partial X_2}$	$\frac{\partial\sigma_6}{\partial X_2}$	$\frac{\partial\sigma_1}{\partial X_3}$	$\frac{\partial\sigma_2}{\partial X_3}$	$\frac{\partial\sigma_3}{\partial X_3}$	$\frac{\partial\sigma_4}{\partial X_3}$	$\frac{\partial\sigma_5}{\partial X_3}$	$\frac{\partial\sigma_6}{\partial X_3}$
Δg_1	0	0	0	β_{141}	0	0	0	0	0	0	β_{152}	0	0	0	0	0	0	β_{163}
Δg_2	0	0	0	$-\beta_{152}$	0	0	0	0	0	0	$-\beta_{141}$	0	0	0	0	0	0	$-\beta_{163}$
Δg_3	0	0	0	β_{341}	0	0	0	0	0	0	$-\beta_{341}$	0	0	0	0	0	0	0
Δg_4	β_{411}	β_{421}	β_{431}	0	0	0	0	0	0	0	0	β_{462}	0	0	0	0	β_{453}	0
Δg_5	0	0	0	0	0	$-\beta_{462}$	$-\beta_{421}$	$-\beta_{411}$	$-\beta_{431}$	0	0	0	0	0	0	$-\beta_{453}$	0	0
Δg_6	0	0	0	0	β_{651}	0	0	0	0	$-\beta_{651}$	0	0	$-\beta_{163}$	β_{163}	0	0	0	0

*Notice that $\beta_{152} = \beta_{141} + \beta_{651}$ and $\beta_{421} = \beta_{561} + \beta_{411}$.

Table 5. Structure of tensor with the internal symmetry $\varepsilon[\mathrm{V}^2]^2\mathrm{V}$ for the groups ∞/m, ∞, 6/m, 6 and $\bar{6}$ **.

	$\frac{\partial\sigma_1}{\partial X_1}$	$\frac{\partial\sigma_2}{\partial X_1}$	$\frac{\partial\sigma_3}{\partial X_1}$	$\frac{\partial\sigma_4}{\partial X_1}$	$\frac{\partial\sigma_5}{\partial X_1}$	$\frac{\partial\sigma_6}{\partial X_1}$	$\frac{\partial\sigma_1}{\partial X_2}$	$\frac{\partial\sigma_2}{\partial X_2}$	$\frac{\partial\sigma_3}{\partial X_2}$	$\frac{\partial\sigma_4}{\partial X_2}$	$\frac{\partial\sigma_5}{\partial X_2}$	$\frac{\partial\sigma_6}{\partial X_2}$	$\frac{\partial\sigma_1}{\partial X_3}$	$\frac{\partial\sigma_2}{\partial X_3}$	$\frac{\partial\sigma_3}{\partial X_3}$	$\frac{\partial\sigma_4}{\partial X_3}$	$\frac{\partial\sigma_5}{\partial X_3}$	$\frac{\partial\sigma_6}{\partial X_3}$
Δg_1	0	0	0	β_{141}	β_{151}	0	0	0	0	β_{142}	β_{152}	0	β_{113}	β_{123}	β_{133}	0	0	β_{163}
Δg_2	0	0	0	$-\beta_{152}$	β_{142}	0	0	0	0	β_{151}	$-\beta_{141}$	0	β_{123}	β_{113}	β_{133}	0	0	$-\beta_{163}$
Δg_3	0	0	0	β_{341}	β_{351}	0	0	0	0	β_{351}	$-\beta_{341}$	0	β_{313}	β_{313}	β_{333}	0	0	0
Δg_4	β_{411}	β_{421}	β_{431}	0	0	β_{461}	β_{412}	β_{422}	β_{432}	0	0	β_{462}	0	0	0	β_{443}	β_{453}	0
Δg_5	β_{422}	β_{412}	β_{432}	0	0	$-\beta_{462}$	$-\beta_{421}$	$-\beta_{411}$	$-\beta_{431}$	0	0	β_{461}	0	0	0	$-\beta_{453}$	β_{443}	0
Δg_6	0	0	0	β_{641}	β_{651}	0	0	0	0	$-\beta_{651}$	β_{641}	0	$-\beta_{163}$	β_{163}	0	0	0	β_{663}

**Notice that $\beta_{663} = (\beta_{113} - \beta_{123})/2$, $\beta_{641} = (\beta_{151} - \beta_{521})/2$, $\beta_{461} = (\beta_{511} - \beta_{521})/2$, $\beta_{152} = \beta_{141} + \beta_{651}$, and $\beta_{421} = \beta_{561} + \beta_{411}$.

Table 6. Structure of tensor with the internal symmetry $\varepsilon[\mathrm{V}^2]^2\mathrm{V}$ for the groups 4/mmm, 422, 4mm and $\bar{4}2\mathrm{m}$.

	$\frac{\partial\sigma_1}{\partial X_1}$	$\frac{\partial\sigma_2}{\partial X_1}$	$\frac{\partial\sigma_3}{\partial X_1}$	$\frac{\partial\sigma_4}{\partial X_1}$	$\frac{\partial\sigma_5}{\partial X_1}$	$\frac{\partial\sigma_6}{\partial X_1}$	$\frac{\partial\sigma_1}{\partial X_2}$	$\frac{\partial\sigma_2}{\partial X_2}$	$\frac{\partial\sigma_3}{\partial X_2}$	$\frac{\partial\sigma_4}{\partial X_2}$	$\frac{\partial\sigma_5}{\partial X_2}$	$\frac{\partial\sigma_6}{\partial X_2}$	$\frac{\partial\sigma_1}{\partial X_3}$	$\frac{\partial\sigma_2}{\partial X_3}$	$\frac{\partial\sigma_3}{\partial X_3}$	$\frac{\partial\sigma_4}{\partial X_3}$	$\frac{\partial\sigma_5}{\partial X_3}$	$\frac{\partial\sigma_6}{\partial X_3}$
Δg_1	0	0	0	β_{141}	0	0	0	0	0	0	β_{152}	0	0	0	0	0	0	β_{163}
Δg_2	0	0	0	$-\beta_{152}$	0	0	0	0	0	0	$-\beta_{141}$	0	0	0	0	0	0	$-\beta_{163}$
Δg_3	0	0	0	β_{341}	0	0	0	0	0	0	$-\beta_{341}$	0	0	0	0	0	0	0
Δg_4	β_{411}	β_{421}	β_{431}	0	0	0	0	0	0	0	0	β_{462}	0	0	0	0	β_{453}	0
Δg_5	0	0	0	0	0	$-\beta_{462}$	$-\beta_{421}$	$-\beta_{411}$	$-\beta_{431}$	0	0	β_{431}	0	0	0	$-\beta_{453}$	0	0
Δg_6	0	0	0	0	β_{651}	0	0	0	0	$-\beta_{651}$	0	0	β_{613}	$-\beta_{613}$	0	0	0	0

Table 7. Structure of tensor with the internal symmetry $\varepsilon[V^2]^2V$ for the groups 4/m, 4 and $\bar{4}$.

	$\frac{\partial\sigma_1}{\partial X_1}$	$\frac{\partial\sigma_2}{\partial X_1}$	$\frac{\partial\sigma_3}{\partial X_1}$	$\frac{\partial\sigma_4}{\partial X_1}$	$\frac{\partial\sigma_5}{\partial X_1}$	$\frac{\partial\sigma_6}{\partial X_1}$	$\frac{\partial\sigma_1}{\partial X_2}$	$\frac{\partial\sigma_2}{\partial X_2}$	$\frac{\partial\sigma_3}{\partial X_2}$	$\frac{\partial\sigma_4}{\partial X_2}$	$\frac{\partial\sigma_5}{\partial X_2}$	$\frac{\partial\sigma_6}{\partial X_2}$	$\frac{\partial\sigma_1}{\partial X_3}$	$\frac{\partial\sigma_2}{\partial X_3}$	$\frac{\partial\sigma_3}{\partial X_3}$	$\frac{\partial\sigma_4}{\partial X_3}$	$\frac{\partial\sigma_5}{\partial X_3}$	$\frac{\partial\sigma_6}{\partial X_3}$
Δg_1	0	0	0	β_{141}	β_{151}	0	0	0	0	β_{142}	β_{152}	0	β_{113}	β_{123}	β_{133}	0	0	β_{163}
Δg_2	0	0	0	$-\beta_{152}$	β_{142}	0	0	0	0	β_{151}	$-\beta_{141}$	0	β_{123}	β_{113}	β_{133}	0	0	$-\beta_{163}$
Δg_3	0	0	0	β_{341}	β_{351}	0	0	0	0	β_{351}	$-\beta_{341}$	0	β_{313}	β_{313}	β_{333}	0	0	0
Δg_4	β_{411}	β_{421}	β_{431}	0	0	β_{461}	β_{412}	β_{422}	β_{432}	0	0	β_{462}	0	0	0	β_{443}	β_{453}	0
Δg_5	β_{422}	β_{412}	β_{432}	0	0	$-\beta_{462}$	$-\beta_{421}$	$-\beta_{411}$	$-\beta_{431}$	0	0	β_{461}	0	0	0	$-\beta_{453}$	β_{443}	0
Δg_6	0	0	0	β_{641}	β_{651}	0	0	0	0	$-\beta_{651}$	β_{641}	0	β_{613}	$-\beta_{613}$	0	0	0	β_{663}

From the viewpoint of bending-induced optical activity, the most interesting are crystals described by the trigonal symmetry, i.e. those belonging to the groups 32, 3m, $\bar{3}$m, 3, and $\bar{3}$ (see Tables 8 and 9). Spatial distributions of the stress component σ_2 along the X_1 axis in these crystals ($\partial\sigma_2/\partial X_1 \neq 0$) would induce the optical rotation along the axis X_3 (i.e., the optic axis). We would remind that these symmetry groups embrace such well-known crystalline materials as $LiNbO_3$, $LiTaO_3$ and SiO_2. Moreover, we are to comment that the optical rotation can be measured along the optic axis using the simplest and the most reliable direct method based upon measuring a polarization plane rotation.

Table 8. Structure of tensor with the internal symmetry $\varepsilon[V^2]^2V$ for the groups 32, 3m and $\bar{3}$m [+].

	$\frac{\partial\sigma_1}{\partial X_1}$	$\frac{\partial\sigma_2}{\partial X_1}$	$\frac{\partial\sigma_3}{\partial X_1}$	$\frac{\partial\sigma_4}{\partial X_1}$	$\frac{\partial\sigma_5}{\partial X_1}$	$\frac{\partial\sigma_6}{\partial X_1}$	$\frac{\partial\sigma_1}{\partial X_2}$	$\frac{\partial\sigma_2}{\partial X_2}$	$\frac{\partial\sigma_3}{\partial X_2}$	$\frac{\partial\sigma_4}{\partial X_2}$	$\frac{\partial\sigma_5}{\partial X_2}$	$\frac{\partial\sigma_6}{\partial X_2}$	$\frac{\partial\sigma_1}{\partial X_3}$	$\frac{\partial\sigma_2}{\partial X_3}$	$\frac{\partial\sigma_3}{\partial X_3}$	$\frac{\partial\sigma_4}{\partial X_3}$	$\frac{\partial\sigma_5}{\partial X_3}$	$\frac{\partial\sigma_6}{\partial X_3}$
Δg_1	0	0	β_{131}	β_{141}	0	0	0	0	0	0	β_{152}	0	0	0	0	0	β_{153}	β_{163}
Δg_2	0	β_{221}	$-\beta_{131}$	$-\beta_{152}$	0	0	0	0	0	0	β_{141}	β_{262}	0	0	0	0	$-\beta_{153}$	$-\beta_{163}$
Δg_3	β_{311}	$-\beta_{311}$	0	β_{341}	0	0	0	0	0	0	$-\beta_{341}$	$-\beta_{311}$	0	0	0	0	0	0
Δg_4	β_{411}	β_{421}	β_{431}	β_{441}	0	0	0	0	0	0	β_{441}	β_{462}	0	0	0	0	β_{453}	β_{463}
Δg_5	0	0	0	0	β_{551}	$-\beta_{462}$	$-\beta_{421}$	$-\beta_{411}$	$-\beta_{431}$	β_{551}	0	0	β_{463}	$-\beta_{463}$	0	$-\beta_{453}$	0	0
Δg_6	0	0	0	0	β_{651}	0	0	β_{622}	$-\beta_{131}$	$-\beta_{651}$	0	0	$-\beta_{613}$	β_{613}	0	$-\beta_{153}$	0	0

[+]Notice that $\beta_{221} = -2\beta_{622} - 2\beta_{262}$, $\beta_{241} = \beta_{141} + \beta_{651}$, and $\beta_{421} = \beta_{561} + \beta_{411}$.

Table 9. Structure of tensor with the internal symmetry $\varepsilon[V^2]^2V$ for the groups 3 and $\bar{3}$ [++].

	$\frac{\partial\sigma_1}{\partial X_1}$	$\frac{\partial\sigma_2}{\partial X_1}$	$\frac{\partial\sigma_3}{\partial X_1}$	$\frac{\partial\sigma_4}{\partial X_1}$	$\frac{\partial\sigma_5}{\partial X_1}$	$\frac{\partial\sigma_6}{\partial X_1}$	$\frac{\partial\sigma_1}{\partial X_2}$	$\frac{\partial\sigma_2}{\partial X_2}$	$\frac{\partial\sigma_3}{\partial X_2}$	$\frac{\partial\sigma_4}{\partial X_2}$	$\frac{\partial\sigma_5}{\partial X_2}$	$\frac{\partial\sigma_6}{\partial X_2}$	$\frac{\partial\sigma_1}{\partial X_3}$	$\frac{\partial\sigma_2}{\partial X_3}$	$\frac{\partial\sigma_3}{\partial X_3}$	$\frac{\partial\sigma_4}{\partial X_3}$	$\frac{\partial\sigma_5}{\partial X_3}$	$\frac{\partial\sigma_6}{\partial X_3}$
Δg_1	β_{111}	0	β_{131}	β_{141}	β_{151}	β_{161}	β_{112}	0	β_{132}	β_{142}	β_{152}	0	β_{113}	β_{123}	β_{133}	β_{143}	β_{153}	β_{163}
Δg_2	0	β_{221}	$-\beta_{131}$	$-\beta_{152}$	β_{142}	0	0	β_{222}	$-\beta_{132}$	β_{151}	β_{141}	β_{262}	β_{123}	β_{113}	β_{133}	$-\beta_{143}$	$-\beta_{153}$	$-\beta_{163}$
Δg_3	β_{311}	$-\beta_{311}$	0	β_{341}	β_{351}	β_{361}	β_{361}	$-\beta_{361}$	0	β_{351}	$-\beta_{341}$	$-\beta_{311}$	β_{313}	β_{313}	β_{333}	0	0	0
Δg_4	β_{411}	β_{421}	β_{431}	β_{441}	β_{451}	β_{461}	β_{412}	β_{422}	β_{432}	$-\beta_{451}$	β_{441}	β_{462}	β_{413}	$-\beta_{413}$	0	β_{443}	β_{453}	β_{463}
Δg_5	β_{422}	β_{412}	β_{432}	β_{451}	β_{551}	$-\beta_{462}$	$-\beta_{421}$	$-\beta_{411}$	$-\beta_{431}$	β_{551}	β_{451}	β_{461}	β_{463}	$-\beta_{463}$	0	$-\beta_{453}$	$-\beta_{443}$	β_{413}
Δg_6	β_{161}	0	β_{132}	β_{641}	β_{651}	$-\frac{1}{2}\beta_{222}$	0	β_{622}	$-\beta_{131}$	$-\beta_{651}$	β_{641}	$-\frac{1}{2}\beta_{111}$	$-\beta_{613}$	β_{613}	0	$-\beta_{153}$	β_{143}	β_{663}

[++]Notice that $\beta_{113} = \beta_{123} + 2\beta_{663}$, $\beta_{151} = \beta_{521} + 2\beta_{641}$, $\beta_{511} = \beta_{521} + 2\beta_{461}$, $\beta_{112} = -3\beta_{111} - 2\beta_{622} - 2\beta_{262}$, $\beta_{221} = -3\beta_{111} - 2\beta_{622} - 2\beta_{262}$, $\beta_{241} = \beta_{141} + \beta_{651} + \beta_{451}$, and $\beta_{421} = \beta_{561} + \beta_{541} + \beta_{411}$.

In the crystals of orthorhombic groups, no spatial distribution of the bending stresses can induce the optical rotation along the principal crystallographic axes (see Table 10).

In the monoclinic crystals belonging to the groups of symmetry 2/m, 2 and m (with $2 \parallel Y$ and $m \perp Y$ – see Table 11), the bending stresses $\partial\sigma_\mu/\partial X_2$ ($\mu = 1, 3$) should lead to the appearance of optical rotation along the principal crystallographic axes. Of course, the torsion-induced optical activity that appears along the torsion axis is a property of all the point symmetry groups, including the continuous-symmetry Curie groups. Finally, the matrix of the fifth-rank axial tensor for the crystals of triclinic system consists of 108 independent no zero components.

Table 10. Structure of tensor with the internal symmetry $\varepsilon[V^2]^2V$ for the groups mmm, 222 and mm2.

	$\frac{\partial\sigma_1}{\partial X_1}$	$\frac{\partial\sigma_2}{\partial X_1}$	$\frac{\partial\sigma_3}{\partial X_1}$	$\frac{\partial\sigma_4}{\partial X_1}$	$\frac{\partial\sigma_5}{\partial X_1}$	$\frac{\partial\sigma_6}{\partial X_1}$	$\frac{\partial\sigma_1}{\partial X_2}$	$\frac{\partial\sigma_2}{\partial X_2}$	$\frac{\partial\sigma_3}{\partial X_2}$	$\frac{\partial\sigma_4}{\partial X_2}$	$\frac{\partial\sigma_5}{\partial X_2}$	$\frac{\partial\sigma_6}{\partial X_2}$	$\frac{\partial\sigma_1}{\partial X_3}$	$\frac{\partial\sigma_2}{\partial X_3}$	$\frac{\partial\sigma_3}{\partial X_3}$	$\frac{\partial\sigma_4}{\partial X_3}$	$\frac{\partial\sigma_5}{\partial X_3}$	$\frac{\partial\sigma_6}{\partial X_3}$
Δg_1	0	0	0	β_{141}	0	0	0	0	0	0	β_{152}	0	0	0	0	0	0	β_{163}
Δg_2	0	0	0	β_{241}	0	0	0	0	0	0	β_{252}	0	0	0	0	0	0	β_{263}
Δg_3	0	0	0	β_{341}	0	0	0	0	0	0	β_{352}	0	0	0	0	0	0	β_{363}
Δg_4	β_{411}	β_{421}	β_{431}	0	0	0	0	0	0	0	0	β_{462}	0	0	0	0	β_{453}	0
Δg_5	0	0	0	0	0	β_{561}	β_{512}	β_{522}	β_{532}	0	0	0	0	0	0	β_{543}	0	0
Δg_6	0	0	0	0	β_{651}	0	0	0	0	β_{642}	0	0	β_{613}	β_{623}	β_{633}	0	0	0

Table 11. Structure of tensor with the internal symmetry $\varepsilon[V^2]^2V$ for the groups 2/m, 2 and m ($2 \parallel Y, \mathrm{m} \perp Y$).

	$\frac{\partial\sigma_1}{\partial X_1}$	$\frac{\partial\sigma_2}{\partial X_1}$	$\frac{\partial\sigma_3}{\partial X_1}$	$\frac{\partial\sigma_4}{\partial X_1}$	$\frac{\partial\sigma_5}{\partial X_1}$	$\frac{\partial\sigma_6}{\partial X_1}$	$\frac{\partial\sigma_1}{\partial X_2}$	$\frac{\partial\sigma_2}{\partial X_2}$	$\frac{\partial\sigma_3}{\partial X_2}$	$\frac{\partial\sigma_4}{\partial X_2}$	$\frac{\partial\sigma_5}{\partial X_2}$	$\frac{\partial\sigma_6}{\partial X_2}$	$\frac{\partial\sigma_1}{\partial X_3}$	$\frac{\partial\sigma_2}{\partial X_3}$	$\frac{\partial\sigma_3}{\partial X_3}$	$\frac{\partial\sigma_4}{\partial X_3}$	$\frac{\partial\sigma_5}{\partial X_3}$	$\frac{\partial\sigma_6}{\partial X_3}$
Δg_1	0	0	0	β_{141}	0	β_{161}	β_{112}	β_{122}	β_{132}	0	β_{152}	0	0	0	0	β_{143}	0	β_{163}
Δg_2	0	0	0	β_{241}	0	β_{261}	β_{212}	β_{222}	β_{232}	0	β_{252}	0	0	0	0	β_{243}	0	β_{263}
Δg_3	0	0	0	β_{341}	0	β_{361}	β_{312}	β_{322}	β_{332}	0	β_{352}	0	0	0	0	β_{343}	0	β_{363}
Δg_4	β_{411}	β_{421}	β_{431}	0	β_{451}	0	0	0	0	β_{442}	0	β_{462}	β_{413}	β_{423}	β_{433}	0	β_{453}	0
Δg_5	0	0	0	β_{541}	0	β_{561}	β_{512}	β_{522}	β_{532}	0	β_{552}	0	0	0	0	β_{543}	0	β_{563}
Δg_6	β_{611}	β_{621}	β_{631}	0	β_{651}	0	0	0	0	β_{642}	0	β_{662}	β_{613}	β_{623}	β_{633}	0	β_{653}	0

As shown in our work [12], the rank of the tensor β_{lnkmv} can be lowered down to four for the both cases of torsion and bending. Then Eq. (1) may be rewritten as

$$\Delta g_{ln} = \beta_{lnkmv}\partial\sigma_{km} / \partial X_v = \Omega_{lnim} \frac{\delta_{kiv}(\mathrm{Rot}\sigma)_{im}}{2-\delta_{km}} = \Omega_{lnim} M_{im} , \quad (2)$$

where δ_{km} implies the Kronecker delta, δ_{kiv} the Levi–Civita tensor, M_{im} the second-rank axial tensor associated with the torque and bending, and Ω_{lnim} the fourth-rank polar tensor with the internal symmetry $[V^2]V^2$. Our analysis of the structures of matrices of the tensors β_{lnkmv} and Ω_{lnim} has shown that the two alternative descriptions of the gradient piezogyration effect in terms of these tensors for the both cases of bending and torsion applied to crystals or textures agree fully with each other.

3. Conclusions

In the present work we have derived the fifth-rank axial tensor with the internal symmetry $\varepsilon[V^2]^2V$ that describes the gradient piezogyration effect for all of the point symmetry groups and the Curie groups of continuous symmetry. We have found that there exist twelve different structures of such a tensor that include the following groups: (1) $\infty/\infty/\mathrm{mmm}$ and $\infty/\infty2$; (2) m3m, 432 and $\bar{4}3\mathrm{m}$; (3) m3 and 23; (4) ∞/mmm, $\infty\mathrm{mm}$, $\infty2$, 6/mmm, 622, 6mm and $\bar{6}\mathrm{m}2$; (5) ∞/m, ∞, 6/m, 6 and $\bar{6}$; (6) 4/mmm, 422, 4mm and $\bar{4}2\mathrm{m}$; (7) 4/m, 4 and $\bar{4}$; (8) 32, 3m and $\bar{3}\mathrm{m}$; (9) 3 and $\bar{3}$; (10) mmm, 222 and mm2; (11) 2/m, 2 and m; and (12) 1 and $\bar{1}$.

We have demonstrated that the torsion stress should induce the optical rotation in crystals and textures of all symmetry groups, whenever the torque moment is applied around the axes of the principal coordinate system and the light beam propagates along the torsion axis. The mechanical-stress inhomogeneity of the bending type would induce the optical activity along the principal crystallographic axes only in a limited number of cases. Namely, this should occur for the symmetry groups ∞/m, ∞, 6/m, 6, $\bar{6}$, 4/m, 4, $\bar{4}$ and 2/m, 2, m, 32, 3m, $\bar{3}\mathrm{m}$, 3, and $\bar{3}$. Only in the trigonal groups the optical rotation would appear along the optic axis under the bending stress gradient $\partial\sigma_2/\partial X_1$. Finally, we have shown that the approaches describing the gradient piezogyration for the cases of bending and torsion in terms of the fifth-rank axial tensor and the fourth-rank polar tensor give rise to the same results and so fully agree with each other.

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Анотація. *Ми отримали матриці аксіального тензора п'ятого рангу із внутрішньою симетрією* $\varepsilon[V^2]^2V$ *, що описує градієнтний п'єзогіраційний ефект для всіх точкових груп симетрії та граничних груп симетрії Кюрі. Встановлено, що існує дванадцять різних структур такого тензора. Градієнтну п'єзогірацію проаналізовано для випадків кручення та згину кристалів і текстур.*