Pancharatnam's phase induced by spin-orbit interaction in weakly guiding twisted elliptical fibres

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Abstract

We derive the expressions for $l = 1$ modes of weakly guiding twisted elliptical fibres and the polarization corrections to the scalar propagation constant of these modes in the framework of Jones matrix formalism in the reflectionless approximation. Using these results, we demonstrate that the topological Pancharatnam's phase in the twisted elliptical fibre appears due to spin-orbit coupling in fibres and is absent in the scalar approximation.

Keywords: twisted fibre, Pancharatnam-Berry's phase, spin-orbit interaction

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1. Introduction

The notion of topological phase has been introduced in physics in 1984 by Berry to describe the phase accompanying a cyclic adiabatic evolution of quantum systems in the parameter space [1]. It has been demonstrated later on that there exist two different types of topological phases in optics. One of them is manifested, for instance, as a rotation of polarization plane of linearly polarized light upon its propagation in a coiled optical fibre [2–6]. This phase is known to be associated with adiabatic variation of the propagation direction of photon and is proportional to the solid angle subtended by the trajectory of wave vector in the momentum space [3]. On the contrary, the other type of topological phase accompanies the rectilinear propagation of light at a cyclic change of its polarization state. The existence of such a phase has been demonstrated by Pancharatnam [7] and its topological nature has been proved by Berry [8]. It is well-known that Pancharatnam-Berry's phase (or, simpler, Pancharatnam's phase abbreviated hereafter as PP) is proportional to the solid angle subtended by the trajectory of polarization on the Poincare′ sphere. Thus, in this case the parameter space is formed by the Stokes parameters. To detect and measure PP, there has been suggested a number of optical systems providing cyclic evolution of the polarization state of light (see, for example, [9–11]). The authors [12] demonstrate the appearance of such the phase in a medium with

Kerr nonlinearity, which provides a cyclic rotation of elliptically polarized beam. Another optical system, which seems to be quite capable of observing the PP, is twisted elliptical optical fibres.

The fibres mentioned above represent a separate class of perturbed fibres with periodical perturbation of their cross-section. Such perturbations can be created while manufacturing fibres and are technologically achieved either by spinning them [13] or with some other methods [14, 15]. The problem of properties of single-mode twisted fibres has been for decades evoking a steady amount of interest of both experimenters and theoreticians. Since its early appearance in the literature [16], great attention has been paid to study the properties of twisted and spun fibres in the presence of a regular twist of birefringence axes [17–19]. One of the main results of these investigations has been establishing the fact that the mode structure of regularly twisted single-mode fibres is presented by the so-called elliptic screw polarization modes. Recent surge of interest in this problem is related to a randomly varying elliptical birefringence [20–23]. Especial attention is being paid to the studies of influence of a spinning function on polarization mode dispersion [24–26]. One of the most important results of these researches is the established fact that spinning fibres reveal reduced polarization mode dispersion. However, the studies mentioned above have been mostly concerned with the fundamental mode behaviour. The problem of the propagation of higher-order modes ($l \geq 1$) in the twisted elliptical fibres, with accounting for vectorial nature of the electromagnetic field, has been solved in [27] in the framework of perturbation theory applied to the vector wave equation. However, the question of the topological phase of higher-order modes has not been considered in this work.

The main purposes of the present work are to establish the expression for PP in the case of $l = 1$ modes and to show that this phase is determined only by the relation between the twist rate and some constant induced by a spin-orbit interaction. Actually, we would like to bring attention to the fact that certain geometric phases in fibre optics are closely related with this specific interaction. We also provide another example of application of the Jones matrix formalism [28] to the problem of twisted fibres, which could serve as an alternative to the rigorous method for solving the vector wave equation for periodically perturbed fibres [27].

2. Generalized Jones matrices for *l=1* modes

Let us consider a weakly guiding twisted elliptical optical fibre. To introduce twisting, let us suppose that the principal axis of elliptic generatrix, which forms the elliptical fibre, uniformly changes its direction in the transverse plane, so that the inclination angle Ψ increases linearly with *z* coordinate of the cross-section (see Fig. 1): $\Psi = 2\pi z / H$, where H is the pitch parameter.

As is known, the most precise classical description of monochromatic light propagating in a medium with the refractive index $n(\vec{r})$ is provided by the vector wave equation, which can be easily derived from the Maxwell equations [29]:

$$\left(\vec{\nabla}^2 + n^2\left(x,y,z\right)k^2\right)\vec{E}\left(x,y,z\right) = -\vec{\nabla}\left(\vec{E}\left(x,y,z\right)\cdot\vec{\nabla}\ln n^2\left(x,y,z\right)\right), \tag{1}$$

where $\vec{\nabla} = \left(\partial/\partial x, \partial/\partial y, \partial/\partial z\right)$, $k = 2\pi/\lambda$, λ is the light wavelength and $\vec{E}$ the electric field. Usually, the refractive index in fibre optics is given by

$$n^2(x,y,z) = n_{co}^2(1 - 2\Delta f(x,y,z)), \tag{2}$$

where $\Delta = \left(n_{co}^2 - n_{cl}^2\right)/2n_{co}^2$ denotes the height of the refractive index profile, and n_{co} and n_{cl} are the values of the refractive index in the core and cladding, respectively. The profile function $f(x,y,z)$ determines the refractive index distribution. For instance, in the case of straight ideal fibre this function is axially symmetric ($f(x,y,z) = f(r)$, with r being the distance from the fibre axis). The term in the r.h.s. of Eq. (1) is conventionally called as the gradient term. It is evident that twisting of the fibre induces a dependence of refractive index distribution on the z coordinate. It also results in the coupling between the longitudinal and transverse components of the electric field vector due to the gradient term. Exactly, this fact brings about a remarkable difference in the propagation of light in straight and twisted fibres. However, it is possible to show that one can disregard such the coupling in a wide area of pitch values H. Indeed, the scalar product in the r.h.s. of Eq. (1) could be written as $\vec{E}_t\vec{\nabla}_t g + E_z\nabla_z g$, where $\vec{\nabla}_t = \left(\partial/\partial x, \partial/\partial y\right)$, $g \equiv \ln n^2\left(x,y,z\right)$ and $\nabla_z = \partial/\partial z$. Here, and throughout the remainder of the text, the subscript t stands for transverse part of a vector. The following estimate takes place for weakly guiding fibres [29]: $\frac{E_z}{E_t} \propto \frac{\lambda}{r_0} << 1$, where r_0 is the core radius. Since the scales of refractive index variations are r_0 and H respectively for the transverse and z directions, one has the following estimates for the derivatives: $\left|\vec{\nabla}_t g\right| \propto \Delta/r_0$ and $\nabla_z g \propto \delta \;\; \Delta/H$, with δ being the parameter related to the degree of ellipticity. Therefore, one can disregard the coupling between E_z and $\vec{E}_t$ at $H >> \lambda\delta$, i.e., for any reasonable value of the pitch.

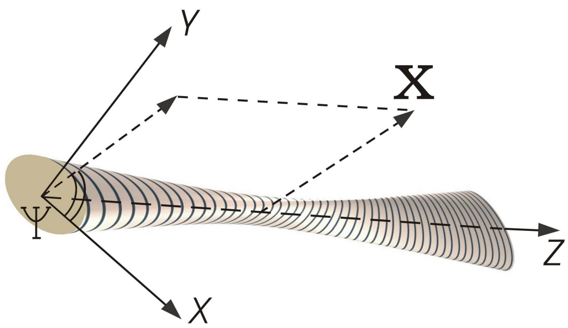

Fig. 1. Model of twisted elliptical optical fibre.

This enables us to write a closed equation in $\vec{E}_t$:

$$\left(\vec{\nabla}^2 + n^2\left(x,y,z\right)k^2\right)\vec{E}_t(x,y,z) = -\vec{\nabla}_t\left(\vec{E}_t(x,y,z)\cdot\vec{\nabla}_t \ln n^2\left(x,y,z\right)\right). \quad (3)$$

In the absence of vector term in the r.h.s., Eq. (3) is similar to the equation for electromagnetic field in a cholesteric liquid crystal. This allows us to use the ideas of Jones matrix formalism [28] when solving Eq. (3). According to the approach, one has to divide the fibre section between $z=0$ and $z=H$ cross-sections into N segments of the same length $\Delta z = H/N$ and consider the evolution of the field $\vec{E}_t$ through the obtained stack of segments, assuming that the orientation of anisotropy axis within each segment is fixed. In what follows, we will consider only strongly elliptical fibres, in which the ellipticity-induced coupling is much greater than the spin-orbit interaction. As is known [29,30], the modes of such fibres are represented by linearly polarized modes. At $l=1$ these modes can be written as

$$\begin{aligned} |1\rangle_L &= \begin{pmatrix} 0 \\ \cos\varphi \end{pmatrix} F_1(R), \qquad |2\rangle_L = \begin{pmatrix} \sin\varphi \\ 0 \end{pmatrix} F_1(R), \\ |3\rangle_L &= \begin{pmatrix} \cos\varphi \\ 0 \end{pmatrix} F_1(R), \qquad |4\rangle_L = \begin{pmatrix} 0 \\ \sin\varphi \end{pmatrix} F_1(R), \end{aligned} \quad (4)$$

where index L stands for the basis of linear polarizations ($\left|\vec{E}_t\right\rangle = \begin{pmatrix} E_x \\ E_y \end{pmatrix}$) and $R = r/r_0$. The radial function $F_l(R)$ satisfies the well-known equation [29]

$$\left[\frac{\partial^2}{\partial R^2} + \frac{1}{R}\frac{\partial}{\partial R} + k^2 n^2(R) - \frac{l^2}{R^2}\right] F_l(R) = \tilde{\beta}_l^2 F_l(R), \quad (5)$$

where $\tilde{\beta}$ is the scalar propagation constant and l an integer. The polarization corrections $\Delta\beta_i$ to the propagation constant $\tilde{\beta}$ ($\beta_i = \tilde{\beta} + \Delta\beta_i$) for the modes referred to in Eq. (4) have the following form [30]:

$$\begin{aligned} \Delta\beta_1 &\approx \frac{1}{2\tilde{\beta}}\left(|D_1| + 0.5A_1\right), \qquad & \Delta\beta_2 &\approx \frac{1}{2\tilde{\beta}}\left(-|D_1| + 0.5A_1\right), \\ \Delta\beta_3 &\approx \frac{1}{2\tilde{\beta}}\left(|D_1| + 0.5A_1 + B_1\right), \qquad & \Delta\beta_4 &\approx \frac{1}{2\tilde{\beta}}\left(-|D_1| + 0.5A_1 + B_1\right), \end{aligned} \quad (6)$$

where one has $A_1 = \frac{\Delta}{Q_1 r_0^2}\left(F_1 F_1' - F_1^2\right)_{R=1}$, $B_1 = \frac{\Delta}{Q_1 r_0^2}\left(F_1^2 + F_1 F_1'\right)_{R=1}$, $D_1 = \frac{\delta}{Q_1 r_0^2}\int_0^\infty \left(R F_1 F_1'' - \frac{F_1^2}{R}\right) dR$ and $Q_1 = \int_0^\infty R F_1^2(R)\, dR$ for the step-index fibres, in which the profile function is $f(r) = \theta(R-1)$ (with θ being the unit step) [30].

We will describe the field in each segment by the state vector $\left|\psi\right\rangle = (a_1, a_2, a_3, a_4)$, so that $\left|\psi\right\rangle = \sum_{i=1}^{4} a_i \left|i\right\rangle$ and the decomposition is carried out into the modes of Eq. (4). Consider two adjacent segments, whose anisotropy axes make an angle φ_0. In the reflectionless approximation, the relation between the vectors $\left|\Phi\right\rangle$ and $\left|\Phi'\right\rangle$ for the field state in these segments is found from the continuity condition for tangential components of $\vec{E}_t$ at the interface: $\vec{E}_{t1}(\vec{r}_t) \equiv \vec{E}_{t2}(\vec{r}_t)$, where $\vec{r}_t$ is the transverse component of $\vec{r}$. This connection has the form

$$\left|\Phi'\right\rangle = \frac{1}{2}\begin{pmatrix} 1+\cos 2\varphi_0 & -1+\cos 2\varphi_0 & -\sin 2\varphi_0 & \sin 2\varphi_0 \\ -1+\cos 2\varphi_0 & 1+\cos 2\varphi_0 & -\sin 2\varphi_0 & \sin 2\varphi_0 \\ \sin 2\varphi_0 & \sin 2\varphi_0 & 1+\cos 2\varphi_0 & 1-\cos 2\varphi_0 \\ -\sin 2\varphi_0 & -\sin 2\varphi_0 & 1-\cos 2\varphi_0 & 1+\cos 2\varphi_0 \end{pmatrix} \left|\Phi\right\rangle \equiv \Re(\varphi_0)\ \left|\Phi\right\rangle. \quad (7)$$

This matrix $\Re(\varphi)$ has the following composition property: $\Re(\varphi_1)\Re(\varphi_2) = \Re(\varphi_1 + \varphi_2)$. Each segment Δz acts upon the input field as a transforming element. In the laboratory frame, the field $\left|\psi'\right\rangle$ at the output segment end can be found by the input field $\left|\psi\right\rangle$: $\left|\psi'\right\rangle = \Re^{-1}(\varphi_0)\,\hat{\tau}(\Delta z)\Re(\varphi_0)\left|\psi\right\rangle$, where $\hat{\tau} = \mathrm{diag}\big(\exp(i\beta_1\Delta z),\ \exp(i\beta_2\Delta z),\ \exp(i\beta_3\Delta z),\ \exp(i\beta_4\Delta z)\big)$ is the phase matrix and 'diag' stands for diagonal matrix. The fibre section of length H transforms the input field $\left|\psi\right\rangle$ according to

$$\left|\psi'\right\rangle = \hat{T}_N(H)\ \left|\psi\right\rangle \equiv$$
$$e^{i\tilde{\beta}H} \lim_{N\to\infty}\left[\mathrm{diag}\left(\exp\left(i\frac{\Gamma_1}{N}\right), \exp\left(i\frac{\Gamma_2}{N}\right), \exp\left(i\frac{\Gamma_3}{N}\right), \exp\left(i\frac{\Gamma_4}{N}\right)\right)\Re\left(\frac{2\pi}{N}\right)\right]^N \left|\psi\right\rangle, \quad (8)$$

where $\Gamma_i = \Delta\beta_i \mathrm{H}$. The following is worth remarking: though the formalism presented above is a natural generalization of the Jones matrix method, unlike the Jones matrices, the $\Re$ ones operate in the four-dimensional modal subspace built on the eigenmodes $\left|i\right\rangle$, that belong to the same eigenvalue $\tilde{\beta}_l$ of the scalar wave equation. A general scheme of the Jones formalism, nevertheless, remains intact.

3. The *l=1* modes of twisted elliptical fibres

Let us define a mode of the twisted fibre as a state $\left|\psi\right\rangle$, which is transformed by the fibre section H into the state $\mu\left|\psi\right\rangle$, where μ is some numerical factor. Evidently, the vector $\left|\psi\right\rangle$ is eigenvector of the transformation operator $\hat{T}$: $\hat{T}_N(H)\left|\psi\right\rangle = \mu\left|\psi\right\rangle$, where μ is the corresponding eigenvalue. Within a phase factor, the matrix $\hat{T}_N(H)$ has an obvious limit:

$$\hat{\mathrm{T}} \propto \lim_{N\to\infty}\left(1+\frac{\hat{A}}{N}\right)^N = \exp\begin{pmatrix} i\Gamma_1 & 0 & -2\pi & 2\pi \\ 0 & i\Gamma_2 & -2\pi & 2\pi \\ 2\pi & 2\pi & i\Gamma_3 & 0 \\ -2\pi & -2\pi & 0 & i\Gamma_4 \end{pmatrix} \equiv \exp\hat{A}\,. \tag{9}$$

In order to find the eigenvectors and the spectrum of the matrix $\hat{T}$, it is sufficient to solve this problem for the operator $\hat{A}$. While the eigenvectors of $\hat{T}$ and $\hat{A}$ should coincide, the eigenvalue γ of $\hat{A}$ should be related with the eigenvalue μ through $\mu = \exp\gamma$. Since it has turned out to be impossible to provide compact analytical expressions for the eigenvectors of $\hat{A}$ at arbitrary relations among its elements, we study here only the case of relatively weak twist ($\Gamma_i >> 2\pi$), which is of practical importance. Using the perturbation theory with degeneracy [31], one can derive the expressions for the modes of weakly twisted elliptical fibres. In the local linear basis they look like

$$\begin{aligned} &|\psi_1\rangle_L = \sin\theta|1\rangle + i\cos\theta|3\rangle\,, \qquad &|\psi_3\rangle_L = \sin\theta|2\rangle - i\cos\theta|4\rangle\,, \\ &|\psi_2\rangle_L = \cos\theta|1\rangle - i\sin\theta|3\rangle\,, \qquad &|\psi_4\rangle_L = \cos\theta|2\rangle + i\sin\theta|4\rangle\,, \end{aligned} \tag{10}$$

where $\cot 2\theta = -|\tilde{B}_1|/4\pi$, $\pi/4 \le \theta \le \pi/2$ and $\tilde{X} = XH/2\tilde{\beta}$, $X = D_1, A_1$ or B_1. It should be emphasized that the angle φ, which the mode $|i\rangle$ depends on, is defined in the local frame of reference. The polarization corrections to the propagation constants of the modes referred to in Eq. (10) read as

$$\Delta\beta_{1,2} = \frac{1}{H}\left\{|\tilde{D}_1| + \frac{\tilde{A}_1+\tilde{B}_1}{2} \mp \Xi\right\}, \qquad \Delta\beta_{3,4} = \frac{1}{H}\left\{-|\tilde{D}_1| + \frac{\tilde{A}_1+\tilde{B}_1}{2} \mp \Xi\right\}, \tag{11}$$

where $\Xi \equiv \sqrt{(\tilde{B}_1/2)^2 + 4\pi^2}$. As follows from the definition of θ, there are two limiting cases of the weak twisting regime.

At $2\pi << |\tilde{B}_1|$, Eqs. (10), (11) give rise to the following expressions for the modes and polarization corrections:

$$\begin{aligned} &\Delta\beta_1 = \frac{1}{H}\left(|\tilde{D}_1| + \frac{\tilde{A}_1}{2} - \frac{4\pi^2}{\tilde{B}_1}\right), \qquad &|\psi_1\rangle_L \approx |1\rangle; \\ &\Delta\beta_2 = \frac{1}{H}\left(|\tilde{D}_1| + \frac{\tilde{A}_1}{2} + \tilde{B}_1 + \frac{4\pi^2}{\tilde{B}_1}\right), \qquad &|\psi_2\rangle_L \approx |3\rangle; \\ &\Delta\beta_3 = \frac{1}{H}\left(-|\tilde{D}_1| + \frac{\tilde{A}_1}{2} - \frac{4\pi^2}{\tilde{B}_1}\right), \qquad &|\psi_3\rangle_L \approx |2\rangle; \\ &\Delta\beta_4 = \frac{1}{H}\left(-|\tilde{D}_1| + \frac{\tilde{A}_1}{2} + \tilde{B}_1 + \frac{4\pi^2}{\tilde{B}_1}\right) \qquad &|\psi_4\rangle_L \approx |4\rangle\,. \end{aligned} \tag{12}$$

The local fibre modes in this case almost coincide with those of the straight elliptical fibre, and the effect of twisting manifest itself only through the appearance of corrections to the propagation constant. It should be remembered that the mode fields adiabatically trace the direction of the anisotropy axis.

In the case of intermediate twist ($\left|\tilde{D}_1\right| >> 2\pi >> \tilde{B}_1$) one has

$$\begin{aligned} \Delta\beta_{1,2} &= \frac{1}{H}\left(\left|\tilde{D}_1\right| + \frac{\tilde{A}_1 + \tilde{B}_1}{2} \mp 2\pi\right), \qquad \left|\psi_{1,2}\right\rangle_L = \left|1\right\rangle \pm i\left|3\right\rangle, \\ \Delta\beta_{3,4} &= \frac{1}{H}\left(-\left|\tilde{D}_1\right| + \frac{\tilde{A}_1 + \tilde{B}_1}{2} \mp 2\pi\right), \qquad \left|\psi_{3,4}\right\rangle_L = \left|2\right\rangle \mp i\left|4\right\rangle. \end{aligned} \tag{13}$$

The fields of the modes (13) could be written in some other form: e. g., $\left|\psi_1\right\rangle_L \propto F_1(r)\begin{pmatrix}1\\ i\end{pmatrix}\cos\varphi$, etc. One can also demonstrate that in the laboratory frame the mode field in the z cross-section is obtained from the initial field $\left|\psi_i\right\rangle_L$ at $z=0$ as

$$\left|\psi_i(z)\right\rangle_L = \Re^{-1}\left(\frac{2\pi z}{H}\right)\exp\left(\frac{z}{H}\hat{A}\right)\left|\psi_i\right\rangle_L \equiv \hat{T}(z)\left|\psi_i\right\rangle_L. \tag{14}$$

4. Pancharatnam's phase in twisted elliptical fibres

It is easily seen that, in the local rotating frame of reference, the modes given by Eqs. (10) have the same form in an arbitrary cross-section of the twisted elliptical fibre. Hence, the vector of polarization at any point in the cross-section returns to its original state as light has passed the period H, thus making a complete cycle. According to the general theory of PP [8], this cyclic evolution of polarization state has to be accompanied by the appearance of purely geometric phase γ_P. As is well-known, this phase is proportional to the solid angle Ω subtended by the trajectory of the vector of polarization on the Poincare′ sphere:

$$\gamma_P = -\frac{1}{2}\Omega. \tag{15}$$

To calculate Ω, one has to study evolution of the light polarization. Since the field in weakly guiding fibres is almost transverse, one can describe it with the Stokes parameters:

$$S_0 = e_x e_x^* + e_y e_y^*,\; S_1 = e_x e_x^* - e_y e_y^*,\; S_2 = e_x e_y^* + e_x^* e_y,\; S_3 = -i(e_x^* e_y - e_x e_y^*). \tag{16}$$

Consider application of these formulae to the mode $\left|\psi_2\right\rangle_L$. First, it is necessary to write this mode in the laboratory frame of reference, i.e., that corresponds to $z=0$. Using Eq. (14), one can obtain the relation

$$\left|\psi_2(z)\right\rangle_L = \cos(\varphi-\varphi_0)\begin{pmatrix}-\cos\theta\sin\varphi_0 - i\sin\theta\cos\varphi_0\\ \cos\theta\cos\varphi_0 - i\sin\theta\sin\varphi_0\end{pmatrix}F_1(R), \tag{17}$$

where $\varphi_0 = (2\pi/H)z$. Obviously, the mode $|\psi_2\rangle_L$ is completely polarized and has a uniform polarization distribution over the cross-section. The normalized Stokes parameters at the point (r,φ) in the cross-section, in which the principal ellipse axis is inclined at the angle φ_0, is given by

$$S_1 = -\cos 2\theta \cos 2\varphi_0 ,\ S_2 = -\cos 2\theta \sin 2\varphi_0 ,\ S_3 = \sin 2\theta ,\ S_0 = 1 . \tag{18}$$

Trivial analysis of Eqs. (18) shows that the evolution of polarization state of the mode $|\psi_2\rangle_L$ in the case of arbitrary twisting ($\pi/4 < \theta < \pi/2$) is represented by a circle trajectory on the Poincare´ sphere and the position of this trajectory is determined by θ (see Fig. 2). The trajectory degenerates to the north pole at $\theta = \pi/4$, thus describing the right-handed circular polarization, while at $\theta = \pi/2$ the mode is x-polarized. Apparently, no evolution of polarization vector takes place in these cases and the PP should be equal to zero.

At the same time, the solid angle can be found as $\tilde{\Omega} = 2\pi(1-\cos\chi)$ at $\pi/4 < \theta < \pi/2$, where χ is defined in Fig. 2. Taking the relation $\cos\chi = S_3$ and Eqs. (18) into account, one readily obtains the desired expression for the solid angle:

$$\tilde{\Omega} = 2\pi(1-\sin 2\theta) = 2\pi\left(1 - \frac{1}{\sqrt{1+\left(\tilde{B}_1/4\pi\right)^2}}\right). \tag{19}$$

To get the final expression for the topological phase γ_P, one should notice that the vector $\vec{S}(S_1, S_2, S_3)$ traces twice the depicted trajectory provided that the light has passed the period H. This fact follows immediately from Eq. (17) and means that the solid angle subtended by the total trajectory is obtained by doubling the solid angle given by Eq. (19). In this way, on the basis of Eqs. (15) and (19) we get

$$\gamma_P = 2\pi\left(\left(1+\left(\tilde{B}_1/4\pi\right)^2\right)^{-1/2} - 1\right). \tag{20}$$

In what follows, we will not take into consideration an unobservable phase 2π in γ_P and so will use the following expression for the PP:

$$\gamma_P = 2\pi / \sqrt{1+\left(\tilde{B}_1/4\pi\right)^2} . \tag{21}$$

Let us now calculate the total phase increment $\Delta\Phi_2 = \beta_2 H$ for the cycle H for the mode $|\psi_2\rangle_L$. Using Eq. (11), one can readily obtain

$$\Delta\Phi_2 = \gamma_d^{st} + 2\pi\sqrt{1+\left(\tilde{B}_1/4\pi\right)^2} - \tilde{B}_1/2 , \tag{22}$$

where $\gamma_d^{st} = \tilde{\beta} H + |\tilde{D}_1| + \frac{1}{2}(\tilde{A}_1 + 2\tilde{B}_1)$ denotes the phase increment, which would have taken place in a straight elliptical fibre of the length H [30]. It is easily seen that twisting-induced second and third terms in Eq. (22) do not coincide with the pure topological phase (21). This is explained by the fact that the twisting-induced phase correction in Eq. (22) is actually composed of two different types of phase: the PP and the phase induced by direct interaction between the light and twisted birefringent medium. Such a situation is typical for a gyrotropic medium, where the state of polarization accomplishes a complete cycle [12]. If we present the mode $|\psi_2\rangle_L$ in the form $|\psi_2\rangle \propto \cos\varphi \begin{pmatrix} 1 \\ i\varepsilon \end{pmatrix} F_1(r)$ (with $\varepsilon = \cot\theta$ being the ellipticity of the polarization ellipse), the total phase $\Delta\Phi_2$ can be rewritten with Eqs. (21) and (22) as

$$\Delta\Phi_2 = \gamma_d^{st} + \gamma_d^{tw} + \gamma_P, \tag{23}$$

where $\gamma_d^{tw} = 2\pi \frac{1-\varepsilon^2}{\varepsilon(1+\varepsilon^2)}$ and $\gamma_P = 4\pi \frac{\varepsilon}{1+\varepsilon^2}$. In this way, both the PP and the phase γ_d^{tw} are determined only by degree of ellipticity ε of the mode. This result is in a good agreement with the corresponding statements of the work [12].

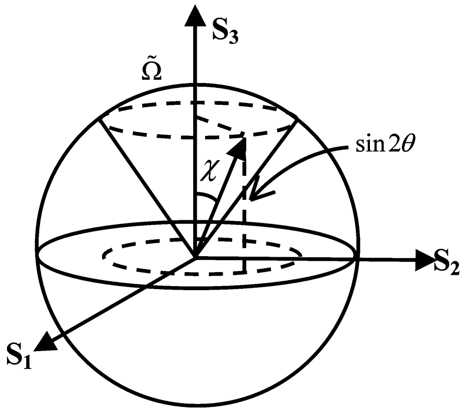

Fig. 2. Evolution of polarization state on the Poincare´ sphere.

However, one should emphasize that ε cannot acquire arbitrary values in the case of twisted elliptical fibre, as in the work [12]. In our case, it is rather determined by the material constants of fibre. In a sense, the topological PP depends therefore not only on the geometric factors, as the Berry's phase in the coiled fibre does, but also on the material parameters of twisted fibres through the initial position of the point on the Poincare´ sphere. It is also interesting to note that for strongly elliptical fibres ($|D_1| >> |A_1|, |B_1|, 2\pi\beta/H$), which are a subject of our consideration, neither the phase γ_d^{tw}, nor the PP depend on the degree of ellipticity of the fibre cross-section. They are determined only by the relation between the twisting rate and the spin-orbit interaction constants. Moreover, as follows from Eq. (21), the PP vanishes whenever the spin-orbit interaction tends to zero or is suppressed by the twisting. This is caused by the fact that in this limiting case the modes become circularly polarized (see Eqs. (13)) and no evolution of the polarization vector on the Poincare′ sphere takes place. This indicates some connection between the geometrical phase and the spin-orbit interaction. As has been pointed out in [32], the phase of higher-order modes in parabolic-index fibres can be attributed to the influence of Berry's phase associated with the helix-like motion of skew rays, which form those modes in the geometric optics approximation. The polarization correction to the scalar propagation constant has been shown in [33] to appear as a result of topological phase arising due to a helical form of the Poynting vector trajectories. Since such the corrections have been treated as manifestation of the spin-orbit interaction, there is a relation between this interaction and the geometrical phase [33]. Our example also demonstrates that a relation exists between the geometrical PP and the spin-orbit coupling. Being understood literally, Eq. (21) conveys just a dependence of the PP on the constant B_1 induced by spin-orbit coupling.

Conclusion

We have shown in the present paper that, in order to recognize the presence of the topological phase for *l*=1 modes in the twisted elliptical fibres, it is necessary to take into account the gradient term in the vector wave equation. Otherwise, in the scalar approximation one would not be able to obtain the correct expression for the PP at arbitrary values of the pitch.

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